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Abstract

In the following report we investigate stochastic particle motion in electron-positron storage rings in the framework of a Fokker-Planck treatment. The motion is described by using the canonical variables x, p_x, z, p_z, $\sigma = s - c \cdot t$, $p_\sigma = \Delta E/E_0$ of the fully six-dimensional formalism. Thus synchrotron- and betatron-oscillations are treated simultaneously taking into account all kinds of coupling (synchro-betatron coupling and the coupling of the betatron oscillations by skew quadrupoles and solenoids). In order to set up the Fokker-Planck equation, action-angle variables of the linear coupled motion are introduced. The averaged dimensions of the bunch, resulting from radiation damping of the synchro-betatron oscillations and from an excitation of these oscillations by quantum fluctuations, are calculated by solving the Fokker-Planck equation. The surfaces of constant density in the six-dimensional phase space, given by six-dimensional ellipsoids, are determined. It is shown that the motion of such an ellipsoid under the influence of external fields can be described by six generating orbit vectors which may be combined into a six-dimensional matrix $\underline{B}(s)$. This "bunch-shape matrix", $\underline{B}(s)$, contains complete information about the configuration of the bunch.

Classical spin diffusion in linear approximation has also been included so that the dependence of the polarization vector on the orbital phase space coordinates can be studied and another derivation of the linearized depolarization time obtained.

Contents

1 Introduction

In an earlier paper [1] we studied the influence of synchrotron radiation on spin-orbit motion within the framework of stochastic differential equations and calculated the damping constants, the beam emittance matrix and the depolarization time (τ_D) of the spin motion.

In this report we represent an alternative way of investigating orbital and classical spin motion, namely on the basis of a Fokker-Planck treatment. This has the advantage that, besides the calculation of the damping constants, the beam emittance matrix and the depolarization time, it is also possible to study the charge distribution in the phase space of the orbit motion. Furthermore, in this way additional insights into the spin depolarization process can be obtained.

To set up the Fokker-Planck equation, we follow an approach similar to one used by J.M. Jowett [2] but in addition concentrate on a fully coupled treatment of synchro-betatron motion.

To achieve this, we introduce in addition to the variables x, p_x, z, p_z describing the (transverse) betatron oscillations, the small and oscillating variables $\sigma = s - c \cdot t$ and $p_\sigma = \Delta E/E_0$ which describe the longitudinal motion.

With the complete set, x, p_x, z, p_z, σ, p_σ, we are then in a position to provide, in the framework of this six-dimensional formalism, a linear analytical technique which handles the external magnetic forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities.

The starting point of our investigation is the Lorentz equation and the BMT equation (chapter 2) applied to the motion of classical spins.

In chapter 3 the coordinate system of the orbital motion is introduced.

The stochastic equations of motion, taking into account the synchrotron radiation, are derived in chapters 2 - 4.

In chapter 5 a new reference orbit, the six-dimensional closed orbit, is introduced and the spin-orbit equations with respect to this new orbit are presented in a combined form.

The linear equations of spin-orbit motion, neglecting non-symplectic orbital terms (i.e. the unperturbed problem), are investigated in chapter 6 by defining the 8-dimensional transfer matrix and by studying the eigenvalue spectrum of the revolution matrix. Furthermore, action-angle variables for the coupled orbital motion are introduced.

The perturbed problem, taking into account the non-symplectic terms, is investigated in chapter 7.

We are then in a position to rewrite the equations of stochastic motion in terms of the orbital action-angle variables and the corresponding variables for the classical spin motion (chapter 8). These equations are the basis for a Fokker-Planck treatment of stochastic spin-orbit motion.

The Fokker-Planck equation for spin-orbit motion is finally discussed in chapter 9. Furthermore, analytical expressions for the damping constants of the (coupled) synchro-betatron oscillations are derived in this chapter and an alternative way of obtaining these constants is given in Appendix D where a simple proof of the well-known Robinson theorem is also presented.

A solution of the Fokker-Planck equation is derived in chapter 10.

For the orbital motion we only consider the stationary distribution and we show that the surfaces of constant density in the six- dimensional phase space are given by six-dimensional ellipsoids and that the motion of such an ellipsoid under the influence of external fields can be described by six generating orbit vectors which may be combined into a six-dimensional matrix $\underline{B}(s)$. This "bunch-shape matrix", $\underline{B}(s)$, contains complete information about the configuration of the bunch. The projections of this six-dimensional ellipsoid onto the different phase planes determine the beam envelopes (chapter 10.1).

For the spin motion, the linearized depolarization time (chapter 10.2) obtained by these methods agrees with the expression obtained previously but these methods allow the phase space dependence of the polarization axis to be calculated in addition [1] .

In Appendix E we prove that the stationary solution of the Fokker-Planck equation for the orbital motion is unique.

A summary of the results is presented in chapter 11.

Finally we remark that much of the contents of chapters 2 - 7 has already been derived in Ref. [1]. These results are only mentioned here again in order to present a complete set of definitions for setting up the Fokker-Planck equations. Readers familiar with the notation can begin at chapter 6.

2 Spin-Orbit Motion in an Electromagnetic Field

We begin the description of classical spin-orbit motion in electron-positron storage rings with a statement of the Lorentz- and the BMT-equations.

2.1 Orbital Motion (Lorentz-Equation)

The equation of motion for a relativistic charged particle in an electromagnetic field, the Lorentz-equation, is:

$$e \cdot \vec{\varepsilon} + \frac{e}{c} \cdot \dot{\vec{r}} \times \vec{B} + \vec{R} = \frac{d}{dt}\left(\frac{E}{c^2} \cdot \dot{\vec{r}}\right) \tag{2.1}$$

with

$$E = \frac{m_0 c^2}{\sqrt{1 - (\dot{\vec{r}})^2/c^2}} = \gamma \cdot m_0 c^2 \tag{2.2}$$

(energy of the particle)

and with the following definitions:

- e = charge of the particle (electron or positron) ;
- m_0 = rest mass of the particle ;
- c = velocity of light;
- $\vec{\varepsilon}$ = electric field;
- $\vec{B}$ = magnetic field ;

[1] A full semiclassical description of the polarization process in storage rings has been given by S.R. Mane [3] and by Ya.S. Derbenev and A.M. Kondratenko [4]. Part of the object for our investigation is to expose similarities between the classical and semiclassical treatments.

- $\vec{R}$ = radiation reaction force;
- $\vec{r}$ = radius vector of the particle;
- $\gamma = E/m_0c^2$.

We adopt an "ansatz" in which the radiation force in (2.1) is separated into two parts:

$$\vec{R} = \vec{R}^D + \delta\vec{R} \ , \tag{2.3}$$

a continuous part $\vec{R}^D$ describing the smoothed radiation process and a discontinuous part $\delta\vec{R}$ describing the quantum fluctuations. The explicit expression for $\vec{R}^D$ is given by [5,6,7]:

$$\vec{R}^D = -\frac{2}{3}\cdot\frac{e^2}{c^5}\gamma^4\cdot\dot{\vec{r}}\cdot\left[(\ddot{\vec{r}})^2 + \frac{\gamma^2}{c^2}(\dot{\vec{r}}\cdot\ddot{\vec{r}})^2\right] \tag{2.4}$$

and we model $\delta\vec{R}$ by a white noise process [2,8] with

$$< \delta\vec{R} > \;=\; 0 \ ; \tag{2.5a}$$

$$< \delta R_i(t)\delta R_j(t') > \;=\; C_{ij}(t)\cdot\delta(t-t') \tag{2.5b}$$

where $< \ >$ indicates a statistical average [9,10].

We also introduce the radiation power

$$P \equiv \vec{R}\cdot\dot{\vec{r}} = P^D + \delta P \tag{2.6}$$

of a (ultrarelativistic) particle in a purely magnetic field. For the case where

$$\dot{\vec{r}}\cdot\vec{B} = 0 \tag{2.7}$$

(a good approximation in storage rings) one then may write [2]:

$$P^D \;=\; E^2\cdot\frac{2\cdot r_e}{3\cdot(m_0c)^3}\cdot\left(\frac{e}{c}\cdot|\vec{B}|\right)^2 \tag{2.8}$$

and

$$< \delta P(s)\cdot\delta P(s') > \;=\; E^4\cdot\frac{55 r_e\hbar}{24\sqrt{3}\cdot(m_0c)^6}\cdot\left(\frac{e}{c}\cdot|\vec{B}|\right)^3\cdot\delta(s-s') \tag{2.9}$$

with

$$r_e = \frac{e^2}{m_0c^2} \tag{2.10}$$

where s designates the arc length of a reference orbit (see chapter 3).

Note that in purely magnetic fields and where (2.7) applies, (2.8) can be derived directly from (2.4).

The photons are emitted in the direction of the momentum of the particle with an opening angle of order

$$\Theta \approx \frac{m_0c^2}{E} \tag{2.11}$$

so that at high energy we are justified in taking the radiation reaction force to be collinear with $\dot{\vec{r}}$ [11,12].

With this assumption the coefficients $C_{ij}(t)$ in (2.5) are then determined by eqn. (2.9).

If we neglect the radiation force, eqn. (2.1) can be written in Lagrangian form:

$$\frac{\partial \mathcal{L}}{\partial \vec{r}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} = 0 \tag{2.12}$$

with the Lagrangian (see e.g. [13]) :

$$\mathcal{L} = -m_0 c^2 \cdot \sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c} \cdot (\dot{\vec{r}} \cdot \vec{A}) - e \cdot \phi \; ; \tag{2.13}$$

$$(v = |\dot{\vec{r}}|)$$

where $\vec{A}$ and ϕ are the vector and scalar potentials from which the electric field $\vec{\epsilon}$ and the magnetic field $\vec{B}$ are derived as

$$\vec{\epsilon} \;=\; -\text{grad}\;\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \; ; \tag{2.14a}$$

$$\vec{B} \;=\; \text{curl}\;\vec{A} \; . \tag{2.14b}$$

2.2 Spin Motion (BMT-Equation)

The equation of relativistic classical spin motion, the BMT-equation, reads as [14,15]:

$$\frac{d}{dt}\vec{\xi} \;=\; \vec{\Omega}_0 \times \vec{\xi} \tag{2.15a}$$

where

$$\frac{m_0 \gamma c}{e} \cdot \vec{\Omega}_0 \;=\; -(1 + \gamma a) \cdot \vec{B} + \frac{a\gamma^2}{1-\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}}\vec{B}) \cdot \dot{\vec{r}} + \left(a\gamma + \frac{\gamma}{1+\gamma}\right) \dot{\vec{r}} \times \frac{\vec{\epsilon}}{c} \; . \tag{2.15b}$$

The following abbreviations have been used:

- $\vec{\xi}$ = classical spin vector in the rest frame of the particle;
- a = $(g-2)/2$ which quantifies the anomalous electron g factor.

In this paper we do not include the spin polarizing effect of synchrotron radiation [11] but only a classical model for the depolarizing effect of orbit excitation. Therefore the Lorentz-equation (2.1) and the BMT-equation (2.15) together with the equations (2.8–10) for the radiation force cover the whole of the physics we need here for a Fokker-Planck treatment of stochastic orbit and classical spin motion.

3 Reference Trajectory and Coordinate Frame

The position vector $\vec{r}$ in eqns. (2.1) and (2.15) refers to a fixed coordinate system. However, in accelerator physics, it is useful to introduce the natural coordinates x, z, s in a suitable curvilinear coordinate system. With this in mind we assume that an ideal closed design orbit exists which describes the path of a particle of constant energy E_0, i.e. we neglect energy variations due to cavities and to radiation loss. In addition, to define the design orbit we ignore field errors and correction magnets. We also require that the design orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has (piecewise) no torsion. The design orbit which will be used as the reference system will, in the following, be described by the vector $\vec{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\vec{r}(s)$ is then described by the deviation $\delta\vec{r}(s)$ of the particle orbit $\vec{r}(s)$ from the design orbit $\vec{r}_0(s)$:

$$\vec{r}(s) = \vec{r}_0(s) + \delta\vec{r}(s) \ . \tag{3.1}$$

The vector $\delta\vec{r}$ can as usual [16] be described using an orthogonal coordinate system ("dreibein") accompanying the particles which travels along the design orbit and comprises

$$\begin{array}{rlcl} \text{the unit tangent vector} & \vec{e}_s(s) & = & \dfrac{d}{ds}\vec{r}_0(s) \ \equiv \ \vec{r}_0{}'(s) \ , \\ \text{a unit vector} & \vec{e}_x(s) & & \end{array}$$

which lies perpendicular to $\vec{e}_s$ in the horizontal plane [1]

$$\text{and the unit vector} \qquad \vec{e}_z(s) \ = \ \vec{e}_s(s) \times \vec{e}_x(s) \ .$$

In this natural coordinate system we may represent $\delta\vec{r}(s)$ as:

$$\delta\vec{r}(s) = (\delta\vec{r}\cdot\vec{e}_x)\cdot\vec{e}_x + (\delta\vec{r}\cdot\vec{e}_z)\cdot\vec{e}_z$$

(since the "dreibein" accompanies the particle the $\vec{e}_s$ - component of $\delta\vec{r}$ is always zero by definition).

Thus, the orbit-vector $\vec{r}(s)$ can be written in the form

$$\vec{r}(x,z,s) = \vec{r}_0(s) + x(s)\cdot\vec{e}_x(s) + z(s)\cdot\vec{e}_z(s) \tag{3.2}$$

and the Serret-Fresnet formulae for the dreibein $(\vec{e}_s, \vec{e}_x, \vec{e}_z)$ read as:

$$\frac{d}{ds}\,\vec{e}_x(s) \ = \ +K_x(s)\cdot\vec{e}_s(s) \ ; \tag{3.3a}$$

$$\frac{d}{ds}\,\vec{e}_z(s) \ = \ +K_z(s)\cdot\vec{e}_s(s) \ ; \tag{3.3b}$$

$$\frac{d}{ds}\,\vec{e}_s(s) \ = \ -K_x(s)\cdot\vec{e}_x(s) - K_z(s)\cdot\vec{e}_z(s) \tag{3.3c}$$

where we assume that

$$K_x(s)\cdot K_z(s) = 0 \tag{3.4}$$

(piecewise no torsion) and where $K_x(s), K_z(s)$ designate the curvatures in the x-direction and in the z-direction respectively.

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by eqns. (3.3).

From eqns. (3.2) and (3.3) one then has

$$\begin{aligned}\dot{\vec{r}} &= \dot{s}\cdot\left[\frac{d\vec{r}_0}{ds}+x\cdot\frac{d\vec{e}_x}{ds}+z\cdot\frac{d\vec{e}_z}{ds}\right]+\dot{x}\cdot\vec{e}_x+\dot{z}\cdot\vec{e}_z \\ &= \vec{e}_s\cdot\dot{s}\cdot(1-x\cdot K_x+z\cdot K_z)+\dot{x}\cdot\vec{e}_x+\dot{z}\cdot\vec{e}_z\end{aligned}$$

so that for the expressions

$$\sqrt{1-\frac{v^2}{c^2}} \quad \text{and} \quad (\dot{\vec{r}}\cdot\vec{A})$$

in eqn. (2.13) we have

$$\begin{aligned}\sqrt{1-\frac{v^2}{c^2}} &= \left\{1-\frac{1}{c^2}\cdot\left[\dot{x}^2+\dot{z}^2+(1+K_x\cdot x+K_z\cdot z)^2\cdot\dot{s}^2\right]\right\}^{1/2}\;; \\ (\dot{\vec{r}}\cdot\vec{A}) &= \dot{x}\cdot A_x+\dot{z}\cdot A_z+\dot{s}\,(1+K_x\cdot x+K_z\cdot z)\cdot A_s\end{aligned}$$

with

$$\vec{A}=A_x\cdot\vec{e}_x+A_z\cdot\vec{e}_z+A_s\cdot\vec{e}_s\;.$$

In the new coordinate system x, z, s, the Lagrangian in eqn. (2.13) then becomes

$$\begin{aligned}\mathcal{L}(x,z,s,\dot{x},\dot{z},\dot{s},t) &= -m_0c^2\left\{1-\frac{1}{c^2}\left[\dot{x}^2+\dot{z}^2+(1+K_x\cdot x+K_z\cdot z)^2\cdot\dot{s}^2\right]\right\}^{1/2} \\ &\quad+\frac{e}{c}\cdot\{\dot{x}\cdot A_x+\dot{z}\cdot A_z+\dot{s}\,(1+K_x\cdot x+K_z\cdot z)\cdot A_s\}-e\phi\end{aligned} \tag{3.5}$$

and eqn. (2.14) leads to

$$\varepsilon_x = -\frac{\partial\phi}{\partial x}-\frac{1}{c}\cdot\frac{\partial A_x}{\partial t}\;; \tag{3.6a}$$

$$\varepsilon_z = -\frac{\partial\phi}{\partial z}-\frac{1}{c}\cdot\frac{\partial A_z}{\partial t}\;; \tag{3.6b}$$

$$\varepsilon_s = -\frac{\partial\phi}{\partial s}-\frac{1}{c}\cdot\frac{\partial A_s}{\partial t} \tag{3.6c}$$

(cavities in the straight sections only) and

$$B_x = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial z}[(1+K_x\cdot x+K_z\cdot z)\cdot A_s]-\frac{\partial}{\partial s}A_z\right\}\;; \tag{3.7a}$$

$$B_z = \frac{1}{(1+K_x\cdot x+K_z\cdot z)}\cdot\left\{\frac{\partial}{\partial s}A_x-\frac{\partial}{\partial x}[(1+K_x\cdot x+K_z\cdot z)\cdot A_s]\right\}\;; \tag{3.7b}$$

$$B_s = \frac{\partial}{\partial x}A_z-\frac{\partial}{\partial z}A_x\;. \tag{3.7c}$$

Finally, the equations of motion

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \; ; \tag{3.8a}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{y}} - \frac{\partial\mathcal{L}}{\partial y} = 0 \; ; \tag{3.8b}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{s}} - \frac{\partial\mathcal{L}}{\partial s} = 0 \tag{3.8c}$$

take the form

$$\begin{aligned}\frac{d}{dt}[m_0\gamma\cdot\dot{x}] &= e\cdot\varepsilon_x + m_0\gamma\cdot(1+K_x\cdot x+K_z\cdot z)\cdot\dot{s}^2\cdot K_x \\ &\quad +\frac{e}{c}\cdot\{\dot{z}\cdot B_s - \dot{s}\cdot(1+K_x\cdot x+K_z\cdot z)\cdot B_z\} \; ;\end{aligned} \tag{3.9a}$$

$$\begin{aligned}\frac{d}{dt}[m_0\gamma\cdot\dot{z}] &= e\cdot\varepsilon_z + m_0\gamma\cdot(1+K_x\cdot x+K_z\cdot z)\cdot\dot{s}^2\cdot K_z \\ &\quad +\frac{e}{c}\cdot\{-\dot{x}\cdot B_s + \dot{s}\cdot(1+K_x\cdot x+K_z\cdot z)\cdot B_x\} \; ;\end{aligned} \tag{3.9b}$$

$$\begin{aligned}\frac{d}{dt}\left[m_0\gamma\cdot(1+K_x\cdot x+K_z\cdot z)\cdot\dot{s}\right] &= e\cdot\varepsilon_s - m_0\gamma\,(K_x\cdot\dot{x}+K_z\cdot\dot{z})\cdot\dot{s} \\ &\quad +\frac{e}{c}\cdot\{\dot{x}\cdot B_z - \dot{z}\cdot B_x\} \; .\end{aligned} \tag{3.9c}$$

4 The Equations of Motion

The variables $x(s)$ and $z(s)$ introduced in eqn. (3.2) can now be used to describe the transverse (betatron) oscillations. In order to describe the longitudinal (synchrotron) oscillations we introduce the additional variables $\sigma = s - c\cdot t$ and $p_\sigma = \Delta E/E_0$, where the quantity σ defines the longitudinal separation of the particle from the equilibrium particle and p_σ describes the energy deviation of the particle.

Using the arc length, s, as independent variable the equations of motion for spin and orbit then can be rewritten using the results of Appendix A, where the Hamiltonian $\mathcal{H}$ of the orbital motion is derived.

In order to include the radiation effects we note that by eqn. (2.11), $x'(s)$ and $z'(s)$ in (A.33) remain essentially unchanged during a photon emission. i.e. we neglect transverse recoil effects [11]. Then by energy conservation only a change in $p_\sigma(s)$ need be considered [2]. We do this via the relationship [2]

$$\begin{aligned}\Delta p_\sigma = -\frac{\partial\mathcal{H}}{\partial\sigma}\cdot\Delta s \quad\longrightarrow\quad & -\frac{\partial\mathcal{H}}{\partial\sigma}\cdot\Delta s - \frac{1}{E_0}\cdot\Delta t\cdot P(s) \\ &= \Delta s\cdot\left[-\frac{\partial\mathcal{H}}{\partial\sigma} - \frac{t'(s)}{E_0}\cdot P(s)\right] \\ &= \Delta s\cdot\left[-\frac{\partial\mathcal{H}}{\partial\sigma} - \frac{1}{E_0\cdot c}\cdot(1-\sigma')\cdot P(s)\right]\end{aligned}$$

which leads to the replacement of the relation

$$p'_\sigma(s) = -\frac{\partial\mathcal{H}}{\partial\sigma}$$

[2] See also the discussion in Ref. [2], Section 6.

of Appendix A by the relation

$$p'_\sigma = -\frac{\partial\mathcal{H}}{\partial\sigma} - \frac{P(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] \ . \tag{4.1}$$

Using eqns. (2.8) and (2.9), the radiation term in (4.1) becomes in first order:

$$\begin{aligned}
-\frac{P(s)}{E_0\cdot c}\cdot\left[1-\frac{\partial\mathcal{H}}{\partial p_\sigma}\right] \;=\; & -C_1\cdot[K_x^2+K_z^2] \\
& -C_1\cdot\left[(K_x^2+K_z^2)\cdot K_x+2K_x\cdot g\right]\cdot x \\
& -C_1\cdot\left[(K_x^2+K_z^2)\cdot K_z-2K_z\cdot g\right]\cdot z \\
& -2C_1\cdot[K_x^2+K_z^2]\cdot p_\sigma \\
& +\delta c
\end{aligned} \tag{4.2a}$$

with

$$\delta c = \mathcal{P}(s)\cdot\sqrt{(|K_x|^3+|K_z|^3)\cdot C_2} \tag{4.2b}$$

where the coefficents C_1 and C_2 are given by

$$C_1 = \frac{2}{3}\epsilon^2\cdot\frac{\gamma_0^4}{E_0}\ ; \tag{4.3a}$$

$$C_2 = \frac{55\cdot\sqrt{3}}{48}\cdot C_1\cdot\Lambda\cdot\gamma_0^2 \quad \text{with} \quad \Lambda=\frac{\hbar}{m_0c} \tag{4.3b}$$

and where by eqns. (2.9) and (A.8a, b) the factor $\mathcal{P}(s)$ of δc in eqn. (4.2) obeys the equation:

$$\langle\mathcal{P}(s),\mathcal{P}(s')\rangle = \delta(s-s')\ . \tag{4.4}$$

4.1 Orbital Motion

Combining now the equations (4.1), (4.2), (A.32), (A.33) and (A.36) we thus obtain:

$$\frac{d}{ds}\vec{y} = (\underline{A}+\delta\underline{A})\cdot\vec{y}+\vec{c}_0+\vec{c}_1+\delta\vec{c} \tag{4.5}$$

with

$$\vec{y} = \begin{pmatrix} x \\ p_x \\ z \\ p_z \\ \sigma \\ p_\sigma \end{pmatrix}\ ; \qquad \delta\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \delta c \end{pmatrix}\ ; \tag{4.6a}$$

$$\vec{c}_0^{\,T} = (0,0,0,0,0,\frac{\epsilon V}{E_0}\sin\varphi - C_1\cdot[K_x^2+K_z^2])\ ; \tag{4.6b}$$

$$\vec{c}_1^{\,T} = (0,-\frac{e}{E_0}\cdot\Delta B_z,0,+\frac{e}{E_0}\cdot\Delta B_x,0,0)\ ; \tag{4.6c}$$

$$\underline{A}(s) = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2+H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi & 0 \end{pmatrix} \tag{4.7}$$

and

$$\begin{aligned} \delta\underline{A} &= ((\delta A_{ik}))\ ; \\ \delta A_{22} &= -\frac{eV(s)}{E_0}\cdot\sin\varphi\ ; \\ \delta A_{44} &= \delta A_{22}\ ; \\ \delta A_{61} &= -C_1\cdot\left[(K_x^2+K_z^2)\cdot K_x + 2K_x\cdot g\right]\ ; \\ \delta A_{63} &= -C_1\cdot\left[(K_x^2+K_z^2)\cdot K_z - 2K_z\cdot g\right]\ ; \\ \delta A_{66} &= -2C_1\cdot(K_x^2+K_z^2)\ ; \\ \delta A_{ik} &= 0 \quad \text{otherwise}\ , \end{aligned} \tag{4.8}$$

where the lens functions $G_1,\ G_2,\ g,\ N,\ H,\ K_x,\ K_z$ are defined as follows:

$$g = \frac{e}{E_0}\cdot\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}\ ; \tag{4.9a}$$

$$N = \frac{1}{2}\cdot\frac{e}{E_0}\cdot\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_z}{\partial z}\right)_{x=z=0}\ ; \tag{4.9b}$$

$$H = \frac{1}{2}\cdot\frac{e}{E_0}\cdot B_s\ ; \tag{4.9c}$$

$$K_x = \frac{e}{E_0}\cdot B_z^{(0)}\ ;\quad K_z = -\frac{e}{E_0}\cdot B_x^{(0)}\ ; \tag{4.9d}$$

$$G_1 = K_x^2+g\ ;\quad G_2 = K_z^2-g\ . \tag{4.9e}$$

Here the matrix $\underline{A}(s)$ results from the Hamiltonian $\mathcal{H}_0$ (see eqn. (A.32a)), the vector $\vec{c}_1$ from the last two terms in the Hamiltonian $\mathcal{H}_1$ (see eqn. (A.32b)) and the vector $\vec{c}_0$ results from the first term of $\mathcal{H}_1$ (induced by the cavities) and the constant term in (4.2) (averaged energy loss by radiation).

The matrix $\delta\underline{A}(s)$ contains non-symplectic terms representing damping effects which are caused by acceleration fields (see eqn. (A.36)) and by radiation loss described by the linear terms in eqn. (4.2).

Finally we remark that the cavity phase φ is to be determined by the condition that the average energy radiated away in the bending magnets [3] :

$$E_0\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot C_1\left[K_x^2+K_z^2\right] = U_0$$

[3] Note that in this linearized treatment we ignore radiation originating in quadrupole and sextupole fields.

must be compensated by the average energy gain in the cavities:

$$E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{e\hat{V}}{E_0} \sin\varphi = U_{Cav} \ ,$$

i.e. we require that

$$E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \left[K_x^2 + K_z^2\right] = E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{e\hat{V}}{E_0} \sin\varphi = U_0 \ . \tag{4.10}$$

4.2 Spin Motion

Introducing again the arc length s of the design orbit as independent variable and using eqn. (3.4) and the relationship

$$\frac{d}{dt} = \frac{dl}{dt} \cdot \frac{d}{dl} = v \cdot \frac{ds}{dl} \cdot \frac{d}{ds} = v \cdot \frac{1}{l'} \cdot \frac{d}{ds} \tag{4.11}$$

where l is the length of the particle orbit with

$$dl = |\vec{r}\,'| \cdot ds,$$

we obtain:

$$l' = +\sqrt{(x')^2 + (z')^2 + (1 + K_x \cdot x + K_z \cdot z)^2} \tag{4.12}$$

so that the Thomas-BMT-equation (2.15) (with $v \approx c$) becomes:

$$\frac{d}{ds}\vec{\xi} \ = \ \frac{1}{c}\, l' \cdot (\vec{\Omega}_0 \times \vec{\xi}) \ . \tag{4.13}$$

Representing the spin vector $\vec{\xi}$ in the form

$$\vec{\xi} = \xi_s \cdot \vec{e}_s + \xi_x \cdot \vec{e}_x + \xi_z \cdot \vec{e}_z \tag{4.14}$$

and using eqn. (3.3) we have:

$$\begin{aligned} \frac{d}{ds}\vec{\xi} &= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z + \xi_x \cdot \frac{d}{ds}\,\vec{e}_x + \xi_s \cdot \frac{d}{ds}\,\vec{e}_s + \xi_z \cdot \frac{d}{ds}\,\vec{e}_z \\ &= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \xi_s \cdot (K_x \cdot \vec{e}_x + K_z \cdot \vec{e}_z) + \xi_x \cdot K_x \vec{e}_s + \xi_z \cdot K_z \vec{e}_s \\ &= \xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z - \vec{\xi} \times (K_z \cdot \vec{e}_x - K_x \cdot \vec{e}_z) \ . \end{aligned} \tag{4.15}$$

Thus eqn. (4.13) can be rewritten as:

$$\xi_s' \cdot \vec{e}_s + \xi_x' \cdot \vec{e}_x + \xi_z' \cdot \vec{e}_z = \vec{\Omega} \times \vec{\xi} \tag{4.16a}$$

with

$$\vec{\Omega} = \frac{l'}{c} \cdot \vec{\Omega}_0 - K_z \cdot \vec{e}_x - K_x \cdot \vec{e}_z \ . \tag{4.16b}$$

It follows from eqns. (4.15) and (4.16a) that for two spins $\vec{\xi}_1$ and $\vec{\xi}_2$ with the same $\vec{\Omega}(s)$ (i.e. the same position in the $(x - p_x - z - p_z - \sigma - p_\sigma)$ phase space) the scalar product

$$\vec{\xi}_1(s) \cdot \vec{\xi}_2(s)$$

is a constant of motion:

$$\begin{aligned} \frac{d}{ds}\left(\vec{\xi}_1(s) \cdot \vec{\xi}_2(s)\right) &= 0 \; ; \\ \Longrightarrow \quad \vec{\xi}_1(s) \cdot \vec{\xi}_2(s) &= \text{const} \end{aligned} \tag{4.17}$$

i.e. the modulus of $\vec{\xi}$ and the angle between $\vec{\xi}_1$ and $\vec{\xi}_2$ are invariants:

$$|\vec{\xi}(s)| = \text{const} \; ; \tag{4.18a}$$

$$\sphericalangle\,(\vec{\xi}_1(s), \vec{\xi}_2(s)) = \text{const} \; . \tag{4.18b}$$

In linear approximation $\vec{\Omega}$ is given by (see Appendix B):

$$\begin{aligned} \Omega_s &= -2H \cdot \left[1 + a\frac{\gamma_0}{1+\gamma_0}\right] + 2H \cdot \eta \cdot \left[1 + a\frac{\gamma_0^2}{(1+\gamma_0)^2}\right] \\ &\quad - a\gamma_0 \frac{\gamma_0}{1+\gamma_0} \cdot (x' K_x - z' K_z) \; ; \end{aligned} \tag{4.19a}$$

$$\begin{aligned} \Omega_x &= K_z \cdot a\gamma_0 + [1 + a\gamma_0] \cdot K_z^2 \cdot z - K_z \cdot \eta \\ &\quad -(1 + a\gamma_0) \cdot [(N - H') \cdot x + gz] + a\gamma_0 \frac{\gamma_0}{1+\gamma_0} \cdot 2Hx' \\ &\quad + \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \cdot z' - (1 + a\gamma_0) \cdot \frac{e}{E_0} \Delta B_x \; ; \end{aligned} \tag{4.19b}$$

$$\begin{aligned} \Omega_z &= -K_x \cdot a\gamma_0 - [1 + a\gamma_0] \cdot K_x^2 \cdot x + K_x \cdot \eta \\ &\quad +(1 + a\gamma_0) \cdot [(N + H') \cdot z - gx] + a\gamma_0 \frac{\gamma_0}{1+\gamma_0} \cdot 2Hz' \\ &\quad - \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \cdot x' - (1 + a\gamma_0) \cdot \frac{e}{E_0} \Delta B_z \end{aligned} \tag{4.19c}$$

($-$no solenoid field in the bending magnets and in the cavities $\Longrightarrow$ $K_x \cdot H = K_z \cdot H = 0$; $V \cdot H = 0$) with

$$\eta \equiv p_\sigma = \frac{\Delta E}{E_0} \; .$$

5 Introduction of a New Reference Orbit (Closed Orbit)

5.1 The Determining Equations of the New Orbit (Closed Orbit)

The equations (4.5) of the orbit form a system of linear and inhomogeneous differential equations with inhomogeneous parts $\delta\vec{c}$, $\vec{c}_0$ and $\vec{c}_1$. The term $\delta\vec{c}$ which is proportional to

$\hbar^{1/2}$ describes the quantum fluctuations of the radiation field and $\vec{c}_0$ is due to the variation of the energy of the circulating particles resulting from radiation losses and the presence of accelerating fields. The vector $\vec{c}_1$ originates from fields ΔB_x and ΔB_z which can be interpreted as field errors or perturbing external fields. The term $\delta\underline{A}$ which contains the accelerating fields and the radiation losses (see eqn. (4.8)) will be treated with perturbation theory.

The description of the orbital motion and the spin motion can now be simplified by eliminating the inhomogeneous parts $\vec{c}_0$ and $\vec{c}_1$ in eqn. (4.1). This is achieved in the usual manner by looking for the (unique) periodic solution $\vec{y}_0$ of the inhomogeneous equation

$$\vec{y}\,' = (\underline{A} + \delta\underline{A})\,\vec{y} + \vec{c}_0 + \vec{c}_1$$

namely

$$\begin{aligned} \vec{y}_0\,' &= (\underline{A} + \delta\underline{A})\,\vec{y}_0 + \vec{c}_0 - \vec{c}_1\;; && (5.1a)\\ \vec{y}_0(s_0 + L) &= \vec{y}_0(s_0)\;;\quad \text{(condition of periodicity)}\,. && (5.1b) \end{aligned}$$

Then the general solution of (4.5) can be separated into

$$\vec{y} = \vec{y}_0 + \vec{\tilde{y}} \tag{5.2}$$

where the vector $\vec{\tilde{y}}$ describes the synchro-betatron oscillations around the new closed equilibrium trajectory $\vec{y}_0$, which we call the "six-dimensional closed orbit".

5.2 The Linearized Equations of Orbital Motion with Respect to the Closed Orbit

The six-dimensional closed orbit may be determined by setting up the transfer matrices for the different types of lenses of a storage ring as demonstrated in Ref. [1]. Thus we assume in the following that $\vec{y}_0(s)$ is known.

By inserting (5.2) into (4.5) and using (5.1a) we then obtain:

$$\vec{\tilde{y}}\,'(s) = (\underline{A} + \delta\underline{A})\,\vec{\tilde{y}} + \delta\vec{c} \tag{5.3}$$

where the inhomogeneous parts $\vec{c}_0$ and $\vec{c}_1$ have indeed disappeared as required. Equation (5.3) now describes the free synchro-betatron oscillations around the new reference trajectory $\vec{y}_0(s)$.

Later we will need to use the fact that the orbit equation (5.3) without the radiation terms $\delta\underline{A}$ and $\delta\vec{c}$

$$\frac{d}{ds}\vec{\tilde{y}} = \underline{A}\cdot\vec{\tilde{y}} \tag{5.4}$$

can be written in canonical form

$$\frac{d}{ds}\,\vec{\tilde{y}} \;=\; -\underline{S}\cdot\frac{\partial\tilde{\mathcal{H}}}{\partial\vec{\tilde{y}}} \tag{5.5}$$

with the Hamiltonian

$$\begin{aligned} \tilde{\mathcal{H}} = \frac{1}{2}\cdot\Big\{&[\tilde{p}_x + H\cdot\tilde{z}]^2 + [\tilde{p}_z - H\cdot\tilde{x}]^2 + G_1\cdot\tilde{x}^2 + G_2\cdot\tilde{z}^2 - 2N\cdot\tilde{x}\tilde{z}\Big\}\\ &-\frac{1}{2}\tilde{\sigma}^2\cdot\frac{\epsilon V}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi - [K_x\cdot\tilde{x} + K_z\cdot\tilde{z}]\cdot\tilde{p}_\sigma \end{aligned} \tag{5.6}$$

where the matrix $\underline{S}$ is given by

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix} \; ; \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \, . \tag{5.7}$$

5.3 Spin Motion

5.3.1 Perturbation Theory

In analogy to the separation of the oscillation amplitude $\vec{y}$ into two parts we can divide the vector $\vec{\Omega}$ (see eqn. (4.19)) into two components, namely:

$$\vec{\Omega}\,(\vec{y}) = \vec{\Omega}^{(0)} + \vec{\omega} \tag{5.8}$$

with

$$\vec{\Omega}^{(0)} = \vec{\Omega}\,(\vec{y}_0) \tag{5.9a}$$

and

$$\vec{\omega} \equiv \vec{\Omega} - \vec{\Omega}^{(0)} = \vec{\omega}\,(\vec{\tilde{y}}) \, . \tag{5.9b}$$

The components of the precession vector $\vec{\Omega}^{(0)}$ are given by (eqns. (4.19) and (5.2)):

$$\begin{aligned} \Omega_s^{(0)} &= -2H\left[1 + a\frac{\gamma_0}{1+\gamma_0}\right] + 2H \cdot \eta_0 \cdot \left[1 + a\frac{\gamma_0^2}{(1+\gamma_0)^2}\right] \\ &\quad -a\gamma_0\frac{\gamma_0}{1+\gamma_0}(x_0' K_z - z_0' K_x) \; ; \end{aligned} \tag{5.10a}$$

$$\begin{aligned} \Omega_x^{(0)} &= K_z \cdot a\gamma_0 + [1 + a\gamma_0] \cdot K_z^2 \cdot z_0 - K_z \cdot \eta_0 \\ &\quad -(1 + a\gamma_0)[(N - H') \cdot x_0 + g z_0] + a\gamma_0\frac{\gamma_0}{1+\gamma_0} \cdot 2Hx_0' \\ &\quad + \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \cdot z_0' - (1 + a\gamma_0) \cdot \frac{e}{E_0}\Delta B_x \; ; \end{aligned} \tag{5.10b}$$

$$\begin{aligned} \Omega_z^{(0)} &= -K_x \cdot a\gamma_0 - [1 + a\gamma_0] \cdot K_x^2 \cdot x_0 + K_x \cdot \eta_0 \\ &\quad +(1 + a\gamma_0)[(N + H') \cdot z_0 - g x_0] + a\gamma_0\frac{\gamma_0}{1+\gamma_0} \cdot 2Hz_0' \\ &\quad - \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{e}{E_0} V(s) \sin\varphi \cdot x_0' - (1 + a\gamma_0) \cdot \frac{e}{E_0}\Delta B_z \, . \end{aligned} \tag{5.10c}$$

With the help of this precession vector

$$\vec{\Omega}^{(0)}(s) = \begin{pmatrix} \Omega_s^{(0)}(s) \\ \Omega_x^{(0)}(s) \\ \Omega_z^{(0)}(s) \end{pmatrix}$$

which describes the spin motion along the closed orbit we can introduce a suitable periodic reference frame for spin, $(\vec{n}_0,\ \vec{m},\ \vec{l})$, (see Appendix C) in which the unit spin vector $\vec{\xi}$ may be represented as

$$\vec{\xi} = \sqrt{1 - \alpha^2 - \beta^2} \cdot \vec{n}_0 + \alpha \cdot \vec{m} + \beta \cdot \vec{l} \, . \tag{5.11}$$

In this paper we retrict the discussion to the case where the spins are only slightly tilted with respect to $\vec{n}_0$ (i.e. $\alpha^2+\beta^2 \ll 1$). The formalism is linearized as in Appendix C, and the linearized equation of spin motion with respect to this spin frame is then given by:

$$\frac{d}{ds}\vec{\zeta} = \underline{G}_0 \cdot \vec{y} + \underline{D}_0 \cdot \vec{\zeta} \tag{5.12}$$

with

$$\vec{\zeta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{5.13}$$

and

$$\underline{G}_0 = \begin{pmatrix} l_s & l_x & l_z \\ -m_s & -m_x & -m_z \end{pmatrix} \cdot \underline{F}_{(3\times 6)} \; ; \tag{5.14a}$$

$$\underline{D}_0 = \begin{pmatrix} 0 & \psi' \\ -\psi' & 0 \end{pmatrix} . \tag{5.14b}$$

Here the matrix $\underline{G}_0$ describes the spin-orbit coupling. The function $\psi(s)$ designates the spin phase function and $\underline{F}_{(3\times 6)}$ is given by eqn. (C.26) of the Appendix C.

5.4 The Spin-Orbit Equations in a Combined Form

By combining the orbital part $\vec{y}$ and the spin part $\vec{\zeta}$ into an eight-dimensional vector as first done by A. Chao [11]:

$$\vec{u} = \begin{pmatrix} \vec{y} \\ \vec{\zeta} \end{pmatrix} \tag{5.15}$$

we can rewrite the orbital equation (5.3) and the spin equation (5.12) in a compact matrix notation as follows:

$$\frac{d}{ds}\vec{u} = \left(\underline{\hat{A}} + \delta\underline{\hat{A}}\right) \cdot \vec{u} + \delta\vec{\hat{c}} \tag{5.16}$$

with

$$\underline{\hat{A}} = \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{G}_0 & \underline{D}_0 \end{pmatrix} \; ; \tag{5.17a}$$

$$\delta\underline{\hat{A}} = \begin{pmatrix} \delta\underline{A} & \underline{0} \\ \underline{0} & \underline{0} \end{pmatrix} \; ; \tag{5.17b}$$

$$\delta\vec{\hat{c}} = \begin{pmatrix} \delta\vec{c} \\ 0 \\ 0 \end{pmatrix} . \tag{5.17c}$$

These equations describe the spin-orbit motion in a storage ring under the influence of radiation damping and radiation fluctuation.

In detail, one has:

a)	$g \neq 0$;	$N = H = K_x = K_z = V = 0$:	quadrupole;
b)	$N \neq 0$;	$g = H = K_x = K_z = V = 0$:	skew quadrupole;
c)	$K_x^2 + K_z^2 \neq 0$;	$g = N = H = V = 0$:	bending magnet;
d)	$H \neq 0$;	$g = N = K_x = K_z = V = 0$:	solenoid;
d)	$V \neq 0$;	$g = N = H = K_x = K_z = 0$:	cavity.

In the language of the theory of stochastic differential equations, eqn. (5.16) is a linear Langevin equation [8] with periodic s dependent coefficients and linear drift terms. As such, the equations of motion for the moments of the canonical variables $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$, can be solved exactly and the asymptotic (long time) distribution function over the phase space is a periodic Gaussian function with a correlation matrix given in terms of the asymptotic solution for the first and second moments. A detailed treatment of this problem has been given by [1,17,18]. Further material may be found in [8,19,20].

However, in this paper we wish to anticipate extensions to the work to include non-linear orbit motion and radiation (see for example Refs. [2,21]) and instead of working in the variables $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$, we choose to reexpress (5.16) in terms of action-angle variables. The transformation to action-angle variables will be made in section (6.2.1).

6 The Unperturbed Problem

As a first step in solving the spin-orbit motion it is reasonable to neglect in a first approximation the small terms $\delta\hat{\underline{A}}$ and $\delta\vec{c}$ and to consider only the "unperturbed problem"

$$\frac{d}{ds}\vec{u} = \hat{\underline{A}} \cdot \vec{u} \tag{6.1}$$

with the orbital part

$$\frac{d}{ds}\vec{y} = \underline{A} \cdot \vec{y} \tag{6.2}$$

and the spin part

$$\frac{d}{ds}\vec{\zeta} = \underline{G}_0 \cdot \vec{y} + \underline{D}_0 \cdot \vec{\zeta}\ . \tag{6.3}$$

The radiative perturbations described by $\delta\hat{\underline{A}}$ and $\delta\vec{c}$ will then be treated in a second step with perturbation theory.

6.1 Definition of the Transfer Matrix for Spin-Orbit Motion

Since eqn. (6.1) is linear and homogeneous, the solution can be written in the form:

$$\vec{u}(s) = \hat{\underline{M}}(s, s_0) \cdot \vec{u}(s_0) \tag{6.4}$$

which defines the transfer matrix $\hat{\underline{M}}(s, s_0)$ of spin-orbit motion.

By eqn. (6.1), $\hat{\underline{M}}(s, s_0)$ is determined by the differential equation

$$\begin{aligned} \frac{d}{ds}\ \hat{\underline{M}}(s, s_0) &= \hat{\underline{A}}(s) \cdot \hat{\underline{M}}(s, s_0)\ ; && (6.5a) \\ \hat{\underline{M}}(s_0, s_0) &= \underline{1}\ . && (6.5b) \end{aligned}$$

If we write $\hat{\underline{M}}$ as

$$\hat{\underline{M}} = \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} \tag{6.6}$$

we obtain from eqn. (6.5):

$$\begin{aligned} \frac{d}{ds} \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} &= \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{G}_0 & \underline{D}_0 \end{pmatrix} \begin{pmatrix} \underline{M} & \underline{0} \\ \underline{G} & \underline{D} \end{pmatrix} \\ &= \begin{pmatrix} \underline{AM} & \underline{0} \\ \underline{G}_0\underline{M} + \underline{D}_0\underline{G} & \underline{D}_0\underline{D} \end{pmatrix} \end{aligned} \tag{6.7a}$$

and

$$\begin{pmatrix} \underline{M}(s_0, s_0) & \underline{0} \\ \underline{G}(s_0, s_0) & \underline{D}(s_0, s_0) \end{pmatrix} = \underline{1} \tag{6.7b}$$

or

$$I) \quad \frac{d}{ds}\, \underline{M}(s, s_0) = \underline{A}(s) \cdot \underline{M}(s, s_0) \; ; \quad \underline{M}(s_0, s_0) = \underline{1} \; ; \tag{6.8}$$

$(\underline{M}(s, s_0) =$ transfer matrix for the orbit) ;

$$II) \quad \frac{d}{ds}\, \underline{D}(s, s_0) = \underline{D}_0(s) \cdot \underline{D}(s, s_0) \; ; \quad \underline{D}(s_0, s_0) = \underline{1}$$

$$\Rightarrow \quad \underline{D}(s, s_0) = \begin{pmatrix} \cos\,[\psi(s) - \psi(s_0)] & \sin\,[\psi(s) - \psi(s_0)] \\ -\sin\,[\psi(s) - \psi(s_0)] & \cos\,[\psi(s) - \psi(s_0)] \end{pmatrix} \; ; \tag{6.9}$$

$$III) \quad \frac{d}{ds}\, \underline{G}(s, s_0) = \underline{G}_0(s) \cdot \underline{M}(s, s_0) - \underline{D}_0(s) \cdot \underline{G}(s, s_0) \; ; \; \underline{G}(s_0, s_0) = \underline{0}$$

$$\begin{aligned} \Rightarrow \quad \underline{G}(s, s_0) &= \underline{D}(s, s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}(s_0, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \\ &= \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}(s, \tilde{s}) \cdot \underline{G}_0(\tilde{s}) \cdot \underline{M}(\tilde{s}, s_0) \; . \end{aligned} \tag{6.10}$$

By eqns. (6.8-10) the transfer matrix $\hat{\underline{M}}(s, s_0)$ is determined in a unique way.
In particular, one finds the following expressions for the revolution matrix $\hat{\underline{M}}(s_0 + L, s_0)$:

$$\hat{\underline{M}}(s_0 + L, s_0) = \begin{pmatrix} \underline{M}(s_0 + L, s_0) & \underline{0} \\ \underline{G}(s_0 + L, s_0) & \underline{D}(s_0 + L, s_0) \end{pmatrix} \tag{6.11}$$

with

$$\underline{D}(s_0 + L, s_0) = \begin{pmatrix} \cos\,[2\pi Q_{spin}] & \sin\,[2\pi Q_{spin}] \\ -\sin\,[2\pi Q_{spin}] & \cos\,[2\pi Q_{spin}] \end{pmatrix} \tag{6.12}$$

where the quantity Q_{spin} (the spin tune) is defined by eqns. (C.8) and (C.19a). Note that in this formalism we neglect Stern-Gerlach terms [22] so that the orbital motion is not influenced by the spin motion.

6.2 The Eigenvalue Spectrum of the Revolution Matrix; Floquet-Theorem

In order to define our action-angle variables we need to investigate the eigenvalue spectrum of the revolution matrix.

6.2.1 Orbital Motion alone

Firstly, we investigate the orbital motion alone which is described by the transfer matrix $\underline{M}$. In this case, the revolution matrix is symplectic:

$$\underline{M}^T(s,s_0)\cdot\underline{S}\cdot\underline{M}(s,s_0)=\underline{S} \tag{6.13a}$$

since the equations of motion for the orbit can be written in canonical form (see eqns. (5.4) and (5.5)), i.e. $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ are canonical variables. This condition can directly be derived from (5.5) and (5.7) [23].

Differentiating eqn. (6.13a) with respect to s and using (6.8), one obtains an alternative relation for symplecticity in the form:

$$\underline{A}^T(s)\cdot\underline{S}+\underline{S}\cdot\underline{A}(s)=0 \tag{6.13b}$$

which is equivalent to eqn. (6.13a).

The symplecticity condition (6.13a) or (6.13b) ensures that the transfer matrix, $\underline{M}(s,s_0)$, contains complete information about the stability of the synchro-betatron motion.

The following statements are then valid for the eigenvalue spectrum

$$\begin{aligned}\underline{M}(s_0+L,s_0)\,\vec{v}_\mu(s_0) &= \lambda_\mu\cdot\vec{v}_\mu(s_0)\ ; \\ (\mu &= 1,2,3,4,5,6)\end{aligned} \tag{6.14}$$

of $\underline{M}(s_0+L,s_0)$:

1) The eigenvectors of $\underline{M}$ can be separated into three groups

$$(\vec{v}_k,\ \vec{v}_{-k})\ ;\ k=I,\ II,\ III$$

with the properties

$$\underline{M}\ \vec{v}_k=\lambda_k\cdot\vec{v}_k\ ;\ \ \underline{M}\ \vec{v}_{-k}=\lambda_{-k}\cdot\vec{v}_{-k}\ ;\ \ \lambda_k\cdot\lambda_{-k}=1\ ; \tag{6.15a}$$

$$\left\{\begin{array}{l}[\vec{v}_{-k}(s_0)]^T\cdot\underline{S}\cdot\vec{v}_k(s_0)=-[\vec{v}_k(s_0)]^T\cdot\underline{S}\cdot\vec{v}_{-k}(s_0)\neq 0\ ;\\ \\ [\vec{v}_\mu(s_0)]^T\cdot\underline{S}\cdot\vec{v}_\nu(s_0)=0\ \ \text{otherwise}\ ;\end{array}\right. \tag{6.15b}$$

$$(k=I,II,III)\ .$$

In the following we put :

$$\left\{\begin{array}{l}\lambda_k=e^{-i\cdot 2\pi Q_k}\ ;\\ \\ \lambda_{-k}=e^{-i\cdot 2\pi Q_{-k}}\ ;\end{array}\right. \tag{6.16}$$

$$(k=I,\ II,\ III)\ .$$

Then using eqn. (6.15a) we get

$$Q_{-k} = -Q_k \ , \tag{6.17}$$

where the quantity Q_k can be either real or complex.

2) Eqns.(6.16) and (6.17) imply that the eigenvalues of $\underline{M}(s_0 + L, s_0)$ always appear in reciprocal pairs

$$\begin{gathered}(\lambda_k, \ \lambda_{-k} = 1/\lambda_k) \ ; \\ (k = I, II, III) \ .\end{gathered} \tag{6.18}$$

Since $\underline{M}(s_0 + L, s_0)$ is real, then λ^* as well as λ is an eigenvalue.

For the eigenvalue spectrum of $\underline{M}(s_0 + L, s_0)$ there are then the following possibilities :

a) All 6 eigenvalues are complex with unit absolute value and therefore lie on a unit circle in the complex plane :

$$\begin{gathered}|\lambda_k| = |\lambda_{-k}| = 1 \ ; \\ (k = I, II, III) \ .\end{gathered}$$

Then :

$$\begin{gathered}Q_k \ \ \text{real} \ ; \\ \lambda_k = \lambda^*_{-k} \ ; \ \ \vec{v}_{-k} = (\vec{v}_k)^* \ .\end{gathered} \tag{6.19}$$

b) One reciprocal pair is real and the others lie on a unit circle :

$$\begin{aligned}\lambda_I &= \lambda_I^* \ ; \ \ \lambda_{-I} = \lambda^*_{-I} \ ; \ \ \lambda_{-I} = 1/\lambda_I \ ; \\ \lambda_{-II} &= \lambda_{II}^* \ ; \ \ |\lambda_{II}| = |\lambda_{-II}| = 1 \ ; \\ \lambda_{-III} &= \lambda_{III}^* \ ; \ \ |\lambda_{III}| = |\lambda_{-III}| = 1 \ .\end{aligned}$$

c) Two reciprocal pairs are real and the third pair lies on the unit circle :

$$\begin{aligned}\lambda_I &= \lambda_I^* \ ; \ \ \lambda_{-I} = \lambda^*_{-I} \ ; \ \ \lambda_{-I} = 1/\lambda_I \ ; \\ \lambda_{II} &= \lambda_{II}^* \ ; \ \ \lambda_{-II} = \lambda^*_{-II} \ ; \ \ \lambda_{-II} = 1/\lambda_{II} \ ; \\ \lambda_{-III} &= \lambda_{III}^* \ ; \ \ |\lambda_{III}| = |\lambda_{-III}| = 1 \ .\end{aligned}$$

d) All reciprocal pairs are real :

$$\begin{gathered}\lambda_k = \lambda_k^* \ ; \ \ \lambda_{-k} = \lambda^*_{-k} \ ; \ \ \lambda_{-k} = 1/\lambda_k \ ; \\ (k = I, II, III) \ .\end{gathered}$$

e) One eigenvalue e.g. λ_I is complex and does not lie on the unit circle :

$$|\lambda_I| \neq 1 \ ; \ \ \lambda_I \neq \lambda_I^* \ .$$

Then we must have :

$$\lambda_{-I} = 1/\lambda_I$$

and

$$\lambda_{II} = \lambda_I^* \; ;$$
$$\lambda_{-II} = 1/\lambda_I^*$$

or

$$\lambda_{II} = 1/\lambda_I^* \; ;$$
$$\lambda_{-II} = \lambda_I^* \; .$$

The third, remaining pair must lie on the unit circle or on the real axis.

In the following it will become clear that only case a) leads to stable particle motion.

3) We write:

$$\vec{v}_\mu(s) = \underline{M}(s, s_0)\; \vec{v}_\mu(s_0) \; . \tag{6.20}$$

Then the vector $\vec{v}_\mu(s)$ is an eigenvector of the matrix $\underline{M}(s+L, s)$ with the eigenvalue λ_μ:

$$\underline{M}(s+L, s)\; \vec{v}_\mu(s) \;\; = \;\; \lambda_\mu \cdot \vec{v}_\mu(s) \; . \tag{6.21}$$

Proof:

$$\begin{aligned}
\underline{M}(s+L, s)\; \vec{v}_\mu(s) &= \underline{M}(s+L, s) \cdot \underline{M}(s, s_0) \vec{v}_\mu(s_0) \\
&= \underline{M}(s+L, s_0+L) \cdot \underline{M}(s_0+L, s_0)\; \vec{v}_\mu(s_0) \\
&= \underline{M}(s, s_0) \cdot \underline{M}(s_0+L, s_0)\; \vec{v}_\mu(s_0) \\
&= \lambda_\mu \cdot \underline{M}(s, s_0)\; \vec{v}_\mu(s_0) \\
&= \lambda_\mu \cdot \; \vec{v}_\mu(s) \; ; \quad q.e.d.
\end{aligned}$$

The eigenvector $\vec{v}_\mu(s)$ thus has the same eigenvalue as $\vec{v}_\mu(s_0)$: The eigenvalue is therefore independent of s.

4) We put

$$\vec{v}_\mu(s) = \vec{\hat{v}}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \; . \tag{6.22a}$$

Then :

$$\vec{\hat{v}}_\mu(s+L) = \vec{\hat{v}}_\mu(s) \; . \tag{6.22b}$$

Proof:

We put eqn. (6.22a) into (6.21). Using eqn. (6.16) we obtain :

$$\vec{\hat{v}}_\mu(s+L) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s+L)/L} = e^{-i \cdot 2\pi Q_\mu} \cdot \vec{\hat{v}}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot s/L} \; .$$

One now gets eqn. (6.22b) when one cancels the factor

$$e^{-i\cdot 2\pi Q_\mu\cdot(s+L)/L} = e^{-i\cdot 2\pi Q_\mu}\cdot e^{-i\cdot 2\pi Q_\mu\cdot s/L}$$

on each side.

Eqn. (6.22) is a statement of the Floquet theorem : Vectors $\vec{v}_\mu(s)$ are special solutions of the equations of motion (6.2) which can be expressed as the product of a periodic function $\vec{\tilde{v}}_\mu(s)$ and a harmonic function

$$e^{-i\cdot 2\pi Q_\mu\cdot(s/L)}\ .$$

5) The general solution of the equation of motion (6.2) is a linear combination of the special solutions (6.22a) and can be therefore written in the form

$$\vec{y}(s) = \sum_{k=I,II,III}\left\{A_k\cdot\vec{\tilde{v}}_k(s)\cdot e^{-i\cdot 2\pi Q_k\cdot(s/L)} + A_{-k}\cdot\vec{\tilde{v}}_{-k}(s)\cdot e^{+i\cdot 2\pi Q_{-k}\cdot(s/L)}\right\}. \quad (6.23)$$

We now see that the amplitude of the betatron oscillations only remains limited and the particle motion under control if the Q_k are real, i.e. if all eigenvalues, as already predicted, lie on the unit circle :

$$|\lambda_k| = |\lambda_{-k}| = 1\ ; \quad (k = I,\ II,\ III)\ ; \quad (6.24)$$

(Stability criterion) .

On the contrary, if at least one of the exponents Q_k is complex, according to (6.17) either Q_k or Q_{-k} has a positive imaginary part. In this case the components of $\vec{y}(s)$ grow exponentially and the motion is unstable.

6) In the following, we always assume that the stability condition (6.24) is satisfied.

Then from eqn. (6.19):

$$\vec{v}_{-k} = (\vec{v}_k)^*\ ; \quad (k = I,\ II,\ III), \quad (6.25)$$

and (6.15b) simplifies to $\left(\vec{v}^{+} = (\vec{v}^{\,T})^*\right)$:

$$\left\{\begin{array}{l} \vec{v}_k^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_k(s_0) = -\vec{v}_{-k}^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_{-k}(s_0)\neq 0\ ; \\ \\ \vec{v}_\mu^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_\nu(s_0) = 0 \quad \text{otherwise}\ ; \end{array}\right. \quad (6.26)$$

$$(k = I,\ II,\ III)\ .$$

Because the terms $\vec{v}_\mu^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_\mu(s_0)$ in eqn. (6.26) are purely imaginary :

$$\begin{aligned} \left[\vec{v}_\mu^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_\mu(s_0)\right]^{+} &= \vec{v}_\mu^{+}(s_0)\cdot\underline{S}^{+}\cdot\vec{v}_\mu(s_0) \\ &= -\left[\vec{v}_\mu^{+}(s_0)\cdot\underline{S}\cdot\vec{v}_\mu(s_0)\right] \\ &\quad \left(\text{since}\ \ \underline{S}^{+} = -\underline{S}\right) \end{aligned}$$

the vectors $\vec{v}_k(s_0)$ and $\vec{v}_{-k}(s_0)$ $(k = I,\ II,\ III)$ can be normalised from now on as:

$$\vec{v}_k^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_k(s_0) = -\vec{v}_{-k}^{\,+}(s_0) \cdot \underline{S} \cdot \vec{v}_{-k}(s_0) = i\ ; \qquad (k = I,\ II,\ III)\ . \tag{6.27}$$

From the validity of the symplecticity condition (6.13) it then follows that the vectors $\vec{v}_k(s)$ and $\vec{v}_{-k}(s)$ $(k = I,\ II,\ III)$ satisfy the conditions (6.26), (6.27) also at position s :

$$\begin{cases} \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_k(s) = -\vec{v}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_{-k}(s) = i\ ; \\ \\ \vec{v}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{v}_\nu(s) = 0 \quad \text{otherwise}\ . \end{cases} \tag{6.28}$$

The same relationships are fulfilled by the "Floquet-vectors" $\vec{\hat{v}}_\mu(s)$:

$$\begin{cases} \vec{\hat{v}}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\hat{v}}_k(s) = -\vec{\hat{v}}_{-k}^{\,+}(s) \cdot \underline{S} \cdot \vec{\hat{v}}_{-k}(s) = i\ ; \\ \\ \vec{\hat{v}}_\mu^{\,+}(s) \cdot \underline{S} \cdot \vec{\hat{v}}_\nu(s) = 0 \quad \text{otherwise}\ . \end{cases} \tag{6.29}$$

7) Using these results we are now able to introduce a new set of canonical variables in which to write the Fokker-Planck equation.

To do this we express the coefficients A_k, A_{-k} $(k = I, II, III)$ in eqn. (6.23) as :

$$A_k = \sqrt{J_k} \cdot e^{-i[\Phi_k - 2\pi Q_k \cdot s/L]}\ ; \tag{6.30a}$$

$$A_{-k} = \sqrt{J_k} \cdot e^{+i[\Phi_k - 2\pi Q_k \cdot s/L]}\ . \tag{6.30b}$$

Then eqn. (6.23) takes the form:

$$\vec{y}(s) = \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \vec{\hat{v}}_k(s) \cdot e^{-i\Phi_k} + \vec{\hat{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\}\ . \tag{6.31}$$

From (6.31) we now have:

$$\frac{\partial \vec{y}}{\partial \Phi_k} = -i \cdot \sqrt{J_k} \cdot \left\{ \vec{\hat{v}}_k(s) \cdot e^{-i\Phi_k} - \vec{\hat{v}}_{-k}(s) \cdot e^{+i\Phi_k} \right\}\ ; \tag{6.32a}$$

$$\frac{\partial \vec{y}}{\partial J_k} = +\frac{1}{2\sqrt{J_k}} \cdot \left\{ \vec{\hat{v}}_k(s) \cdot e^{-i\Phi_k} + \vec{\hat{v}}_{-k}(s) \cdot e^{-i\Phi_k} \right\}\ . \tag{6.32b}$$

Taking into account the relations (6.29) we obtain the equations:

$$\frac{\partial \vec{y}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = -\frac{\partial \vec{y}}{\partial \Phi_l}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_k} = \delta_{kl}\ ; \tag{6.33a}$$

$$\frac{\partial \vec{y}}{\partial J_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial J_l} = \frac{\partial \vec{y}}{\partial \Phi_k}^T \cdot \underline{S} \cdot \frac{\partial \vec{y}}{\partial \Phi_l} = 0 \tag{6.33b}$$

which can be combined into the matrix form

$$\underline{\mathcal{J}}^T \cdot \underline{S} \cdot \underline{\mathcal{J}} = \underline{S} \tag{6.34}$$

where $\underline{\mathcal{J}}$ signifies the Jacobian matrix

$$\underline{\mathcal{J}} = \left(\frac{\partial\vec{y}}{\partial\Phi_I}, \frac{\partial\vec{y}}{\partial J_I}, \frac{\partial\vec{y}}{\partial\Phi_{II}}, \frac{\partial\vec{y}}{\partial J_{II}}, \frac{\partial\vec{y}}{\partial\Phi_{III}}, \frac{\partial\vec{y}}{\partial J_{III}}\right) \tag{6.35}$$

which is 6×6-matrix just written as a row of column vectors $(\partial\vec{y}/\partial\Phi_I)$ etc.

Equation (6.34) proves that eqn. (6.31) represents a canonical transformation [13]

$$\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma \longrightarrow \Phi_I,\ J_I,\ \Phi_{II},\ J_{II},\ \Phi_{III},\ J_{III} \tag{6.36}$$

and that $\Phi_k,\ J_k\ (k = I, II, III)$ are indeed canonical variables which can now be interpreted as action-angle variables since

$$\frac{dJ_k}{ds} = 0 \quad \Longrightarrow \quad J_k = const; \tag{6.37a}$$

$$\frac{d\Phi_k}{ds} = \frac{2\pi}{L} Q_k \quad \Longrightarrow \quad \Phi_k = \frac{2\pi}{L} Q_k \cdot s + const \tag{6.37b}$$

in the unperturbed system.

The corresponding orbital Hamiltonian is

$$\mathcal{H} = \frac{2\pi}{L} \sum_k J_k \cdot Q_k \tag{6.38}$$

and eqns. (6.37) are the resulting canonical equations of motion. (In Ref. [24] it is shown how to construct the Hamiltonian (6.38) in a general way starting from the Hamiltonian (5.6) in terms of the variables $\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma$.)

The orbit vector $\vec{y}(s)$ in (6.31) is thus an explicit function of the canonical variables J_k and Φ_k and of the longitudinal variable, s, via the eigenvectors, $\vec{v}_k(s)$.

8) For the limiting case of a vanishing coupling between the synchro-betatron oscillations the revolution-matix, $\underline{M}(s+L, s)$ takes the form:

$$\underline{M}(s+L,s) = \begin{pmatrix} \underline{m}_x(s+L,s) & \underline{0}_2 & \underline{0}_2 \\ \underline{0}_2 & \underline{m}_z(s+L,s) & \underline{0}_2 \\ \underline{0}_2 & \underline{0}_2 & \underline{m}_\sigma(s+L,s) \end{pmatrix} . \tag{6.39}$$

The symplecticity condition (6.13) now reads:

$$\underline{m}_y^T \cdot \underline{S}_2 \cdot \underline{m}_y = \underline{S}_2 \tag{6.40}$$

$(y = \tilde{x},\ \tilde{z},\ \tilde{\sigma})$ or

$$\det\ (\underline{m}_y) = 1\ . \tag{6.41}$$

Generally, the corresponding submatrices

$$\underline{d}_x = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} ;$$

$$\underline{d}_z = \begin{pmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{pmatrix} ;$$

$$\underline{d}_\sigma = \begin{pmatrix} M_{55} & M_{56} \\ M_{65} & M_{66} \end{pmatrix}$$

of the revolution matrix for the coupled synchro-betatron oscillations have determinants differing from 1. Therefore we may consider the difference

$$\det\,(\underline{d}_y) - \det\,(\underline{m}_y) = \det\,(\underline{d}_y) - 1$$

of the determinant, det $(\underline{d}_y)$, from the value 1 as a measure for the coupling strength of the betatron and synchrotron oscillations at the position s [25].

According to Courant-Snyder [16] we now can write for the revolution matrix $\underline{m}_y$:

$$\underline{m}_y(s+L,s) = \cos 2\pi Q_y \cdot \underline{1} + \sin 2\pi Q_y \cdot \underline{K}_y(s) \tag{6.42a}$$

with

$$\underline{K}_y(s) = \begin{pmatrix} \alpha_y(s) & \beta_y(s) \\ -\gamma_y(s) & -\alpha_y(s) \end{pmatrix} \tag{6.42b}$$

and

$$\beta_y \cdot \gamma_y - \alpha_y^2 = 1 \tag{6.43}$$

where in addition we require:

$$\beta_y \geq 0\ . \tag{6.44}$$

Using this representation of $\underline{m}_y(s+L,s)$ we may calculate the normalized eigenvectors of the revolution matrix (6.38):

$$\vec{v}_I = \begin{pmatrix} \vec{w}_x \\ \vec{0}_2 \\ \vec{0}_2 \end{pmatrix}\ ; \tag{6.45a}$$

$$\vec{v}_{II} = \begin{pmatrix} \vec{0}_2 \\ \vec{w}_z \\ \vec{0}_2 \end{pmatrix}\ ; \tag{6.45b}$$

$$\vec{v}_{III} = \begin{pmatrix} \vec{0}_2 \\ \vec{0}_2 \\ \vec{w}_\sigma \end{pmatrix} \tag{6.45c}$$

with the eigenvalues:

$$\begin{aligned} \lambda_I &= e^{-i\cdot 2\pi Q_x}\ ; \\ \lambda_{II} &= e^{-i\cdot 2\pi Q_z}\ ; \\ \lambda_{III} &= e^{-i\cdot 2\pi Q_\sigma} \end{aligned} \tag{6.46}$$

and where the vector $\vec{w}_y$ $(y = \tilde{x},\ \tilde{z},\ \tilde{\sigma})$ is given by

$$\vec{w}_y(s) = \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s)+i] \end{pmatrix} \cdot e^{-i\cdot\psi_y(s)}\ . \tag{6.47}$$

Comparing (6.45) and (6.16) we can make the following identifications for the decoupled case:

$$\begin{aligned} Q_I &\longleftrightarrow Q_x\ ; \\ Q_{II} &\longleftrightarrow Q_z\ ; \\ Q_{III} &\longleftrightarrow Q_\sigma\ . \end{aligned}$$

The stability condition (6.24) then reads:

$$Q_x, \; Q_z, \; Q_\sigma \quad \text{real}$$

or using (6.42):

$$-2 \leq \mathrm{Sp}\;(\underline{m}_y) \;\leq +2\;. \tag{6.48}$$

In order to obtain differential equations for the "Twiss parameters" $\alpha_y(s), \;\beta_y(s), \;\gamma_y(s)$ and the phase function, $\psi_y(s)$, defined by (6.42) and (6.47) we remark that the Hamiltonian for uncoupled synchro-betatron oscillations in its most general form can be written as:

$$\mathcal{H}_0 = \mathcal{H}_{0\tilde{x}} + \mathcal{H}_{0\tilde{z}} + \mathcal{H}_{0\tilde{\sigma}} \tag{6.49}$$

with $(y \equiv \tilde{x}, \tilde{z}, \tilde{\sigma})$

$$\mathcal{H}_{0y} = \frac{1}{2} F_y(s) \cdot p_y^2 + R_y \cdot y \cdot p_y + \frac{1}{2} G_y(s) \cdot y^2 \tag{6.50}$$

from which result the corresponding canonical equations of motion become:

$$\frac{d}{ds}\begin{pmatrix} y \\ p_y \end{pmatrix} = \underline{A}_y \cdot \begin{pmatrix} y \\ p_y \end{pmatrix} \tag{6.51}$$

with

$$\underline{A}_y(s) \;=\; \begin{pmatrix} R_y & F_y \\ -G_y & -R_y \end{pmatrix}\;. \tag{6.52}$$

Furthermore, from the condition

$$\begin{aligned} \underline{m}_y(s+L, s) &= \underline{m}_y(s+L, s_0+L) \cdot \underline{m}_y(s_0+L, s_0) \cdot \underline{m}_y(s_0, s) \\ &= \underline{m}_y(s, s_0) \cdot \underline{m}_y(s_0+L, s_0) \cdot \underline{m}_y^{-1}(s, s_0) \end{aligned} \tag{6.53}$$

and using (6.42a), we obtain:

$$\underline{K}_y(s) \;=\; \underline{m}_y(s, s_0) \cdot \underline{K}_y(s_0) \cdot \underline{m}_y^{-1}(s, s_0)\;. \tag{6.54}$$

For the derivative w.r.t. s

$$\underline{K}'_y(s) \;\equiv\; \begin{pmatrix} \alpha'_y(s) & \beta'_y(s) \\ -\gamma'_y(s) & -\alpha'_y(s) \end{pmatrix} \tag{6.55}$$

we now get:

$$\begin{aligned} \underline{K}'_y(s) &= \frac{1}{\Delta s} \cdot \lim_{\Delta s \to 0} \left\{ \underline{K}_y(s+\Delta s) - \underline{K}_y(s) \right\} \\ &= \frac{1}{\Delta s} \cdot \lim_{\Delta s \to 0} \left\{ \underline{m}_y(s+\Delta s, s) \cdot \underline{K}_y(s) \cdot \underline{m}_y^{-1}(s+\Delta s, s) - \underline{K}_y(s) \right\} \\ &= \frac{1}{\Delta s} \cdot \lim_{\Delta s \to 0} \left\{ \left[\underline{1} + \Delta s \cdot \underline{A}_y(s)\right] \cdot \underline{K}_y(s) \cdot \left[\underline{1} - \Delta s \cdot \underline{A}_y(s)\right] - \underline{K}_y(s) \right\} \\ &= \underline{A}_y(s) \cdot \underline{K}_y(s) - \underline{K}_y(s) \cdot \underline{A}_y(s) \\ &= \begin{pmatrix} [-\gamma_y \cdot F_y + \beta_y(s) \cdot G_y] & 2 \cdot [\beta_y \cdot R_y(s) - \alpha_y \cdot F_y] \\ 2 \cdot [-\alpha_y \cdot G_y + \gamma_y \cdot R_y] & -[-\gamma_y \cdot F_y + \beta_y \cdot G_y] \end{pmatrix}\;. \end{aligned} \tag{6.56}$$

By comparing (6.55) and (6.56) we then find that

$$
\begin{aligned}
\alpha_y'(s) &= -\gamma_y \cdot F_y + \beta_y \cdot G_y \; ; & (6.57a)\\
\beta_y'(s) &= 2 \cdot [\beta_y \cdot R_y - \alpha_y \cdot F_y] \; ; & (6.57b)\\
\gamma_y'(s) &= 2 \cdot [\alpha_y \cdot G_y - \gamma_y \cdot R_y] \; . & (6.57c)
\end{aligned}
$$

Finally, using the fact that the eigenvector $\vec{w}_y$ in (6.47) must be a solution of the equation of motion (6.51) and by taking into account eqn. (6.57) we obtain:

$$
\begin{aligned}
\psi_y'(s) &= \frac{F_y(s)}{\beta_y(s)} & (6.58a)\\
\Longrightarrow \quad \psi_y(s) &= \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} + \psi_{0y} \; . & (6.58b)
\end{aligned}
$$

The Floquet vector

$$\vec{\tilde{w}}_y(s) = \vec{w}_y(s) \cdot e^{+i \cdot 2\pi Q_y \cdot (s/L)}$$

as defined by eqn. (6.22) now reads :

$$\vec{\tilde{w}}_y = \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} \cdot \exp\left\{ i \cdot \left[2\pi Q_y \frac{s}{L} - \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - \psi_{0y} \right] \right\} \qquad (6.59)$$

and the "action-angle representation" (6.31) of the orbital motion with the action variable J_y and the angle variable Φ_y takes the form:

$$
\begin{aligned}
\begin{pmatrix} y \\ p_y \end{pmatrix} &= \sqrt{J_y} \cdot \frac{1}{\sqrt{2\beta_y(s)}} \cdot \begin{pmatrix} \beta_y(s) \\ -[\alpha_y(s) + i] \end{pmatrix} \\
&\quad \times \exp\left\{ -i \cdot \left[\Phi_y(s) + \psi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L} \right] \right\} \\
&\quad + compl.conj. & (6.60)
\end{aligned}
$$

or, written in components:

$$
\begin{aligned}
y(s) &= \sqrt{2J_y \cdot \beta_y} \cdot \cos\left[\Phi_y(s) + \psi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L} \right] \; ; & (6.61a)\\
p_y(s) &= -\sqrt{\frac{2J_y}{\beta_y}} \cdot \left\{ \sin\left[\Phi_y(s) + \psi_{oy} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L} \right] \right. \\
&\quad \left. + \alpha_y \cdot \cos\left[\Phi_y(s) + \psi_{0y} + \int_0^s d\tilde{s} \cdot \frac{F_y(\tilde{s})}{\beta_y(\tilde{s})} - 2\pi Q_y \frac{s}{L} \right] \right\} \; . & (6.61b)
\end{aligned}
$$

This can be the starting point for a canonical perturbation treatment of coupled synchro-betatron oscillations as demonstrated in Ref. [26].

9) From eqn. (6.23) and the relations (6.22a) and (6.28) we obtain

$$A_k = -i \cdot \vec{v}_k^{\,+}(s) \cdot \underline{S} \cdot \vec{\tilde{y}}(s) \qquad (6.62)$$

and from (6.30a,b) we have:

$$J_k(s) = |\vec{v}_k^+(s) \cdot \underline{S} \cdot \vec{y}(s)|^2 \ . \tag{6.63}$$

In the special case, (6.47), of vanishing coupling we may thus write:

$$J_y(s) \ = \ \frac{1}{2\beta_y(s)} \cdot \left\{[\alpha_y \cdot \tilde{y} + \beta_y \cdot \tilde{p}_y]^2 + \tilde{y}^2\right\} \ . \tag{6.64}$$

The terms on the r.h.s. of (6.64) just represent the well known Courant-Snyder invariants [16] for the linear uncoupled case. Therefore the term on the r.h.s. of (6.63) may be interpreted as the generalized Courant-Snyder invariant for the linear coupled case.

10) From eqns. (6.61a, b) one obtains the relation :

$$\epsilon_y = 2J_y \tag{6.65}$$

where ϵ_y designates the emittance of the uncoupled oscillation in the y-direction (note, that at a fixed position s, eqns. (6.61a, b) describe an ellipse in the $(y - p_y)$ phase plane and that the area of this ellipse is given by $F = 2\pi \cdot J_y = \pi \cdot \epsilon_y$ [25]). Generalizing this equation for coupled motion, we may write:

$$\epsilon_k = 2J_k, \tag{6.66}$$

defining ϵ_k as the emittance of the k^{th} (coupled) mode.

6.2.2 Spin-Orbit Motion

The eigenvectors of the whole eight-dimensional revolution matrix $\hat{\underline{M}}(s_0 + L, s_0)$ for spin and orbit [4] which are defined by

$$\hat{\underline{M}}(s_0 + L, s_0) \cdot \vec{q}_\mu = \hat{\lambda}_\mu \cdot \vec{q}_\mu \tag{6.67}$$

can now be written in the form:

$$\vec{q}_k(s_0) \ = \ \begin{pmatrix} \vec{v}_k(s_0) \\ \vec{w}_k(s_0) \end{pmatrix} \ ; \ \vec{q}_{-k}(s_0) = [\vec{q}_k(s_0)]^* \tag{6.68a}$$

$$\text{for } k = I,\ II,\ III$$

and

$$\vec{q}_{IV}(s_0) \ = \ \begin{pmatrix} \vec{0}_6(s_0) \\ \vec{w}_{IV}(s_0) \end{pmatrix} \ ; \ \vec{q}_{-IV}(s_0) = [\vec{q}_{IV}(s_0)]^* \tag{6.68b}$$

$$\text{for } k = IV \ .$$

By combining eqns. (6.67), (6.68), (6.11), (6.12) and (6.15a), the two-dimensional vectors $\vec{w}_k(s_0)$ $(k = I,\ II,\ III)$ and $\vec{w}_{IV}(s_0)$ can be written as:

$$\vec{w}_k(s_0) = -\left[\underline{D}(s_0 + L, s_0) - \hat{\lambda}_k\right]^{-1} \cdot \underline{G}(s_0 + L, s_0) \cdot \vec{v}_k(s_0) \tag{6.69a}$$

[4] A description of a method to determine the eigenvectors of the transfer matrix may be found in Ref. [25].

$$\text{for } k = I,\ II,\ III\ ;$$

$$\vec{w}_{IV}(s_0) \;=\; \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot e^{-i\psi(s_0)} \tag{6.69b}$$

$$\text{for } k = IV$$

and

$$\vec{w}_{-k}(s_0) = [\vec{w}_k(s_0)]^* ; \quad (k = I,\ II,\ III,\ IV) \tag{6.70}$$

($\vec{v}_k(s_0)$ being defined in (6.15a)).

The corresponding eigenvalues are

$$\hat{\lambda}_k = \lambda_k = e^{-i \cdot 2\pi Q_k} ; \quad (k = I,\ II,\ III) \tag{6.71a}$$

and

$$\hat{\lambda}_{IV} = e^{-i \cdot 2\pi Q_{IV}} \quad \text{with} \quad Q_{IV} = Q_{spin} . \tag{6.71b}$$

For the eigenvectors $\vec{q}_\mu(s)$ of the transfer matrix $\hat{\underline{M}}(s + L, s)$ (initial position s) we also have:

$$\vec{q}_\mu(s) = \hat{\underline{M}}(s, s_0)\ \vec{q}_\mu(s_0) \equiv \begin{pmatrix} \vec{v}_k(s) \\ \vec{w}_k(s) \end{pmatrix} . \tag{6.72}$$

In particular we find (see eqns. (6.6) and (6.9)):

$$\vec{q}_{IV}(s) \;=\; \begin{pmatrix} \vec{0}_6 \\ \vec{w}_{IV}(s) \end{pmatrix} ; \quad \vec{q}_{-IV}(s) = [\vec{q}_{IV}(s)]^* \tag{6.73a}$$

with

$$\vec{w}_{IV}(s) \;=\; \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix} \cdot e^{-i\psi(s)} ; \quad \vec{w}_{-IV}(s) = [\vec{w}_{IV}(s)]^* . \tag{6.73b}$$

The eigenvalues remain independent of s:

$$\hat{\lambda}_\mu(s) = \hat{\lambda}_\mu(s_0) . \tag{6.74}$$

The following orthogonality relations for $\vec{w}_{IV}(s)$ are important for our later investigations:

$$\vec{w}_{IV}^{\,+}(s) \cdot \underline{S}_2 \cdot \vec{w}_{IV}(s) = -\vec{w}_{-IV}^{\,+}(s) \cdot \underline{S}_2 \cdot \vec{w}_{-IV}(s) = i ; \tag{6.75a}$$

$$\vec{w}_{IV}^{\,+}(s) \cdot \underline{S}_2 \cdot \vec{w}_{IV}(s) = -\vec{w}_{IV}^{\,+}(s) \cdot \underline{S}_2 \cdot \vec{w}_{-IV}(s) = 0 . \tag{6.75b}$$

These relations resemble those for $\vec{v}_k(s)$ (see eqn. (6.28)).

Defining

$$\vec{q}_\mu(s) = \vec{\hat{q}}_\mu(s) \cdot e^{-i \cdot 2\pi Q_\mu \cdot (s/L)} \tag{6.76a}$$

we find

$$\vec{\hat{q}}_\mu(s + L) = \vec{\hat{q}}_\mu(s) . \tag{6.76b}$$

Eqn. (6.76) is an extension of the Floquet-theorem to the spin-orbit motion and it will play an important role in our further investigations.

We write the spin parts of the $\vec{\hat{q}}_k$ as $\vec{\hat{w}}_k$.

7 The Perturbed Problem

The general solution of the unperturbed equation of motion (6.1) can now be written in the form

$$\vec{u}(s) = \sum_{k=I,II,III,IV} \{A_k \cdot \vec{q}_k(s) + A_{-k} \cdot \vec{q}_{-k}(s)\}$$

where A_k, A_{-k} are constants of integration and $k = I,\ II,\ III,\ IV$. Note that the orbital part of this equation is identical with eqns. (6.23, 31), and that for $k = I, II, III$, the A_k, A_{-k} are given by eqns. (6.23), (6.30), (6.62). In order to solve the perturbed problem (5.16) we make the following "ansatz" where the A_k now depend on s (variation of constants) :

$$\vec{u}(s) = \sum_{k=I,II,III,IV} \{A_k(s) \cdot \vec{q}_k(s) + A_{-k}(s) \cdot \vec{q}_{-k}(s)\} \ . \tag{7.1}$$

Inserting (7.1) into (5.16) we obtain :

$$\begin{aligned}\sum_{k=I,II,III,IV} \left\{A'_k(s) \cdot \vec{q}_k + A'_{-k}(s) \cdot \vec{q}_{-k}\right\} &= \delta\underline{\hat{A}} \cdot \sum_{k=I,II,III,IV} \{A_k(s) \cdot \vec{q}_k + A_{-k}(s) \cdot \vec{q}_{-k}\} \\ &\quad + \delta\vec{\hat{c}} \end{aligned} \tag{7.2}$$

and dividing this equation into its orbital part and spin part we find :

$$\begin{aligned}\sum_{k=I,II,III} \left\{A'_k(s) \cdot \vec{v}_k + A'_{-k}(s) \cdot \vec{v}_{-k}\right\} &= \delta\underline{A} \cdot \sum_{k=I,II,III} \{A_k(s) \cdot \vec{v}_k + A_{-k}(s) \cdot \vec{v}_{-k}\} \\ &\quad + \ \delta\vec{c}\ ; \end{aligned} \tag{7.3a}$$

$$A'_{IV}(s) \cdot \vec{w}_{IV} + A'_{-IV}(s) \cdot \vec{w}_{-IV} = -\sum_{k=I,II,III} \left\{A'_k(s) \cdot \vec{w}_k + A'_{-k}(s) \cdot \vec{w}_{-k}\right\}. \tag{7.3b}$$

Equation (7.3) shows that as expected in this treatment, the spin motion is not directly affected by radiation but only indirectly via the orbital motion.

The orthogonality conditions (6.28) allow eqn. (7.3a) (for $k = I,\ II,\ III$) to be rewritten as:

$$A'_k(s) = X_k(s) - i \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \delta\vec{c}\ ; \tag{7.4a}$$

$$A'_{-k}(s) = [A'_k(s)]^* \tag{7.4b}$$

with

$$\begin{aligned} X_k(s) &= \sum_{l=I,II,III} A_l(s) \cdot (-i) \cdot \vec{v}_k^{+} \cdot \underline{S} \cdot \delta\underline{A} \cdot \vec{v}_l \\ &\quad + \sum_{l=I,II,III} A_{-l}(s) \cdot (-i) \cdot \vec{v}_k^{+} \cdot \underline{S} \cdot \delta\underline{A} \cdot \vec{v}_{-l}\ ; \end{aligned} \tag{7.5a}$$

$$X_{-k}(s) = [X_k(s)]^* \ . \tag{7.5b}$$

Taking into account (4.2) and (5.7) we can write the last term on the r.h.s. of (7.4a) as:

$$\begin{aligned} -i \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \delta\vec{c} = &= +i \cdot v_{k5}^*(s) \cdot \delta c(s) \\ &= +i \cdot v_{k5}^*(s) \cdot \sqrt{\omega(s)} \cdot \mathcal{P}(s) \end{aligned} \tag{7.6}$$

with (see eqn. (4.2))

$$\omega = (|K_x|^3 + |K_z|^3) \cdot C_2 \; . \tag{7.7}$$

Also, from (7.3b) and (6.75):

$$\begin{aligned} A'_{IV}(s) &= i \cdot \sum_{k=I,II,III} \left\{ A'_k(s) \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_k + A'_{-k}(s) \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_{-k} \right\} \\ &= i \cdot \sum_{k=I,II,III} \left\{ X_k(s) \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_k - X_{-k}(s) \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_{-k} \right\} \\ &- \sum_{k=I,II,III} \left\{ v_{k5}^* \cdot \delta c \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_k - v_{k5} \cdot \delta c \cdot \vec{w}_{IV}^{+} \underline{S}_2 \vec{w}_{-k} \right\} \; ; \end{aligned} \tag{7.8a}$$

$$A'_{-IV}(s) = [A'_{IV}(s)]^* \; . \tag{7.8b}$$

Using the relations:

$$\begin{aligned} \vec{w}_{IV}^{+} \underline{S} \vec{w}_k &= \frac{1}{\sqrt{2}} \cdot e^{i\psi} \cdot \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} w_{k1} \\ w_{k2} \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} e^{i\psi} \cdot [-w_{k2} + i \cdot w_{k1}] \; ; \end{aligned} \tag{7.9a}$$

$$\vec{w}_{IV}^{+} \underline{S} \vec{w}_{-k} = \frac{1}{\sqrt{2}} e^{i\psi} \cdot [-w_{k2}^* + i \cdot w_{k1}^*] \tag{7.9b}$$

we obtain from (7.8a):

$$\begin{aligned} A'_{IV}(s) &= i \cdot \frac{1}{\sqrt{2}} e^{i\psi} \cdot \sum_{k=I,II,III} \{ X_k(s) \cdot [-w_{k2} + i \cdot w_{k1}] + X_{-k}(s) \cdot [-w_{k2}^* + i \cdot w_{k1}^*] \} \\ &+ \sqrt{\omega(s)} \cdot \mathcal{P}(s) \cdot \sqrt{2} \cdot e^{i\psi} \cdot \sum_{k=I,II,III} \{ \Im m[v_{k5}^* \cdot w_{k1}] + i \cdot \Im m[v_{k5}^* \cdot w_{k2}] \} \; . \end{aligned} \tag{7.10}$$

8 Stochastic Equations for the Variables $J_k(s)$ and $\Phi_k(s)$

Before preparing the Fokker-Planck equation we define two more action-angle like variables, $J_{IV}(s)$, $\Phi_{IV}(s)$, (for spin) by

$$A_{IV}(s) = \sqrt{J_{IV}(s)} \cdot e^{-i \cdot [\Phi_{IV}(s) - 2\pi Q_{IV} \cdot s/L]} \tag{8.1a}$$

or, using eqns. (C.19b) and (6.71b) :

$$A_{IV}(s) = \sqrt{J_{IV}(s)} \cdot e^{-i \cdot [\Phi_{IV}(s) - \psi(s)]} \; . \tag{8.1b}$$

Then for $k = I,\ II,\ III,\ IV$ it follows that:

$$\begin{aligned} A'_k(s) &= \frac{1}{2}\cdot\frac{J'_k}{\sqrt{J_k}}\cdot e^{-i\cdot[\Phi_k - 2\pi Q_k\cdot s/L]} - i\cdot\left[\Phi'_k - \frac{2\pi}{L}Q_k\right]\cdot A_k \\ &= A_k\cdot\left\{\frac{1}{2}\cdot\frac{J'_k}{J_k} - i\cdot\left[\Phi'_k - \frac{2\pi}{L}Q_k\right]\right\} \end{aligned}$$

and for $J'_k(s)$ and $\Phi'_k(s)$ we get (with $J_k = A_k\cdot A_{-k}$) :

$$\begin{aligned} J'_k(s) &= A'_k(s)\cdot A_{-k}(s) + A_k(s)\cdot A'_{-k}(s) \\ &= 2\cdot\Re e\left\{A'_k(s)\cdot A_{-k}(s)\right\}\ ; \end{aligned} \tag{8.2a}$$

$$\begin{aligned} \Phi'_k(s) - \frac{2\pi}{L}\cdot Q_k &= -\frac{A'_k(s)\cdot A_{-k}(s)}{i\cdot J_k(s)} + \frac{1}{2}\cdot\frac{A'_k(s)\cdot A_{-k}(s) + A_k(s)\cdot A'_{-k}(s)}{i\cdot J_k(s)} \\ &= -\frac{1}{2}\cdot\frac{A'_k(s)\cdot A_{-k}(s) - A_k(s)\cdot A'_{-k}(s)}{i\cdot J_k(s)} \\ &= -\frac{1}{J_k(s)}\cdot\Im m\left\{A'_k(s)\cdot A_{-k}(s)\right\}\ . \end{aligned} \tag{8.2b}$$

Here the terms $(A'_k\cdot A_{-k})$ appearing in (8.2a,b) are given by:

$$A'_k(s)\cdot A_{-k}(s) = Y_k(s) + i\cdot\sqrt{J_k}\cdot\hat{v}^*_{k5}(s)\cdot\sqrt{\omega(s)}\cdot\mathcal{P}(s)\cdot e^{i\cdot\Phi_k(s)} \tag{8.3a}$$

for $k = I,\ II,\ III$

and

$$\begin{aligned} A'_{IV}(s)\cdot A_{-IV}(s) = Y_{IV}(s) + \sqrt{J_{IV}}\cdot\sqrt{\omega(s)}\cdot\mathcal{P}(s)\cdot\sqrt{2}\cdot e^{i\Phi_{IV}}\ \times \\ \sum_{k=I,II,III}\left\{\Im m[v^*_{k5}\cdot w_{k1}] + i\cdot\Im m[v^*_{k5}\cdot w_{k2}]\right\} \end{aligned} \tag{8.3b}$$

for $k = IV$

with

$$\begin{aligned} Y_k(s) &= X_k(s)\cdot A_{-k}(s) \\ &= \sum_{l=I,II,III}\sqrt{J_l}\cdot\sqrt{J_k}\cdot(-i)\cdot\vec{\hat{v}}_k^{\,+}\cdot\underline{S}\cdot\delta\underline{A}\cdot\vec{\hat{v}}_l\cdot e^{i\cdot[\Phi_k - \Phi_l]} \\ &+ \sum_{l=I,II,III}\sqrt{J_l}\cdot\sqrt{J_k}\cdot(-i)\cdot\vec{\hat{v}}_k^{\,+}\cdot\underline{S}\cdot\delta\underline{A}\cdot\vec{\hat{v}}_{-l}\cdot e^{i\cdot[\Phi_k + \Phi_l]} \end{aligned} \tag{8.4a}$$

for $k = I,\ II,\ III$

and

$$\begin{aligned} Y_{IV}(s) &= i\cdot\frac{1}{\sqrt{2}}\sqrt{J_{IV}}\,e^{i\Phi_{IV}}\ \times \\ &\sum_{k=I,II,III}\left\{X_k(s)\cdot[-w_{k2} + i\cdot w_{k1}] + X_{-k}(s)\cdot[-w^*_{k2} + i\cdot w^*_{k1}]\right\} \end{aligned}$$

$$
\begin{aligned}
&= i \cdot \frac{1}{\sqrt{2}} \sqrt{J_{IV}} e^{i\Phi_{IV}} \times \\
&\quad \sum_{k=I,II,III} \{-\left[X_k(s) \cdot w_{k2} + compl.conj.\right] \\
&\qquad\qquad + i \cdot \left[X_k(s) \cdot w_{k1} + compl.conj.\right]\} \qquad (8.4b)
\end{aligned}
$$

for $k = IV$

where for the term $X_k \cdot w_{k\nu}$ $(\nu = 1, 2)$ in (8.4b) we can write:

$$
\begin{aligned}
X_k \cdot w_{k\nu} &= \sum_{l=I,II,III} \sqrt{J_l(s)} \cdot e^{-i \cdot \Phi_l} \cdot (-i) \cdot \vec{\hat{v}}_k^{+} \cdot \underline{S} \cdot \delta \underline{A} \cdot \vec{\hat{v}}_l \cdot \hat{w}_{k\nu} \, . \\
&= \sum_{l=I,II,III} \sqrt{J_l(s)} \cdot e^{+i \cdot \Phi_l} \cdot (-i) \cdot \vec{\hat{v}}_k^{+} \cdot \underline{S} \cdot \delta \underline{A} \cdot \vec{\hat{v}}_{-l} \cdot \hat{w}_{k\nu} \, . \qquad (8.5)
\end{aligned}
$$

If we then write $J_k'(s)$ and $\Phi_k'(s)$ in the form:

$$
\begin{aligned}
J_k'(s) &= K_J^{(k)}(\Phi_k, J_k) + Q_J^{(k)}(\Phi_k, J_k) \cdot \mathcal{P}(s) \, ; \qquad (8.6a) \\
\Phi_k'(s) &= K_\Phi^{(k)}(\Phi_k, J_k) + Q_\Phi^{(k)}(\Phi_k, J_k) \cdot \mathcal{P}(s) \qquad (8.6b)
\end{aligned}
$$

we obtain the drift and diffusion coefficients [8], proportional to C_1 and $\sqrt{C_2}$ respectively. For $k = I,\ II,\ III$ we have:

$$
\begin{aligned}
K_J^{(k)} &= 2 \cdot \Re e \{Y_k(s)\} \; ; \\
Q_J^{(k)} &= i \cdot \sqrt{\omega} \cdot \sqrt{J_k} \cdot \left[\hat{v}_{k5}^* \cdot e^{i \cdot \Phi_k} - \hat{v}_{k5} \cdot e^{-i \cdot \Phi_k}\right] \; ; \qquad (8.7a)
\end{aligned}
$$

$$
\begin{aligned}
K_\Phi^{(k)} &= +\frac{2\pi}{L} \cdot Q_k - \frac{1}{J_k(s)} \cdot \Im m \{Y_k(s)\} \; ; \\
Q_\Phi^{(k)} &= -\sqrt{\omega} \cdot \frac{1}{2\sqrt{J_k}} \cdot \left[\hat{v}_{k5}^* \cdot e^{i \cdot \Phi_k} + \hat{v}_{k5} \cdot e^{-i \cdot \Phi_k}\right] \qquad (8.7b)
\end{aligned}
$$

and for $k = IV$:

$$
\begin{aligned}
K_J^{(IV)} &= 2 \cdot \Re e \{Y_{IV}(s)\} \; ; \\
Q_J^{(IV)} &= \sqrt{\omega} \cdot 2 \cdot \sqrt{2 J_{IV}} \times \\
&\quad \sum_{k=I,II,III} \{\cos \Phi_{IV} \cdot \Im m[v_{k5}^* \cdot w_{k1}] - \sin \Phi_{IV} \cdot \Im m[v_{k5}^* \cdot w_{k2}]\} \, ; \qquad (8.8a) \\
K_\Phi^{(IV)} &= +\frac{2\pi}{L} \cdot Q_{IV} - \frac{1}{J_{IV}(s)} \cdot \Im m \{Y_{IV}(s)\} \; ; \\
Q_\Phi^{(IV)} &= -\sqrt{\omega} \cdot \sqrt{2} \frac{1}{\sqrt{J_{IV}}} \times \\
&\quad \sum_{k=I,II,III} \{\sin \Phi_{IV} \cdot \Im m[v_{k5}^* \cdot w_{k1}] + \cos \Phi_{IV} \cdot \Im m[v_{k5}^* \cdot w_{k2}]\} \, . \qquad (8.8b)
\end{aligned}
$$

In making the transformation from the variables $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$ to (J_k, Φ_k) we have used the usual rules of algebra i.e. in our classical model of photon emission we are interpreting the Langevin equations according to the Stratanovich convention (see for example [8,27]).

A comparison of eqns. (8.6) and (8.8) with eqn. (6.37) shows explicitely how the J_k' and Φ_k' are modified by radiation effects. Note also from (8.6) that the stochastic motions of J_k and Φ_k are driven from a common noise source. Furthermore, it is clear that the variables (J_k, Φ_k) which by construction originally described uncoupled normal modes, have become coupled via the radiation emission. Also, in contrast to the use of $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$ variables in eqn. (5.16), the Langevin equations for the (J_k, Φ_k) are non-linear.

The relations (8.6), (8.7), (8.8), now provide the basis for a Fokker-Planck [8] treatment of spin-orbit motion.

9 The Fokker-Planck Equation of Stochastic Spin-Orbit Motion

With the stochastic differential equations (8.6) of our white noise model for photon emission, the Fokker-Planck (F-P) equation for the spin-orbit phase space density function $W(J, \Phi; s)$ reads as [2]:

$$\begin{aligned} \frac{\partial W}{\partial s} &= \sum_{k=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_k}[D_J^{(k)} \cdot W] - \frac{\partial}{\partial \Phi_k}[D_\Phi^{(k)} \cdot W] \right\} \\ &+ \sum_{k,l=I,II,III,IV} \left\{ \frac{1}{2}\frac{\partial^2}{\partial J_k \partial J_l}[Q_J^{(k)} \cdot Q_J^{(l)} \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[Q_J^{(k)} \cdot Q_\Phi^{(l)} \cdot W] \right. \\ &\left. \qquad + \frac{1}{2}\frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \cdot W] \right\} \end{aligned} \tag{9.1}$$

with the drift coefficients given by

$$D_J^{(k)} = K_J^{(k)} + \tilde{K}_J^{(k)} ; \tag{9.2a}$$

$$D_\Phi^{(k)} = K_\Phi^{(k)} + \tilde{K}_\Phi^{(k)} \tag{9.2b}$$

and where the quantities $\tilde{K}_J^{(k)}$ and $\tilde{K}_\Phi^{(k)}$ are the artificial drift terms which arise when using the Stratanovich interpretation of eqn. (5.16):

$$\tilde{K}_J^{(k)} = \frac{1}{2}\frac{\partial Q_J^{(k)}}{\partial J_k} \cdot Q_J^{(k)} + \frac{1}{2}\frac{\partial Q_J^{(k)}}{\partial \Phi_k} \cdot Q_\Phi^{(k)} ; \tag{9.3a}$$

$$\tilde{K}_\Phi^{(k)} = \frac{1}{2}\frac{\partial Q_\Phi^{(k)}}{\partial J_k} \cdot Q_J^{(k)} + \frac{1}{2}\frac{\partial Q_\Phi^{(k)}}{\partial \Phi_k} \cdot Q_\Phi^{(k)} . \tag{9.3b}$$

(Note that $Q_J^{(k)}$ and $Q_\Phi^{(k)}$ only contain the two variables J_k and Φ_k; see eqns. (8.7, 8).)

From (9.3a,b) we have with the help of (8.7) and (8.8).
for $k = I, II, III$:

$$\tilde{K}_J^{(k)} = |v_{k5}|^2 \cdot \omega(s) ; \tag{9.4a}$$

$$\tilde{K}_\Phi^{(k)} = i \cdot \frac{\omega(s)}{4J_k} \cdot \left[(\hat{v}_{k5}^*)^2 \cdot \epsilon^{i \cdot 2\Phi_k} - (\hat{v}_{k5})^2 \cdot \epsilon^{-i \cdot 2\Phi_k} \right] \tag{9.4b}$$

and for $k = IV$:

$$\tilde{K}_J^{(IV)} = \frac{1}{4 \cdot J_{IV}} \cdot \left[Q_J^{(IV)}\right]^2 + J_{IV} \cdot \left[Q_\Phi^{(IV)}\right]^2 \ ; \tag{9.5a}$$

$$\tilde{K}_\Phi^{(IV)} = -\frac{1}{2 \cdot J_{IV}} \cdot Q_J^{(IV)} Q_\Phi^{(IV)} \tag{9.5b}$$

with $Q_J^{(IV)}, Q_\Phi^{(IV)}$ given by (8.8).

It is clear that the F-P equation (9.1) is very complicated and that the drift and diffusion coefficients are oscillating functions in s. However, in this paper we will be interested in the long time (asymptotic) equilibrium behaviour and therefore it will be sufficient to deal with the distribution of quantities averaged over times on the scale of damping times [2]. Denoting the one-turn averages by the bracket $\langle \quad \rangle$, we therefore write the F-P equation in the form:

$$\begin{aligned}
\frac{\partial W}{\partial s} &= \sum_{k=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_k}[\langle D_J^{(k)} \rangle \cdot W] - \frac{\partial}{\partial \Phi_k}[\langle D_\Phi^{(k)} \rangle \cdot W] \right\} \\
&+ \sum_{k,l=I,II,III,IV} \left\{ \frac{1}{2} \frac{\partial^2}{\partial J_k \partial J_l}[\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle \cdot W] + \frac{\partial^2}{\partial J_k \partial \Phi_l}[\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle \cdot W] \right. \\
&\qquad \left. + \frac{1}{2} \frac{\partial^2}{\partial \Phi_k \partial \Phi_l}[\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle \cdot W] \right\}
\end{aligned} \tag{9.6}$$

whereby oscillating terms due to the (linear) s-dependence of the angle variables Φ_k:

$$\Phi_k(s) \approx \Phi_k(s_0) + (s - s_0) \cdot \frac{2\pi}{L} Q_k$$

(see eqn. (6.37b)) may be neglected since they are approximately averaged away by integration, and only one turn averages of the periodic integrands remain.

To do that, we first introduce the following abbreviation:

$$\begin{aligned}
\delta Q_k &= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{\tilde{v}}_k^{+}(s) \cdot \underline{S} \cdot \delta \underline{A}(s) \cdot \vec{v}_{\tilde{k}}(s) \\
&= \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^{+}(s) \cdot \underline{S} \cdot \delta \underline{A}(s) \cdot \vec{v}_k(s) \ ;
\end{aligned} \tag{9.7}$$

$(k = I,\ II,\ III)$

and the quantities (the "damping constants"; see forward to eqn. (9.17)):

$$\alpha_k = -2\pi \cdot \Im m\{\delta Q_k\} \tag{9.8}$$

so that from (8.4a) and (8.5) :

$$\begin{aligned}
\langle Y_k \rangle &= J_k \cdot (-i) \cdot \frac{2\pi}{L} \delta Q_k \\
&= J_k \cdot (-i) \cdot \frac{2\pi}{L} \cdot \left[\Re e\{\delta Q_k\} - i \cdot \frac{1}{2\pi} \alpha_k \right] \\
&\quad \text{for} \quad k = I,\ II,\ III \ ;
\end{aligned} \tag{9.9a}$$

$$\langle Y_{IV} \rangle = 0. \tag{9.9b}$$

In Appendix B it is shown that δQ_k is just the (complex) Q-shift of the k^{th} oscillation mode ($k = I,\ II,\ III$) caused by the perturbation $\delta\underline{A}$.

Then from (9.2) and using (8.7), (8.8), (9.4) and (9.5) we obtain

for $k = I, II, III$:

$$\begin{aligned}
\langle D_J^{(k)} \rangle &= -J_k \cdot \frac{2}{L}\alpha_k + \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) \ ; && (9.10a)\\
\langle D_\Phi^{(k)} \rangle &= \frac{2\pi}{L} \cdot [Q_k + \Re e\{\delta Q_k\}] \ ; && (9.10b)\\
\langle (Q_J^{(k)})^2 \rangle &= 2J_k \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) \ ; && (9.10c)\\
\langle (Q_\Phi^{(k)})^2 \rangle &= \frac{1}{2J_k} \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) && (9.10d)
\end{aligned}$$

and for $k = IV$:

$$\begin{aligned}
\langle D_J^{(IV)} \rangle &= \frac{2}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}(\tilde{s})] \right)^2 \ ; && (9.11a)\\
\langle D_\Phi^{(IV)} \rangle &= 0 \ ; && (9.11b)\\
\langle (Q_J^{(IV)})^2 \rangle &= 4J_{IV} \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}(\tilde{s})] \right)^2 \ ; && (9.11c)\\
\langle (Q_\Phi^{(IV)})^2 \rangle &= \frac{1}{4J_{IV}} \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}(\tilde{s})] \right)^2 && (9.11d)
\end{aligned}$$

and

$$\begin{aligned}
\langle Q_J^{(k)} \cdot Q_J^{(l)} \rangle &= 0 \ \text{ for } \ k \neq l \ ; && (9.12a)\\
\langle Q_\Phi^{(k)} \cdot Q_\Phi^{(l)} \rangle &= 0 \ \text{ for } \ k \neq l \ ; && (9.12b)\\
\langle Q_J^{(k)} \cdot Q_\Phi^{(l)} \rangle &= 0 \ . && (9.12c)
\end{aligned}$$

Introducing the constants

$$\begin{aligned}
a_k &= \alpha_k \cdot \frac{1}{L} \ ; \quad (k = I, II, III) \ ; && (9.13a)\\
b_k &= 2\pi \cdot \frac{1}{L} \cdot \hat{Q}_k \ ; && (9.13b)\\
M_k &= \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) \ ; \quad (k = I, II, III) \ ; && (9.13c)\\
a_{IV} &= 0 \ ; && (9.13d)\\
b_{IV} &= 2\pi \cdot \frac{1}{L} \cdot Q_{IV} && (9.13e)\\
M_{IV} &= 2 \cdot \frac{1}{L}\int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}] \right)^2 && (9.13f)
\end{aligned}$$

with

$$\hat{Q}_k = Q_k + \Re e\{\delta Q_k\} \qquad (9.14)$$

we may finally write:

$$
\begin{aligned}
\frac{\partial W}{\partial s} &= \sum_{k=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_k} \left[(-2a_k \cdot J_k + M_k) \cdot W \right] - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot W \right] \right. \\
&\qquad \left. + \frac{1}{2} \frac{\partial^2}{\partial J_k^2} \left[2J_k \cdot M_k \cdot W \right] + \frac{1}{2} \frac{\partial^2}{\partial \Phi_k^2} \left[\frac{1}{2J_k} \cdot M_k \cdot W \right] \right\} \\
&= \sum_{k=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_k} \left[(-2a_k \cdot J_k + M_k) \cdot W - \frac{\partial}{\partial J_k} (J_k \cdot M_k \cdot W) \right] \right. \\
&\qquad \left. - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot W - \frac{1}{4J_k} \cdot M_k \cdot \frac{\partial}{\partial \Phi_k} W \right] \right\} \\
&= \sum_{k=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_k} \left[-2a_k \cdot J_k \cdot W - M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right] \right. \\
&\qquad \left. - \frac{\partial}{\partial \Phi_k} \left[b_k \cdot W - \frac{M_k}{4J_k} \cdot \frac{\partial W}{\partial \Phi_k} \right] \right\} .
\end{aligned}
\tag{9.15}
$$

This equation determines the averaged charge and spin distribution of the particles in a bunch. On comparison with eqn. (9.1) we see that the s-dependent coefficients have been replaced by s-independent constants given by the one turn averages and that the r.h.s. has separated into a sum of four terms, one for each pair of action-angle variables.

Remarks:

1) The averaging procedure indicated by the bracket $\langle\ \rangle$ only results in the forms (9.10), (9.11), (9.12) away from the linear resonances

$$Q_k \pm Q_l \approx integer.$$

On resonance the common noise source would cause the modes to be correlated and also extra terms would appear [18]. But on resonance the particle motion can be unstable so that this case is of no interest here.

2) In eqn. (8.6) the first terms on the r.h.s. describe the influence of the continuous emission of synchrotron light on the synchro-betatron oscillations and the second terms the influence of quantum fluctuations of the radiation field (the function $\mathcal{P}(s)$ in (8.6)). If the quantum fluctuation term is neglected and if we take into account eqns. (8.7a) and (9.9a), eqn. (8.6a) may approximated by the form :

$$J_k'(s) \approx -\frac{2}{L} \alpha_k \cdot J_k(s) . \tag{9.16}$$

Eqn. (9.16) can be integrated to give:

$$
\begin{aligned}
J_k(s) &= J_k(s_0) \cdot e^{-2\alpha_k \cdot (s - s_0)/L} \\
\Longrightarrow \quad \sqrt{J_k(s)} &= \sqrt{J_k(s_0)} \cdot e^{-\alpha_k \cdot (s - s_0)/L} .
\end{aligned}
\tag{9.17}
$$

Since $\sqrt{J_k}$ represents the amplitude for the k^{th} mode of the synchro-betatron oscillations, the quantity α_k may be interpreted as the damping constant of the k^{th} mode [1,28]. From (9.7) and (9.8) we have:

$$\alpha_k = \frac{i}{2} \cdot \int_{s_0}^{s_0+L} ds \cdot \vec{v}_k^+(s) \cdot \left[\underline{S} \cdot \delta \underline{A}(s) + \delta \underline{A}^T(s) \cdot \underline{S}\right] \cdot \vec{v}_k(s) \; . \tag{9.18}$$

This formula may be used for a calculation of the damping constants [5] . An alternative way to calculate α_k is described in Appendix D.

Note, that α_k vanishes if $\delta \underline{A}$ contains only symplectic terms, as may be seen by eqn. (6.13b). In this case the tune shift δQ_k of the k^{th} mode induced by the perturbation $\delta \underline{A}$ is a real number. Thus an imaginary part of δQ_k, i.e. a damping of the oscillation modes ($\alpha_k \neq 0$), only appears if the perturbation terms are nonsymplectic.

3) Equation (9.16) can be generalized if we calculate the derivative of the average $< J_k >$ of the action variable J_k, using the Fokker-Planck equation (9.15).

Then we obtain:

$$\begin{aligned}
\frac{d}{ds} < J_k(s) > &= \frac{d}{ds} \int \cdots \int dJ_I \, dJ_{II} \, dJ_{III} \, dJ_{IV} \; d\Phi_I \, d\Phi_{II} \, d\Phi_{III} \, d\Phi_{IV} \cdot J_k \cdot W \\
&= \int \cdots \int dJ_I \, dJ_{II} \, dJ_{III} \, dJ_{IV} \, d\Phi_I \, d\Phi_{II} \, d\Phi_{III} \, d\Phi_{IV} \cdot J_k \cdot \frac{\partial}{\partial s} W \\
&= \int \cdots \int dJ_I \, dJ_{II} \, dJ_{III} \, dJ_{IV} \, d\Phi_I \, d\Phi_{II} \, d\Phi_{III} \, d\Phi_{IV} \cdot J_k \\
&\quad \times \sum_{l=I,II,III,IV} \left\{ -\frac{\partial}{\partial J_l} \left[(-2a_l \cdot J_l + M_l) \cdot W - \frac{\partial}{\partial J_l}(J_l \cdot M_l \cdot W) \right] \right. \\
&\qquad \left. - \frac{\partial}{\partial \Phi_l} \left[b_l \cdot W - \frac{1}{4J_l} \cdot M_l \cdot \frac{\partial}{\partial \Phi_l} W \right] \right\} \\
&= \int \cdots \int dJ_I \, dJ_{II} \, dJ_{III} \, dJ_{IV} \, d\Phi_I \, d\Phi_{II} \, d\Phi_{III} \, d\Phi_{IV} \cdot J_k \\
&\qquad \times \left\{ -\frac{\partial}{\partial J_k} \left[-2a_k \cdot J_k \cdot W - M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right] \right\} \\
&= \int \cdots \int dJ_I dJ_{II} dJ_{III} dJ_{IV} d\Phi_I d\Phi_{II} d\Phi_{III} d\Phi_{IV} \\
&\qquad \times \left[-2a_k \cdot J_k \cdot W - M_k \cdot J_k \cdot \frac{\partial W}{\partial J_k} \right] \\
&= \int \cdots \int dJ_I dJ_{II} dJ_{III} dJ_{IV} d\Phi_I d\Phi_{II} d\Phi_{III} d\Phi_{IV} \\
&\qquad \times \left[-2a_k \cdot J_k \cdot W + M_k \cdot W \right] \\
&= -2a_k \cdot < J_k > \; + M_k
\end{aligned} \tag{9.19}$$

(see also Ref. [2]).

[5] The sum of the three α_k is a measure of the relative decrease in the 6-dimensional phase space volume over one turn [2].

In this equation there appears the same damping term

$$-2a_k \cdot < J_k > \; = \; -\frac{2\,\alpha_k}{L} \cdot < J_k >$$

as on the r.h.s. of (9.16) and an additional term M_k due to the influence of quantum fluctuation on the spin-orbit motion (characterized by the function $\omega(s)$ in eqns. (9.13c, f)). Thus the constant M_k which is proportional to $\hbar$ is a measure of the stochastic excitation rate of spin orbit motion.

In the case of orbital motion alone where the damping constants α_k are different from zero, the stochastic excitation of synchro-betatron motion caused by quantum fluctuation of the radiation field and the damping of the oscillations caused by continuous emission of synchrotron light, can lead to the equilibrium condition

$$\frac{1}{2} < \epsilon_k >^{(stat)} \equiv < J_k >^{(stat)} \; = \; \frac{M_k}{2a_k} \tag{9.20}$$

which represents the stationary solution of eqn. (9.19).

Finally we remark that the relation (9.19) can also be derived by solving the stochastic differential equations (7.4) and (7.8), as is shown in Refs. [1,28].

4) Inspection of eqn. (9.15) shows that for the angle variables there is no analogue of the coefficients α_k which lead to damping of the action variables. Thus the angle variables are only subject to diffusion and we can thus assume that the angles Φ_k are uniformly distributed in $[0, 2\pi]$ [2]. See also Appendix E.

5) Since (eqn. (9.13d)), a_{IV} is zero, there is no spin damping effect. On the contrary, the stochastic orbit motion causes spin diffusion at a rate proportional to M_{IV}. This is discussed in section 10.2.

10 Solution of the Fokker-Planck Equation

10.1 Orbital Motion

In order to solve the Fokker-Planck equation (9.15) we first investigate the orbit motion alone. This is, in our treatment, independent of the spin motion so that for the orbital phase space density, W_{orbit} , and using the separability of (9.15) we have:

$$\begin{aligned}\frac{\partial}{\partial s} W_{orbit} \; = \; \sum_{k=I,II,III} \Bigg\{ & -\frac{\partial}{\partial J_k} \left[-2a_k \cdot J_k \cdot W_{orbit} - M_k \cdot J_k \cdot \frac{\partial}{\partial J_k} W_{orbit} \right] \\ & -\frac{\partial}{\partial \Phi_k} \left[b_k \cdot W_{orbit} - \frac{M_k}{4J_k} \cdot \frac{\partial}{\partial \Phi_k} W_{orbit} \right] \Bigg\} \; .\end{aligned} \tag{10.1}$$

With our assumption that the phases Φ_k are uniformly distributed (see Appendix E), W_{orbit} is independent of the Φ_k and we may write:

$$W_{orbit}(J_k, \Phi_k) \; = \; \left(\frac{1}{2\pi}\right)^3 \cdot \hat{W}_{orbit}(J_I, J_{II}, J_{III}) \; ; \tag{10.2a}$$

$$\frac{\partial}{\partial s}\hat{W}_{orbit} = \sum_{k=I,II,III}\left\{-\frac{\partial}{\partial J_k}\left[-2a_k \cdot J_k \cdot \hat{W}_{orbit} - M_k \cdot J_k \cdot \frac{\partial}{\partial J_k}\hat{W}_{orbit}\right]\right\} . \quad (10.2b)$$

The relation (10.2b) has the form of a continuity equation:

$$\frac{\partial}{\partial s}\ \hat{W}_{orbit} + \sum_{k=I,II,III}\frac{\partial}{\partial J_k}\ \Im_k = 0 \quad (10.3a)$$

with

$$\Im_k = -2a_k \cdot J_k \cdot \hat{W}_{orbit} - M_k \cdot J_k \cdot \frac{\partial}{\partial J_k}\ \hat{W}_{orbit} . \quad (10.3b)$$

Thus $\Im_k$ may be interpreted as a current density for the probability $\hat{W}_{orbit}$.

10.1.1 Stationary Distribution

We are only interested in the "stationary distribution" for which [6] :

$$\frac{\partial}{\partial s}\,W_{orbit} = 0 . \quad (10.4)$$

Equation (10.4) is fulfilled with the condition:

$$\Im_k = 0 \implies -2a_k \cdot \hat{W}_{orbit} - M_k \cdot \frac{\partial}{\partial J_k}\ \hat{W}_{orbit} = 0 \quad (10.5)$$

leading to the solution [29]:

$$\hat{W}_{orbit} = \hat{W}_I \cdot \hat{W}_{II} \cdot \hat{W}_{III} ; \quad (10.6a)$$

$$\hat{W}_k = C_k \cdot e^{-J_k \cdot (2a_k/M_k)} ; \quad (k = I, II, III) . \quad (10.6b)$$

Here the factor C_k is fixed by the normalization condition [7]

$$\int_0^\infty dJ_k \cdot \hat{W}_k(J_k) = 1$$

[6]The asymptotic solution of the full F-P equation (9.1) can be expected to exhibit the periodicity of the lattice so that $W(J,\Phi;s) = W(J,\Phi;s+L)$ (see also [18]). In general $\frac{\partial}{\partial s} W_{orbit} \neq 0$. However, in the absence of radiation $\frac{d}{ds} W_{orbit} = 0$ so that $\partial W/\partial s = \{\mathcal{H}, W\}$. Then in (J,Φ) variables and when the phases are uniformly distributed we have $\partial W/\partial s = 0$. Although radiation clearly has a decisive effect on the asymptotic form of W, the radiation effects act slowly. as can be seen from the fact that the damping times are usually equivalent to hundreds or thousands of turns. Thus in the presence of radiation effects, W_{orb}^{asymp} is expected to differ only slightly from a form for which $\partial W/\partial s = 0$. In fact it can be shown [18] that the difference is of $O(\delta)$ where δ is the relative energy loss per turn. Thus even in the presence of radiation it is still a good approximation to put $\partial W_{orb}^{asymp}/\partial s = 0$. In our formalism, the F-P equation has s-independent coefficients given by one turn averages and for such an equation, the asymptotic (periodic) solution will indeed have $\partial W_{orb}^{asymp}/\partial s = 0$. And we can expect that this solution differs from the true W_{orb}^{asymp} only by terms of $O(\delta)$. Note that in the variables $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$ and even in the absence of radiation, we may only require periodicity and that $\partial W/\partial s \neq 0$ even for uniformly distributed phases. Thus our use of action-angle variables leads to non-linear Langevin equations but together with averaging results in simplified equilibrium conditions.

[7]We assume that $a_k \equiv \alpha_k/L > 0$ for $k = I, II, III$. Since M_k are positive definite (see eqn. (9.13c)) then with $a_k < 0$ (antidamping; see eqn. (9.16)) the density funktion $\hat{W}_k$ cannot be normalized, i e. a stationary solution of the Fokker-Planck equation (10.2b) does not exist.

which leads to:

$$C_k = \frac{1}{2\pi} \cdot \frac{2a_k}{M_k} .$$

Thus we have for W_{orbit} :

$$W_{orbit}^{(stat)} = \frac{1}{(2\pi)^3} \cdot \frac{1}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}_{III}} \cdot e^{-\left[J_I/\hat{J}_I + J_{II}/\hat{J}_{II} + J_{III}/\hat{J}_{III}\right]} \tag{10.7}$$

with

$$\begin{aligned} \hat{J}_k &= \frac{M_k}{2a_k} \\ &= \frac{1}{2\alpha_k} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot |v_{k5}(\tilde{s})|^2 \cdot \omega(\tilde{s}) \ ; \quad (k = I,\ II,\ III) \end{aligned} \tag{10.8}$$

(see eqns. (9.13a, b)). In Appendix **E** it is shown that this solution is unique.

From eqn. (10.7) the average $< J_k >$ of J_k $(k = I, II, III)$ is

$$\begin{aligned} < J_k > &\equiv \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^{\infty} dJ_I \int_0^{\infty} dJ_{II} \int_0^{\infty} dJ_{III} \\ & \quad \times W_{orbit}^{(stat)}(J_l, \Phi_l) \cdot J_k \ = \ \hat{J}_k \end{aligned} \tag{10.9a}$$

which agrees with eqn. (9.20). Using this result and the expression (6.31) for $\vec{\tilde{y}}(s)$, the beam emittance matrix

$$\begin{aligned} < \tilde{y}_m \tilde{y}_n > &\equiv \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^{\infty} dJ_I \int_0^{\infty} dJ_{II} \int_0^{\infty} dJ_{III} \\ & \quad \times W_{orbit}^{(stat)}(J_k, \Phi_k) \cdot \tilde{y}_m(s) \tilde{y}_n(s) \end{aligned}$$

is given by:

$$\begin{aligned} < \tilde{y}_m \tilde{y}_n > &= \frac{1}{(2\pi)^3} \cdot \frac{1}{\hat{J}_I \cdot \hat{J}_{II} \cdot \hat{J}_{III}} \cdot \int_0^{2\pi} d\Phi_I \int_0^{2\pi} d\Phi_{II} \int_0^{2\pi} d\Phi_{III} \int_0^{\infty} dJ_I \int_0^{\infty} dJ_{II} \int_0^{\infty} dJ_{III} \\ & \quad \times e^{-\left[J_I/\hat{J}_I + J_{II}/\hat{J}_{II} + J_{III}/\hat{J}_{III}\right]} \\ & \quad \times \sum_{k=I,II,III} \sqrt{J_k} \cdot \left\{ \hat{v}_{km}(s) \cdot e^{-i\Phi_k} + \hat{v}_{km}^*(s) \cdot e^{+i\Phi_k} \right\} \\ & \quad \times \sum_{l=I,II,III} \sqrt{J_l} \cdot \left\{ \hat{v}_{ln}(s) \cdot e^{-i\Phi_l} + \hat{v}_{ln}^*(s) \cdot e^{+i\Phi_l} \right\} \\ &= 2 \cdot \sum_{k=I,II,III} \hat{J}_k \cdot \Re e \left\{ \hat{v}_{km} \cdot \hat{v}_{kn}^* \right\} \\ &= 2 \cdot \sum_{k=I,II,III} \hat{J}_k \cdot \Re e \left\{ v_{km} \cdot v_{kn}^* \right\} . \end{aligned} \tag{10.9b}$$

This formula was already derived in Ref. [1] and (within the framework of a dispersion formalism) in Ref. [17] (see also Ref. [18], where the emittance matrix is parametrized in terms of products of eigenvectors from the outset).

Furthermore, the density of the particle distribution in the $(\tilde{x}-\tilde{p}_x-\tilde{z}-\tilde{p}_z-\tilde{\sigma}-\tilde{p}_\sigma)$ phase space is given by:

$$\rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma) = W_{orbit}^{(stat)} \cdot |\det(\underline{\mathcal{J}})|^{-1} \tag{10.10}$$

where $\underline{\mathcal{J}}$ signifies the Jacobian matrix (6.35).

But from eqn. (6.34) it follows that:

$$|\det(\underline{\mathcal{J}})| = 1$$

so that eqn. (10.10) finally takes the form:

$$\begin{aligned}
\rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma) &= \frac{1}{(2\pi)^3}\cdot\frac{1}{\hat{J}_I\cdot\hat{J}_{II}\cdot\hat{J}_{III}}\cdot e^{-\left[J_I/\hat{J}_I + J_{II}/\hat{J}_{II} + J_{III}/\hat{J}_{III}\right]} \\
&= \frac{1}{(2\pi)^3}\cdot\frac{1}{\hat{J}_I\cdot\hat{J}_{II}\cdot\hat{J}_{III}} \times \\
&\quad e^{-\left[|\vec{v}_I^{+}\underline{S}\vec{y}|^2/\hat{J}_I + |\vec{v}_{II}^{+}\underline{S}\vec{y}|^2/\hat{J}_{II} + |\vec{v}_{III}^{+}\underline{S}\vec{y}|^2/\hat{J}_{III}\right]} .
\end{aligned} \tag{10.11}$$

Here we have used the relationship

$$J_k = |\vec{v}_k^{+}\underline{S}\,\vec{y}|^2$$

which may be derived from eqns. (6.31) and (6.28).

10.1.2 The Surfaces of Constant Density in the $(\tilde{x}-\tilde{p}_x-\tilde{z}-\tilde{p}_z-\tilde{\sigma}-\tilde{p}_\sigma)$ Phase Space

We now look for the surfaces of constant density in the $(\tilde{x}-\tilde{p}_x-\tilde{z}-\tilde{p}_z-\tilde{\sigma}-\tilde{p}_\sigma)$ phase space i.e. the surfaces for which

$$\rho(\tilde{x},\tilde{p}_x,\tilde{z},\tilde{p}_z,\tilde{\sigma},\tilde{p}_\sigma) = \text{const}$$

$$\Longrightarrow \quad \frac{J_I}{\hat{J}_I} + \frac{J_{II}}{\hat{J}_{II}} + \frac{J_{III}}{\hat{J}_{III}} = \text{const} \tag{10.12}$$

(see eqn. (10.11)).

With the constraint (10.12) the variables $J_I,\ J_{II},\ J_{III}$ may be parameterized as

$$\begin{aligned}
\sqrt{J_I} &= C\cdot\cos\chi_1\cos\chi_2\cdot\sqrt{\hat{J}_I}\ ; \\
\sqrt{J_{II}} &= C\cdot\cos\chi_1\sin\chi_2\cdot\sqrt{\hat{J}_{II}}\ ; \\
\sqrt{J_{III}} &= C\cdot\sin\chi_1\cdot\sqrt{\hat{J}_{III}}
\end{aligned}$$

and then from (6.31) (with $\delta_k = \Phi_k - 2\pi Q_k\cdot s/L = const$) :

$$\begin{aligned}
\vec{y}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= C\sqrt{\hat{J}_I}\cdot\cos\chi_1\cos\chi_2\cdot\left[\vec{v}_I(s)\cdot e^{i\delta_I} + \vec{v}_I^{\,*}(s)\cdot e^{-i\delta_I}\right] \\
&+ C\sqrt{\hat{J}_{II}}\cdot\cos\chi_1\sin\chi_2\cdot\left[\vec{v}_{II}(s)\cdot e^{i\delta_{II}} + \vec{v}_{II}^{\,*}(s)\cdot e^{-i\delta_{II}}\right] \\
&+ C\sqrt{\hat{J}_{III}}\cdot\sin\chi_1\cdot\left[\vec{v}_{III}(s)\cdot e^{i\delta_{III}} + \vec{v}_{III}^{\,*}(s)\cdot e^{-i\delta_{III}}\right].
\end{aligned} \tag{10.13}$$

Equation (10.13) defines a six-dimensional ellipsoid in the $(\tilde{x}-\tilde{p}_x-\tilde{z}-\tilde{p}_z-\tilde{\sigma}-\tilde{p}_\sigma)$ phase space [30,31] which by (6.21) is periodic with period L [30] :

$$\vec{\tilde{y}}(s+L;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) = \vec{\tilde{y}}(s;\chi_1,\chi_2,\delta_I-2\pi Q_I,\delta_{II}-2\pi Q_{II},\delta_{III}-2\pi Q_{III}) .$$

By decomposing the vectors

$$C\cdot\sqrt{J_k}\cdot\vec{v}_k$$

into a real and imaginary part:

$$\begin{aligned} C\cdot\sqrt{\hat{J}_I}\cdot\vec{v}_I &= \frac{1}{2}\cdot[\vec{y}_1-i\cdot\vec{y}_2] ; \\ C\cdot\sqrt{\hat{J}_{II}}\cdot\vec{v}_{II} &= \frac{1}{2}\cdot[\vec{y}_3-i\cdot\vec{y}_4] ; \\ C\cdot\sqrt{\hat{J}_{III}}\cdot\vec{v}_{III} &= \frac{1}{2}\cdot[\vec{y}_5-i\cdot\vec{y}_6] \end{aligned}$$

eqn. (10.13) takes the form:

$$\begin{aligned} \vec{\tilde{y}}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cos\chi_2\cdot[\vec{y}_1(s)\cdot\cos\delta_I+\vec{y}_2(s)\cdot\sin\delta_I] \\ &+ \cos\chi_1\sin\chi_2\cdot[\vec{y}_3(s)\cdot\cos\delta_{II}+\vec{y}_4(s)\cdot\sin\delta_{II}] \\ &+ \sin\chi_1\cdot[\vec{y}_5(s)\cdot\cos\delta_{III}+\vec{y}_6(s)\cdot\sin\delta_{III}] . \end{aligned} \tag{10.14}$$

It follows from eqn. (10.14) that the motion of this ellipsoid under the influence of the external fields can be described by six generating orbit vectors $\vec{y}_k$:

$$\vec{y}_k(s) = \underline{M}(s,s_0)\ \vec{y}_k(s_0) ; \quad (k=1,2,3,4,5,6) . \tag{10.15}$$

Combining these vectors into a six-dimensional matrix $\underline{B}(s)$:

$$\underline{B}(s) = (\vec{y}_1(s),\ \vec{y}_2(s),\ \vec{y}_3(s),\ \vec{y}_4(s),\ \vec{y}_5(s),\ \vec{y}_6(s)) \tag{10.16}$$

we have:

$$\underline{B}(s) = \underline{M}(s,s_o)\ \underline{B}(s_o) . \tag{10.17}$$

This "bunch -shape matrix", $\underline{B}(s)$, now contains complete information about the configuration of the bunch and enables projection of the ellipsoid (10.14) onto the individual phase planes [30,31,32] as discussed in the next section.

10.1.3 The Projections of the 6-dimensional Ellipsoid. Beam Envelopes

In order to determine the projections of the six dimensional ellipsoid which characterize the beam envelopes [32] we first of all write eqn. (10.14) in component form:

$$\begin{aligned} \tilde{x}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) = &\cos\chi_1\cdot\cos\chi_2\cdot[y_{11}(s)\cdot\cos\delta_I+y_{21}(s)\cdot\sin\delta_I]+ \\ &\cos\chi_1\cdot\sin\chi_2\cdot[y_{31}(s)\cdot\cos\delta_{II}+y_{41}(s)\cdot\sin\delta_{II}]+ \\ &\sin\chi_1\cdot[y_{51}(s)\cdot\cos\delta_{III}+y_{61}(s)\cdot\sin\delta_{III}] ; \end{aligned} \tag{10.18a}$$

$$\begin{aligned}\tilde{p}_x(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cdot\cos\chi_2\cdot[y_{12}(s)\cdot\cos\delta_I + y_{22}(s)\cdot\sin\delta_I] + \\ &\quad \cos\chi_1\cdot\sin\chi_2\cdot[y_{32}(s)\cdot\cos\delta_{II} + y_{42}(s)\cdot\sin\delta_{II}] + \\ &\quad \sin\chi_1\cdot[y_{52}(s)\cdot\cos\delta_{III} + y_{62}(s)\cdot\sin\delta_{III}] \ ; \qquad (10.18b)\end{aligned}$$

$$\begin{aligned}\tilde{z}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cdot\cos\chi_2\cdot[y_{13}(s)\cdot\cos\delta_I + y_{23}(s)\cdot\sin\delta_I] + \\ &\quad \cos\chi_1\cdot\sin\chi_2\cdot[y_{33}(s)\cdot\cos\delta_{II} + y_{43}(s)\cdot\sin\delta_{II}] + \\ &\quad \sin\chi_1\cdot[y_{53}(s)\cdot\cos\delta_{III} + y_{63}(s)\cdot\sin\delta_{III}] \ ; \qquad (10.18c)\end{aligned}$$

$$\begin{aligned}\tilde{p}_z(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cdot\cos\chi_2\cdot[y_{14}(s)\cdot\cos\delta_I + y_{24}(s)\cdot\sin\delta_I] + \\ &\quad \cos\chi_1\cdot\sin\chi_2\cdot[y_{34}(s)\cdot\cos\delta_{II} + y_{44}(s)\cdot\sin\delta_{II}] + \\ &\quad \sin\chi_1\cdot[y_{54}(s)\cdot\cos\delta_{III} + y_{64}(s)\cdot\sin\delta_{III}] \ ; \qquad (10.18d)\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cdot\cos\chi_2\cdot[y_{15}(s)\cdot\cos\delta_I + y_{25}(s)\cdot\sin\delta_I] + \\ &\quad \cos\chi_1\cdot\sin\chi_2\cdot[y_{35}(s)\cdot\cos\delta_{II} + y_{45}(s)\cdot\sin\delta_{II}] + \\ &\quad \sin\chi_1\cdot[y_{55}(s)\cdot\cos\delta_{III} + y_{65}(s)\cdot\sin\delta_{III}] \ ; \qquad (10.18e)\end{aligned}$$

$$\begin{aligned}\tilde{p}_\sigma(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \cos\chi_1\cdot\cos\chi_2\cdot[y_{16}(s)\cdot\cos\delta_I + y_{26}(s)\cdot\sin\delta_I] + \\ &\quad \cos\chi_1\cdot\sin\chi_2\cdot[y_{36}(s)\cdot\cos\delta_{II} + y_{46}(s)\cdot\sin\delta_{II}] + \\ &\quad \sin\chi_1\cdot[y_{56}(s)\cdot\cos\delta_{III} + y_{66}(s)\cdot\sin\delta_{III}] \ . \qquad (10.18f)\end{aligned}$$

The computation of the single projections is then similar to that in Ref. [30] in which the functional relationship between pairs of components was investigated.

Since the details of the method have already been given in Refs. [30] and [31] only a summary will be needed here.

1) Projection on the $x - z$ plane.

We first investigate the projection on the $x - z$ plane. This describes the beam cross section. We will need the maximum amplitude in the x and z directions.

a) Maximum oscillation amplitude in the x direction:

Using the relation

$$Max_{(\delta)}\ \{A\cdot\cos\delta + B\cdot\sin\delta\} = \sqrt{A^2+B^2}$$

and eqn. (10.18a), the largest possible $\tilde{x}$ amplitude is

$$\begin{aligned}Max_{(\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III})}\ \tilde{x}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) &= \sqrt{y_{11}^2+y_{21}^2+y_{31}^2+y_{41}^2+y_{51}^2+y_{61}^2} \\ &= E_x(s)\ . \qquad (10.19)\end{aligned}$$

This occurs for the values:

$$\cos\delta_I = \frac{y_{11}}{\sqrt{y_{11}^2+y_{21}^2}}\ ; \quad \sin\delta_I = \frac{y_{21}}{\sqrt{y_{11}^2+y_{21}^2}}\ ;$$

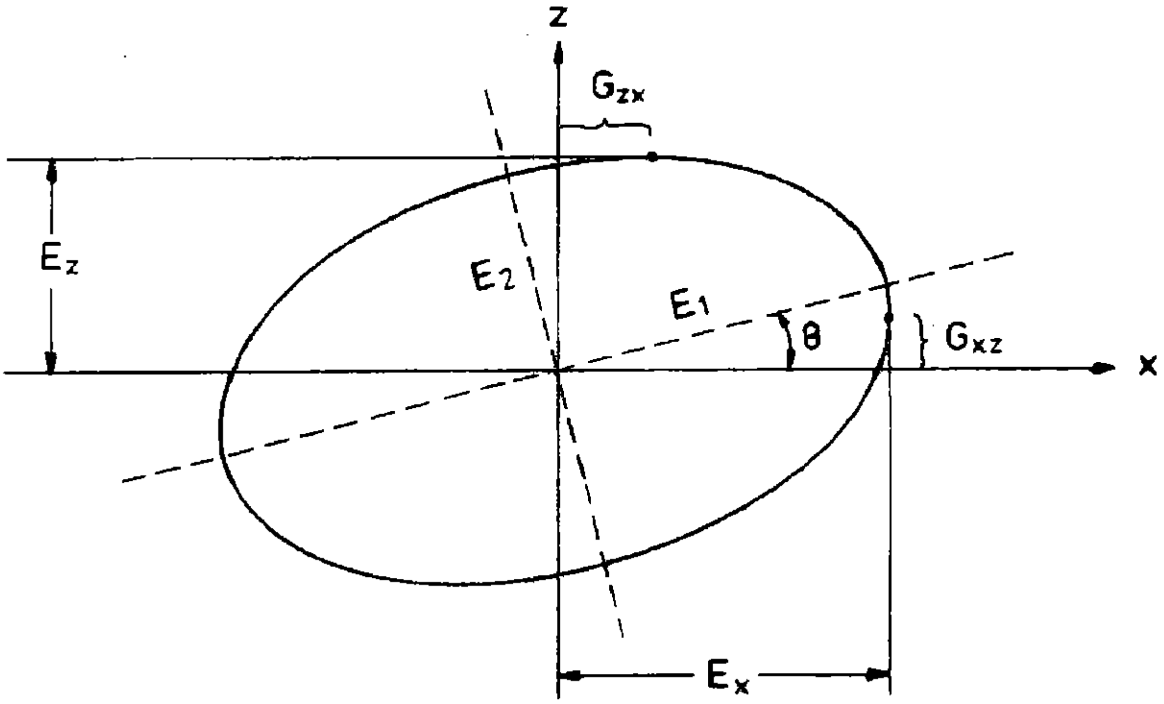

Figure 1: Projection on the $x - z$ plane (beam cross section)

$$\cos\delta_{II} = \frac{y_{31}}{\sqrt{y_{31}^2 + y_{41}^2}} \; ; \quad \sin\delta_{II} = \frac{y_{41}}{\sqrt{y_{31}^2 + y_{41}^2}} \; ;$$

$$\cos\delta_{III} = \frac{y_{51}}{\sqrt{y_{51}^2 + y_{61}^2}} \; ; \quad \sin\delta_{III} = \frac{y_{61}}{\sqrt{y_{51}^2 + y_{61}^2}} \; ;$$

$$\cos\chi_2 = \frac{\sqrt{y_{11}^2 + y_{21}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}} \; ;$$

$$\sin\chi_2 = \frac{\sqrt{y_{31}^2 + y_{41}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}} \; ;$$

$$\cos\chi_1 = \frac{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2 + y_{51}^2 + y_{61}^2}} \; ;$$

$$\sin\chi_1 = \frac{\sqrt{y_{51}^2 + y_{61}^2}}{\sqrt{y_{11}^2 + y_{21}^2 + y_{31}^2 + y_{41}^2 + y_{51}^2 + y_{61}^2}} \; . \tag{10.20}$$

The corresponding $\tilde{z}$ -coordinate is given by eqn. (10.18c) together with eqn. (10.20):

$$G_{xz} = \frac{1}{E_x(s)} \cdot \{y_{11} \cdot y_{13} + y_{21} \cdot y_{23} + y_{31} \cdot y_{33} + y_{41} \cdot y_{43} + y_{51} \cdot y_{53} + y_{61} \cdot y_{63}\} \; . \tag{10.21}$$

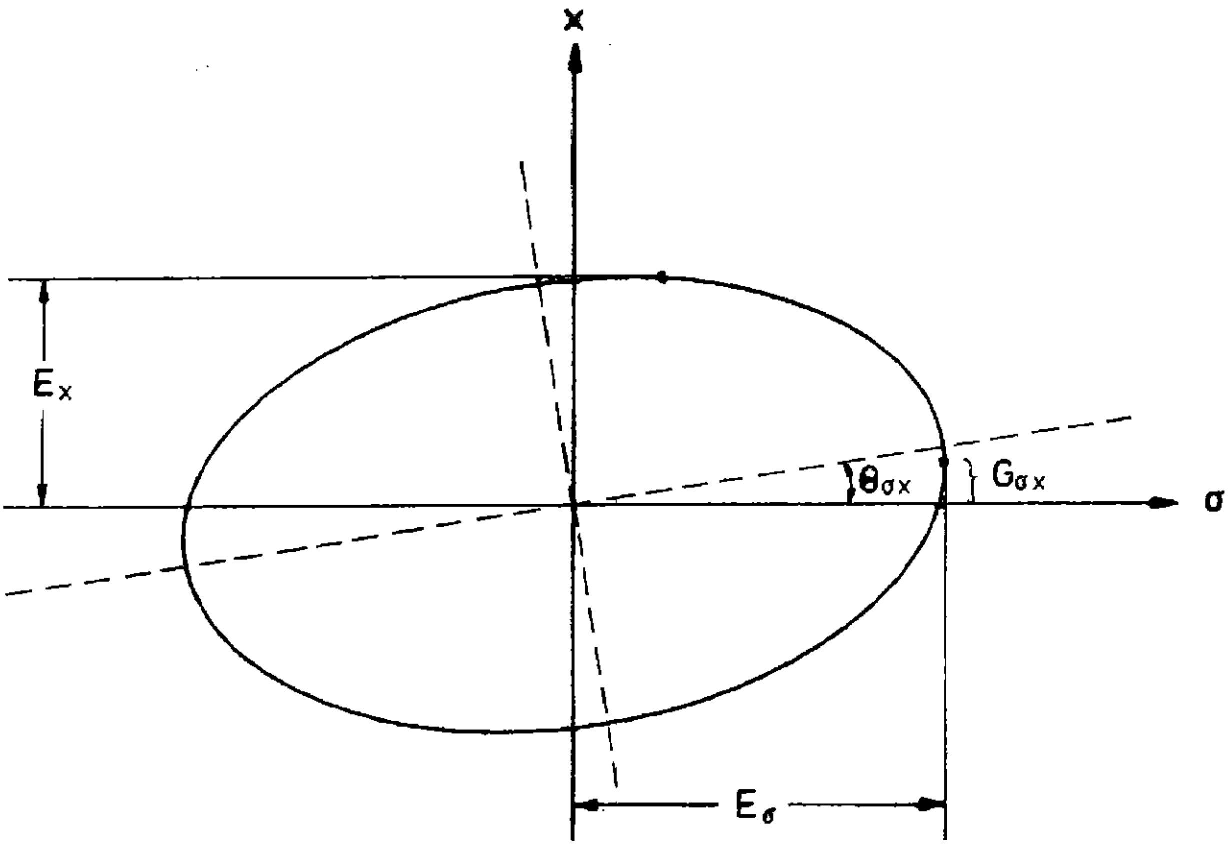

Figure 2: Projection on the $x - \sigma$ plane

b) Maximum oscillation amplitude in the z direction:
Correspondingly, the maximum amplitude in the z direction is obtained from (10.18c):

$$Max_{(\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III})}\ \tilde{z}(s;\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) = \sqrt{y_{13}^2+y_{23}^2+y_{33}^2+y_{43}^2+y_{53}^2+y_{63}^2}$$

$$= E_z(s)\ . \tag{10.22}$$

The accompanying $\tilde{x}$ -coordinate is then:

$$G_{zx} = \frac{1}{E_z(s)}\cdot\{y_{11}\cdot y_{13}+y_{21}\cdot y_{23}+y_{31}\cdot y_{33}+y_{41}\cdot y_{43}+y_{51}\cdot y_{53}+y_{61}\cdot y_{63}\}\ . \tag{10.23}$$

Thus

$$E_x\cdot G_{xz} = E_z\cdot G_{zx}\ . \tag{10.24}$$

c) The boundary curve of the beam cross section.
The projection of the ellipsoid (10.14) is an ellipse, and this is described by the three

independent quantities E_x, G_{xz}, E_z. The parameter G_{zx} depends on the other three (see eqn. (10.24)). In terms of E_x, G_{xz}, E_z, the ellipse can be written as:

$$E_z^2 \cdot \tilde{x}^2 - 2E_x G_{xz} \cdot \tilde{x}\tilde{z} + E_x^2 \cdot \tilde{z}^2 = \epsilon_{xz}^2 \tag{10.25a}$$

with

$$\epsilon_{xz} = E_x \cdot \sqrt{E_z^2 - G_{xz}^2} \tag{10.25b}$$

and where $\pi\epsilon_{xz}$ is the area of the ellipse.

The half axes E_1 and E_2 of the elliptical beam cross section are:

$$E_{1,2} = \frac{1}{2}\left\{\left[E_x^2 + E_z^2\right] \pm \sqrt{[E_x^2 - E_z^2]^2 + 4E_x^2 \cdot G_{xz}^2}\right\} \tag{10.26}$$

and the twist angle θ of the beam is given by:

$$\tan\ 2\theta = 2 \cdot \frac{E_x \cdot G_{xz}}{E_x^2 - E_z^2}\ . \tag{10.27}$$

2) <u>Projection on the $x - \sigma$ plane.</u>

To find the projection of the ellipsoid (10.14) onto the $x - \sigma$ plane we need equations (10.18a, e). Since these have the same general form as eqns. (10.18a, c), we can obtain the projection using exactly the same methods as in the previous section.

The boundary curve of the elliptical projection on the $x - \sigma$ plane is:

$$E_x^2 \cdot \tilde{\sigma}^2 - 2E_\sigma G_{\sigma x} \cdot \tilde{\sigma}\tilde{x} + E_\sigma^2 \cdot \tilde{x}^2 = \epsilon_{\sigma x}^2 \tag{10.28}$$

with

$$\begin{aligned}
E_\sigma &= \sqrt{y_{15}^2 + y_{25}^2 - y_{35}^2 + y_{45}^2 + y_{55}^2 + y_{65}^2}\ ; && (10.29a)\\
G_{\sigma x} &= \frac{1}{E_\sigma}\{y_{11} \cdot y_{15} + y_{21} \cdot y_{25} + y_{31} \cdot y_{35} + y_{41} \cdot y_{45} + y_{51} \cdot y_{55} + y_{61} \cdot y_{65}\}\ ; && (10.29b)\\
\epsilon_{\sigma x} &= E_\sigma \cdot \sqrt{E_x^2 - (G_{\sigma x})^2}\ . && (10.29c)
\end{aligned}$$

The meaning of E_σ and $G_{\sigma x}$ is explained by Fig.2. $\pi\epsilon_{\sigma x}$ is the area of the ellipse (10.28).

3) <u>Projection on the $z - \sigma$ plane.</u>

Finally, the projection of the ellipsoid on the $z - \sigma$ plane (see Fig.3) has the boundary curve:

$$E_z^2 \cdot \tilde{\sigma}^2 - 2E_\sigma G_{\sigma z} \cdot \tilde{\sigma}\tilde{z} + E_\sigma^2 \cdot \tilde{z}^2 = \epsilon_{\sigma z}^2 \tag{10.30}$$

with

$$\begin{aligned}
G_{\sigma z} &= \frac{1}{E_\sigma}\{y_{13} \cdot y_{15} + y_{23} \cdot y_{25} - y_{33} \cdot y_{35} + y_{43} \cdot y_{45} + y_{53} \cdot y_{55} + y_{63} \cdot y_{65}\}\ ; && (10.31a)\\
\epsilon_{\sigma z} &= E_\sigma \cdot \sqrt{E_z^2 - (G_{\sigma z})^2} && (10.31b)
\end{aligned}$$

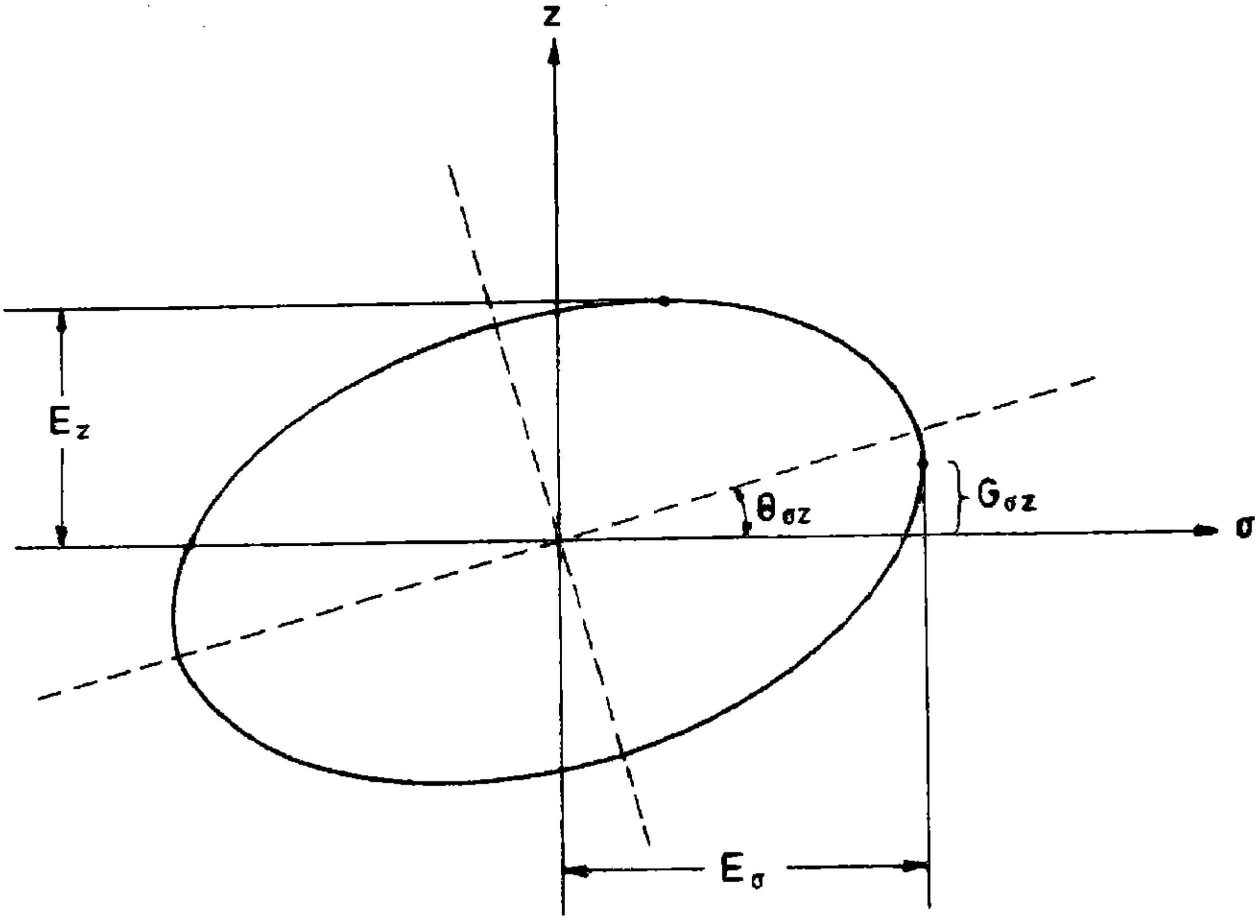

Figure 3: Projection on the $z - \sigma$ plane

($\pi\epsilon_{\sigma z}$ is the area of the ellipse (10.30)).

The projection on the $y - p_y$ plane ($y = x,\ z,\ \sigma$) can be found in a similar way [30,31]. We obtain:

4) Projection on the $x - p_x$ plane.

For the projection of the ellipsoid (10.14) onto the $x - p_x$ plane the corresponding equations are (10.18a, b). Since these two relations have the same form as eqns. (10.18a, c), we obtain an elliptical projection onto the $x - p_x$ plane by analogy with eqn. (10.25). We write the ellipse in the form:

$$A_x^2 \cdot \tilde{x}^2 - 2E_x G_{xp_x} \cdot \tilde{x}\tilde{p}_x + E_x^2 \cdot \tilde{p}_x^2 = \varepsilon_{xp_x}^2 \tag{10.32}$$

with

$$\begin{aligned} A_x(s) &= Max_{(\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III})}\ \tilde{p}_x(s,\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) \\ &= \sqrt{y_{12}^2 + y_{22}^2 + y_{32}^2 + y_{42}^2 + y_{52}^2 + y_{62}^2}\ ; \end{aligned} \tag{10.33}$$

$$G_{xp_x}(s) = \frac{1}{E_x(s)} \cdot \{y_{11} \cdot y_{12} + y_{21} \cdot y_{22} + y_{31} \cdot y_{32} + y_{41} \cdot y_{42} + y_{51} \cdot y_{52} + y_{61} \cdot y_{62}\}\ ; \tag{10.34}$$

$$\pi\varepsilon_{xp_x} = \pi \cdot E_x\sqrt{A_x^2 - E_{p_x}^2}\ ; \tag{10.35}$$

(area of the ellipse(10.32)).

Here, the function $A_x(s)$ represents the maximum amplitude of the momentum p_x and could be called the momentum envelope for the $x - p_x$ plane. $\pi\varepsilon_{xp_x}$ gives the area of the ellipse (10.32) and the meaning of E_{p_x} is indicated in Fig. 4.

5) Projection on the $z - p_z$ plane.

A similar treatment can be used to describe the projection on the $z - p_z$ plane. We write

$$A_z^2 \cdot \tilde{z}^2 - 2E_z G_{zp_z} \cdot \tilde{z}\tilde{p}_z + E_z^2 \cdot \tilde{p}_z^2 = \varepsilon_{zp_z}^2 \tag{10.36}$$

where

$$\begin{aligned} A_z(s) &= Max_{(\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III})}\ \tilde{p}_z(s,\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) \\ &= \sqrt{y_{14}^2 + y_{24}^2 + y_{34}^2 + y_{44}^2 + y_{54}^2 + y_{64}^2}\ ; \end{aligned} \tag{10.37}$$

$$G_{zp_z}(s) = \frac{1}{E_z(s)} \cdot \{y_{13} \cdot y_{14} + y_{23} \cdot y_{24} + y_{33} \cdot y_{34} + y_{43} \cdot y_{44} + y_{53} \cdot y_{54} + y_{63} \cdot y_{64}\}\ ; \tag{10.38}$$

$$\pi\varepsilon_{zp_z} = \pi \cdot E_z\sqrt{A_z^2 - E_{p_z}^2} \tag{10.39}$$

(area of the ellipse (10.36)).

6) Projection on the $\sigma - p_\sigma$ plane.

For the projection on the $\sigma - p_\sigma$ plane we may write:

$$A_\sigma^2 \cdot \tilde{\sigma}^2 - 2E_\sigma G_{\sigma p_\sigma} \cdot \tilde{\sigma}\tilde{p}_\sigma + E_\sigma^2 \cdot \tilde{p}_\sigma^2 = \varepsilon_{\sigma p_\sigma}^2 \tag{10.40}$$

where

$$\begin{aligned} A_\sigma(s) &= Max_{(\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III})}\ \tilde{p}_\sigma(s,\chi_1,\chi_2,\delta_I,\delta_{II},\delta_{III}) \\ &= \sqrt{y_{16}^2 + y_{26}^2 + y_{36}^2 + y_{46}^2 + y_{56}^2 + y_{66}^2}\ ; \end{aligned} \tag{10.41}$$

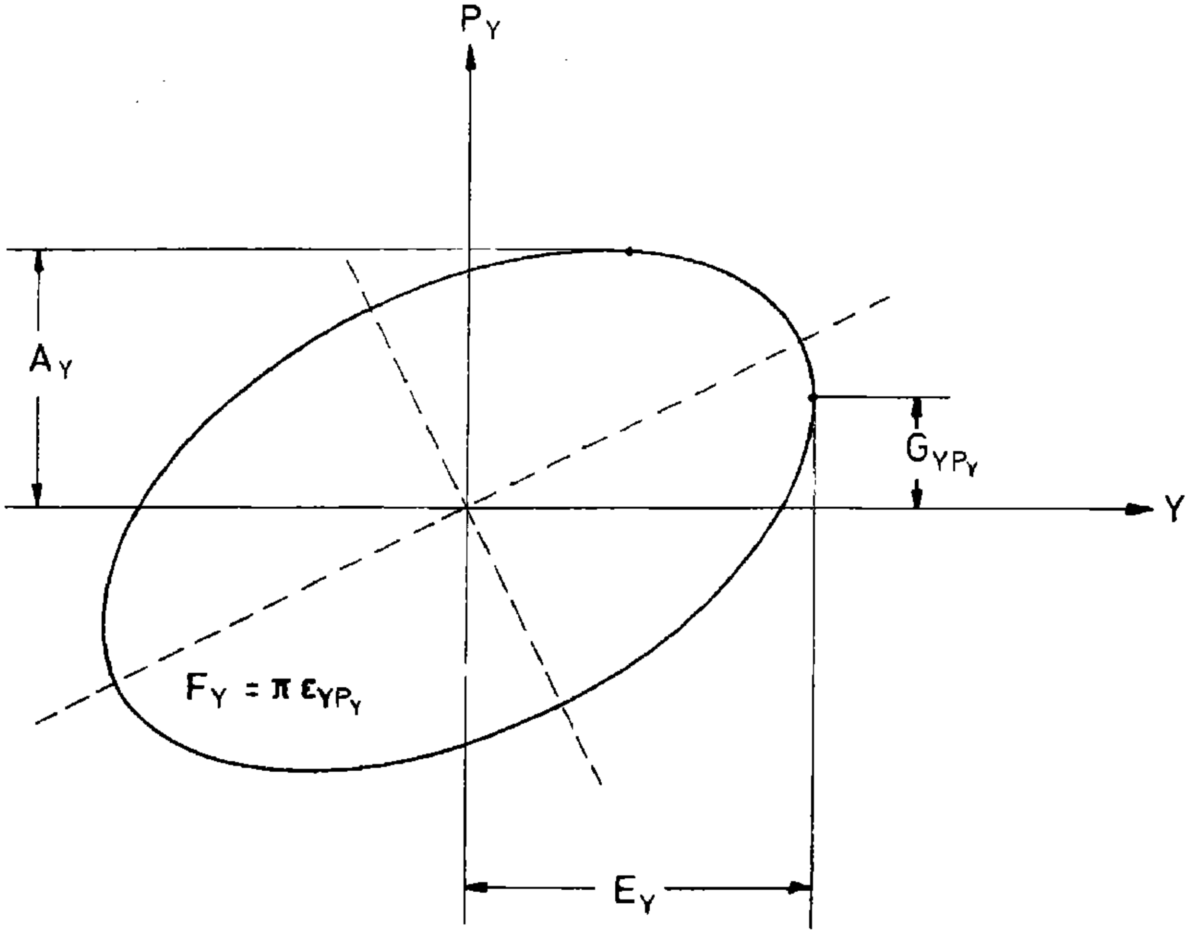

Figure 4: Projection on the $y - p_y$ plane; $(y \equiv x, z, \sigma)$

$$G_{\sigma p_\sigma}(s) = \frac{1}{E_\sigma(s)} \cdot \{y_{15} \cdot y_{16} + y_{25} \cdot y_{26} + y_{35} \cdot y_{36} + y_{45} \cdot y_{46} + y_{55} \cdot y_{56} + y_{65} \cdot y_{66}\} \; ; \qquad (10.42)$$

$$\pi \varepsilon_{\sigma p_\sigma} = \pi \cdot E_\sigma \sqrt{A_\sigma^2 - E_{p_\sigma}^2} \qquad (10.43)$$

(area of the ellipse (10.40)).

This is all represented by the ellipse in Fig. 4.

Finally we mention that all the projections of the ellipsoid already considered are included in the formula

$$E_l^2 \cdot \tilde{y}_k^2 - 2E_k G_{kl} \cdot \tilde{y}_k \tilde{y}_l + E_k^2 \cdot \tilde{y}_l^2 = \epsilon_{kl}^2 \qquad (10.44)$$

with

$$E_k(s) \;=\; \sqrt{y_{1k}^2 + y_{2k}^2 + y_{3k}^2 + y_{4k}^2 + y_{5k}^2 + y_{6k}^2} \; ; \qquad (10.45a)$$

$$E_l(s) = \sqrt{y_{1l}^2 + y_{2l}^2 + y_{3l}^2 + y_{4l}^2 + y_{5l}^2 + y_{6l}^2}\ ; \tag{10.45b}$$

$$G_{kl}(s) = \frac{1}{E_k}\{y_{1k}\cdot y_{1l} + y_{2k}\cdot y_{2l} + y_{3k}\cdot y_{3l} + y_{4k}\cdot y_{4l} + y_{5k}\cdot y_{5l} + y_{6k}\cdot y_{6l}\}\ ; \tag{10.45c}$$

$$\epsilon_{kl} = E_k \cdot \sqrt{E_l^2 - G_{kl}^2} \tag{10.45d}$$

describing the projection onto the $y_k - y_l$ plane (see Fig. 5) where we have used the notation:

$$\begin{aligned}
\tilde{y}_1 &\equiv \tilde{x}\ ; \\
\tilde{y}_2 &\equiv \tilde{p}_x\ ; \\
\tilde{y}_3 &\equiv \tilde{z}\ ; \\
\tilde{y}_4 &\equiv \tilde{p}_z\ ; \\
\tilde{y}_5 &\equiv \tilde{\sigma}\ ; \\
\tilde{y}_6 &\equiv \tilde{p}_\sigma
\end{aligned}$$

and

$$\begin{aligned}
E_2 &\equiv A_x\ ; \\
E_4 &\equiv A_z\ ; \\
E_6 &\equiv A_\sigma\ .
\end{aligned}$$

($\pi\epsilon_{kl}$ is the area of the ellipse (10.44)).

10.2 Spin Motion

In order to describe the spin motion we adopt the ansatz:

$$W = W_{orbit}^{(stat)}(J_I, \Phi_I, J_{II}, \Phi_{II}, J_{III}, \Phi_{III}) \cdot W_{spin}(J_{IV}, \Phi_{IV}, s) \tag{10.46}$$

i. e. we assume a stationary distribution for the orbital motion and that W_{spin} is independent of the orbital variables J_k, Φ_k ($k = I, II, III$).

Then, putting (10.46) into eqn. (9.15), and using (10.7):

$$\begin{aligned}
\frac{\partial}{\partial s} W_{spin} &= +\frac{\partial}{\partial J_{IV}}\left[M_{IV}\cdot J_{IV}\cdot \frac{\partial}{\partial J_{IV}} W_{spin}\right] \\
&\quad -\frac{\partial}{\partial \Phi_{IV}}\left[b_{IV}\cdot W_{spin} - \frac{M_{IV}}{4J_{IV}}\cdot\frac{\partial}{\partial\Phi_{IV}} W_{spin}\right]\ .
\end{aligned} \tag{10.47}$$

which is valid at orbital equilibrium.

In the case that the spin phase Φ_{IV} is uniformly distributed (see also Appendix E) eqn. (10.47) reduces to

$$W_{spin}(J_{IV}, \Phi_{IV}, s) = \frac{1}{2\pi}\cdot \hat{W}(J_{IV}, s) \tag{10.48}$$

and we may write (with $J \equiv J_{IV}$):

$$\frac{\partial}{\partial s}\hat{W} = M_{IV}\cdot\frac{\partial}{\partial J}\left[J\cdot\frac{\partial}{\partial J}\hat{W}\right]\ . \tag{10.49}$$

In contrast to the case of orbital motion, $a_{IV} = 0$ i.e. the spin motion is not damped (eqns. 9.13) and we do not expect that the spin distribution reaches equilibrium. Thus here, we make no attempt to find a solution for which $\frac{\partial}{\partial s} W_{spin} = 0$. If we had included a phenomenological classical description of polarization build up via a model of spin damping, we would have been able to find equilibrium.

Now, writing out the spin part of eqn. (7.1), and using (5.13), (5.15) and (6.72):

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= \sum_{k=I,II,III,IV} \{A_k(s) \cdot \vec{w}_k(s) + A_{-k}(s) \cdot \vec{w}_{-k}(s)\} \\
&= \vec{n}(s) + \{A_{IV}(s) \cdot \vec{w}_{IV}(s) + A_{-IV}(s) \cdot \vec{w}_{-IV}(s)\} \qquad (10.50)
\end{aligned}$$

with

$$\begin{aligned}
\vec{n} &= \sum_{k=I,II,III} \{A_k(s) \cdot \vec{w}_k(s) + A_{-k}(s) \cdot \vec{w}_{-k}(s)\} \\
&= \sum_{k=I,II,III} \sqrt{J_k(s)} \cdot \left\{ e^{-i \cdot \Phi_k(s)} \cdot \vec{w}_k(s) + e^{+i \cdot \Phi_k(s)} \cdot \vec{\bar{w}}_k(s) \right\} . \qquad (10.51)
\end{aligned}$$

The vector $\vec{n}$ represents a solution of the linearized BMT equation on the trajectory

$$(J_k, \Phi_k) \; ; \quad k = I, \ II, \ III$$

which depends explicitely on the orbital phase space point and on s [8]. Under Fourier analysis $\vec{n}$ contains no spin frequency component. This property is equivalent to the periodicity conditions [4]:

$$\begin{aligned}
\vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s) &= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, s) \\
&= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, s) \\
&= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, s) \\
&= \vec{n}(J_I, J_{II}, J_{III}, \Phi_I, \Phi_{II}, \Phi_{III}, s + L) . \qquad (10.52)
\end{aligned}$$

The general non-linearized form for $\vec{n}$, is an important quantity in the quantum theory of the polarization process [4,3]. The linearized form in terms of eigenvectors used here was first given by S.R. Mane [33] who also suggested a way to calculate $\vec{n}$ to higher order [3].

Introducing now the spin vector

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} - \vec{n}(s) = A_{IV}(s) \cdot \vec{w}_{IV}(s) + compl.conj. \qquad (10.53)$$

which describes the spin motion around the $\vec{n}$-axis we obtain from (6.73b) and (8.1b):

$$\begin{aligned}
\tilde{\alpha} &= \frac{1}{\sqrt{2}} \sqrt{J_{IV}} \cdot e^{-i \cdot \Phi_{IV}} + compl.conj. \\
&= \sqrt{2} \sqrt{J_{IV}} \cdot \cos \Phi_{IV} \; ; \\
\tilde{\beta} &= \frac{1}{\sqrt{2}} \sqrt{J_{IV}} \cdot \frac{1}{i} e^{-i \cdot \Phi_{IV}} + compl.conj. \\
&= -\sqrt{2} \sqrt{J_{IV}} \cdot \sin \Phi_{IV}
\end{aligned}$$

[8] Equation (10.51) describes the $\vec{n}$ axis in terms of its components in the $(\vec{m}, \vec{l})$-plane perpendicular to $\vec{n}_0$. For the complete expression for $\vec{n}$ including the $\vec{n}_0$ component see eqn. (5.11).

and therefore

$$J = \frac{1}{2}\left[\tilde{\alpha}^2 + \tilde{\beta}^2\right] \; . \tag{10.54}$$

Thus spins at the same point in the orbital phase space $(\tilde{x},\ \tilde{p}_x,\ \tilde{z},\ \tilde{p}_z,\ \tilde{\sigma},\ \tilde{p}_\sigma)$ and s, can be considered to precess around a common axis $\vec{n}$ with a tilt angle w.r.t. $\vec{n}$ proportional to J. The quantity J describes the spin component perpendicular to the $\vec{n}$ axis [9] .

We now consider the case where at each point in phase space the spins are initially all parallel to the respective $\vec{n}$-axes and that orbital equilibrium has already been established. As in Remark 4 in chapter 9 we can assume that the subsequent distribution of Φ_{IV} will be uniform at all points in the orbital phase space. Thus for the polarization vector $\vec{P}(J_k, \Phi_k;\ s)$ $(k = I, II, III)$ defined as the vector average of the spins at the point

$$(J_k, \Phi_k; s);\ k = I, II, III,$$

we have (see Fig. 5 and eqn. (5.11))

$$\begin{aligned} \vec{P}(J_k, \Phi_k; s) &= \vec{n} \cdot < \sqrt{1 - (\tilde{\alpha}^2 + \tilde{\beta}^2)} > \\ &= \vec{n} \cdot < 1 - \frac{1}{2}(\tilde{\alpha}^2 + \tilde{\beta}^2) > \\ &= \vec{n} \cdot (1 - < J >) \end{aligned} \tag{10.55}$$

in our approximation that $(\tilde{\alpha}^2 + \tilde{\beta}^2) \ll 1$. Therefore $\vec{P}$ remains parallel to $\vec{n}$ [10] .

Furthermore for the expectation value of J [11] :

$$\langle\ J\ \rangle = \int_0^\infty dJ \cdot J \cdot \hat{W}(J, s)$$

eqn. (10.49) gives:

$$\begin{aligned} \frac{d}{ds}\langle\ J\ \rangle &= \int_0^\infty dJ \cdot J \cdot \frac{\partial}{\partial s}\hat{W}(J, s) \\ &= M_{IV} \cdot \int_0^\infty dJ \cdot \left\{ J^2 \cdot \frac{\partial^2 \hat{W}}{\partial J^2} + J \cdot \frac{\partial \hat{W}}{\partial J} \right\} \end{aligned}$$

and using the relations

$$\begin{aligned} \int_0^\infty dJ \cdot J \cdot \frac{\partial \hat{W}}{\partial J} &= J \cdot \left[\hat{W}\right]_0^\infty - \int_0^\infty dJ \cdot 1 \cdot \hat{W} = -1 \; ; \\ \int_0^\infty dJ \cdot J^2 \cdot \frac{\partial^2 \hat{W}}{\partial J^2} &= J^2 \cdot \left[\frac{\partial \hat{W}}{\partial J}\right]_0^\infty - \int_0^\infty dJ \cdot 2J \cdot \frac{\partial \hat{W}}{\partial J} = +2 \end{aligned}$$

we may write

$$\frac{d}{ds}\langle\ J\ \rangle \;=\; M_{IV} \tag{10.56}$$

which is in agreement with eqn. (9.19).

[9] Recall that the working point is far from spin orbit resonances (see Remark 1 in chapter 9 where we discuss averaging). Therefore the spin components of the 8-dimensional eigenvectors $\vec{q}_k$ $(k = I,\ II,\ III)$ are small [1] and $\vec{n}$ is approximately parallel to $\vec{n}_0$. For the same reason $(\tilde{\alpha}^2 + \tilde{\beta}^2) \ll 1$ in eqn. (10.55).

[10] The proposition that $\vec{P}$ and $\vec{n}$ are parallel is an ingredient in the formalisms of references [4] and [3]. At the level of linearized spin orbit motion eqn.(10.55) provides an example of this notion.

[11] In this treatment we assume that J is small. Thus the distribution of $\hat{W}$ peaks at small J and the contribution to the integral from large J can be neglected.

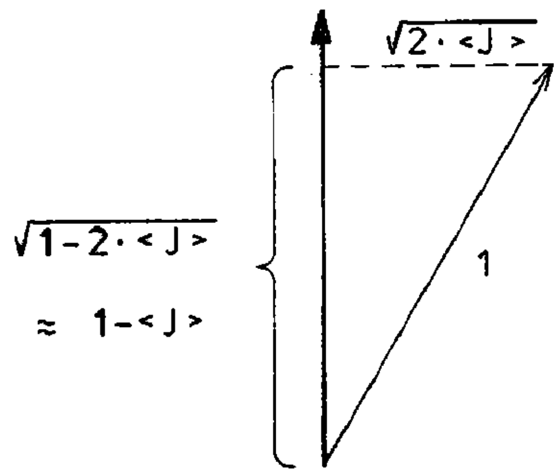

Figure 5: Degree of Polarization

Starting with

$$P = 1 \implies \langle\ J\ \rangle = 0$$

(all spins parallel to $\vec{n}$) we therefore have from (10.55) and (10.56) [1]:

$$\begin{aligned} \frac{d}{ds}\ P &= \frac{d}{ds}\ [1 - \langle\ J\ \rangle] = -M_{IV} = -M_{IV} \cdot P \\ \implies \quad \frac{d}{dt}\ P &= -c \cdot M_{IV} \cdot P\ . \end{aligned} \tag{10.57}$$

Then using eqn. (9.13f) we can calculate the characteristic spin depolarization time τ_D for the diffusion of spins resulting from stochastic orbit motion:

$$\tau_D^{-1} = 2 \cdot \frac{c}{L} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \omega(\tilde{s}) \cdot \sum_{\mu=1}^{2} \left(\Im m \sum_{k=I,II,III} [v_{k5}^*(\tilde{s}) \cdot w_{k\mu}] \right)^2 \tag{10.58}$$

This result has already been derived in a different manner by A. Chao [11]), and serves as the basis of various schemes for maximizing the polarization [34,35,36,37,38,39]. The same picture emerges by analysing the physical content of eqn (9.33) in Ref. [1].

As can be seen from eqn. (10.58), the depolarization time τ_D is independent of J_k, Φ_k ($k = I, II, III$) and thus has (in linear order) the same value at every point of the orbital phase space.

Note that if we look at a fixed point in phase space and s, the depolarization takes place with respect to $\vec{n}$. But taking an average over the phase space we find in our linearized spin treatment that the depolarization takes place along the $\vec{n}_0$ axis.

Even after the spins have reached large tilt angles, we can apply these concepts to the spin components parallel to $\vec{n}$ by noting that the component perpendicular to $\vec{n}$, due to spins in a small region of the equilibrium orbital phase space, will average to zero. Then by applying the above considerations to the spin components parallel to $\vec{n}$ we obtain

$$\frac{d}{dt} P = -c \cdot M_{IV} \cdot P . \tag{10.59}$$

where $0 \leq |P| \leq 1$.

The solution of (10.59) reads as :

$$P(t) = P(0) \cdot c^{-(t/\tau_D)} \tag{10.60}$$

and leads to an exponential decrease of the polarization. The value of P will be the same at all points in the orbital phase space.

11 Summary

We have investigated the influence of radiation damping and quantum fluctuations on the motion of charged particles in storage rings using the Fokker-Planck equation and have included classical spin motion in linear approximation.

The motion was described in terms of the fully six-dimensional formalism with the canonical variables $x,\ p_x,\ z,\ p_z,\ \sigma = s - c \cdot t,\ p_\sigma = \Delta E/E_0$.

With this set of variables we were then able to treat the betatron and synchrotron oscillations simultaneously and canonically, i.e. to provide an analytical technique which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities.

In order to derive the Fokker-Planck equation, action-angle variables were introduced via a canonical transformation taking into account all kinds of coupling (synchro-betatron coupling and coupling of the betatron oscillations by skew quadrupoles and solenoids).

The Fokker-Planck equation was solved for the stationary case and expressions for the average dimensions of the bunch in six dimensions were found.

By investigating the surfaces of constant density in the $(x,\ p_x,\ z,\ p_z,\ \sigma,\ p_\sigma)$ phase space we were able to describe the shape of the bunch in terms of a 6-dimensional ellipsoid. This ellipsoid was represented by the "bunch-shape matrix", $\underline{B}(s)$, which contains as columns six independent orbit vectors.

For the spin motion we presented an alternative way to calculate the linear depolarization time and obtained the usual result.

In this paper we have only considered the case of ultrarelativistic particles. To study the case of arbitrary velocity the variable $\sigma = s - v_0 \cdot t$ (v_0 =average speed of the particles) as described in Ref. [40,22] would be used.

Finally, we remark that starting from the variables $\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$ and using analytical techniques as described in Refs. [17,41] one can also develop an 8-dimensional dispersion formalism.

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Appendix A: Derivation of the Equations of Linearized Orbital Motion

A.1 The Hamiltonian

Starting from the Lagrangian (3.5) and introducing the longitudinal coordinate s as the independent variable (instead of the time t), one can construct the Hamiltonian of the orbit motion by a succession of canonical transformations. Choosing a gauge with $\phi = 0$, one then obtains in the ultrarelativistic case with $v \approx c$ [42] :

$$\begin{aligned} \mathcal{H}(x, p_x, z, p_z, \sigma, p_\sigma; s) &= (1 + p_\sigma) - (1 - p_\sigma) \cdot (1 + K_x \cdot x + K_z \cdot z) \times \\ & \qquad \left[1 - \frac{(p_x - \frac{e}{E_0} A_x)^2}{(1 + p_\sigma)^2} - \frac{(p_z - \frac{e}{E_0} A_z)^2}{(1 + p_\sigma)^2}\right]^{1/2} \\ & \quad - (1 + K_x \cdot x + K_z \cdot z) \cdot \frac{e}{E_0} A_s \ . \end{aligned} \tag{A.1}$$

The corresponding canonical equations read as

$$\begin{aligned} \frac{d}{ds}\, x &= +\frac{\partial \mathcal{H}}{\partial p_x} \ ; \quad \frac{d}{ds}\, p_x = -\frac{\partial \mathcal{H}}{\partial x} \ ; \\ \frac{d}{ds}\, z &= +\frac{\partial \mathcal{H}}{\partial p_z} \ ; \quad \frac{d}{ds}\, p_z = -\frac{\partial \mathcal{H}}{\partial z} \ ; \\ \frac{d}{ds}\, \sigma &= +\frac{\partial \mathcal{H}}{\partial p_\sigma} \ ; \quad \frac{d}{ds}\, p_\sigma = -\frac{\partial \mathcal{H}}{\partial \sigma} \ . \end{aligned}$$

Here the variables σ and p_σ describing the longitudinal motion are defined by

$$\begin{aligned} \sigma &= s - c \cdot t \ ; \\ p_\sigma \equiv \eta &= \frac{\Delta E}{E_0} \ . \end{aligned}$$

Since $\mathcal{H}$ also contains the transverse coordinates x, p_x, z, p_z we are thus able to handle synchrotron oscillations (longitudinal motion) and betatron oscillations (transverse motion) simultaneously.

In order to utilize this Hamiltonian, the magnetic field $\vec{B}$ and the corresponding vector potential,

$$\vec{A} = \vec{A}(x, y, s), \tag{A.2}$$

(eqn. (2.14)) for commonly occurring types of accelerator magnet must be given. Once $\vec{A}$ is known, the fields $\vec{\epsilon}$ and $\vec{B}$ can be found using eqns. (3.6, 7). In the variables x, z, s, σ these become (with $\phi = 0$):

$$\vec{\epsilon} = \frac{\partial}{\partial \sigma} \vec{A} \tag{A.3}$$

and

$$B_x = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial z}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] - \frac{\partial}{\partial s} A_z \right\} ; \tag{A.4a}$$

$$B_z = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x}[(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \right\} ; \tag{A.4b}$$

$$B_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x . \tag{A.4c}$$

A.2 Description of the Electromagnetic Field

Using the freedom to select a gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity [42].

A.2.1 Bending Magnet

If the curvatures K_x and K_z of the design orbit are given, the magnetic bending field on the design orbit, $B_x^{(0)}(s)$ and $B_z^{(0)}(s)$:

$$B_x^{(0)}(s) = B_x(0, 0, s) ; \tag{A.5a}$$
$$B_z^{(0)}(s) = B_z(0, 0, s) \tag{A.5b}$$

can be easily calculated from eqn. (3.9) if we notice that the design orbit

$$x(s) = z(s) \equiv 0 \tag{A.6}$$

is, by definition, a solution of the equations of motion for

$$\vec{\epsilon} = 0 ; \quad E \equiv m_0 \gamma c^2 = E_0 . \tag{A.7}$$

Thus we get (with $\dot{s} = c$) :

$$\frac{e}{E_0} \cdot B_x^{(0)} = -K_z ; \tag{A.8a}$$
$$\frac{e}{E_0} \cdot B_z^{(0)} = +K_x . \tag{A.8b}$$

The corresponding vector potential can be written as

$$\frac{\epsilon}{E_0} \cdot A_s = -\frac{1}{2}(1 + K_x \cdot x - K_z \cdot z) ; \tag{A.9a}$$

$$A_x = A_z = 0 . \tag{A.9b}$$

A.2.2 Quadrupole

The quadrupole fields are

$$B_x = z \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} ; \tag{A.10a}$$

$$B_z = x \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} , \tag{A.10b}$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}\left(z^2 - x^2\right) ; \tag{A.11a}$$

$$A_x = A_z = 0 . \tag{A.11b}$$

In the following we rewrite the term $(e/E_0) \cdot A_s$ in (A.1) as

$$\frac{e}{E_0} A_s = \frac{1}{2} g \cdot (z^2 - x^2) ; \tag{A.12a}$$

$$g = \frac{e}{E_0} \cdot \left(\frac{\partial B_z}{\partial x}\right)_{x=z=0} . \tag{A.12b}$$

A.2.3 Skew Quadrupole

The fields are

$$B_x = -\frac{1}{2} \cdot \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot x ; \tag{A.13a}$$

$$B_z = +\frac{1}{2} \cdot \left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot z . \tag{A.13b}$$

Thus we may use

$$A_s = -\frac{1}{2}\left(\frac{\partial B_z}{\partial z} - \frac{\partial B_x}{\partial x}\right)_{x=z=0} \cdot xz ; \tag{A.14a}$$

$$A_x = A_z = 0 , \tag{A.14b}$$

and we write

$$\frac{e}{E_0} A_s = N \cdot xz ; \tag{A.15a}$$

$$N = \frac{1}{2} \cdot \frac{e}{E_0} \left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0} . \tag{A.15b}$$

A.2.4 Solenoid Fields

The field components in the current free region are given by [42,43]:

$$B_x(x,z,s) = x \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + z^2)^\nu ; \tag{A.16a}$$

$$B_z(x,z,s) = z \cdot \sum_{\nu=0}^{\infty} b_{2\nu+1} \cdot (x^2 + z^2)^\nu ; \tag{A.16b}$$

$$B_s(x,z,s) = \sum_{\nu=0}^{\infty} b_{2\nu} \cdot (x^2 + z^2)^\nu \tag{A.16c}$$

where for consistency with Maxwell's equations the coefficients b_μ obey the recursion equations:

$$\begin{aligned} b_{2\nu+1}(s) &= -\frac{1}{(2\nu+2)} \cdot b'_{2\nu}(s) \; ; && \text{(A.17a)} \\ b_{2\nu+2}(s) &= +\frac{1}{(2\nu+2)} \cdot b'_{2\nu+1}(s) \; ; && \text{(A.17b)} \end{aligned}$$

$$(\nu = 0, 1, 2, ...)$$

and where

$$b_0(s) \equiv B_s(0,0,s) \; . \tag{A.18}$$

The vector potential leading to the solenoid field of eqn. (A.16) is then:

$$\begin{aligned} A_x(x,z,s) &= -z \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; && \text{(A.19a)} \\ A_z(x,z,s) &= +x \cdot \sum_{\nu=0}^{\infty} \frac{1}{(2\nu+2)} \cdot b_{(2\nu)}(s) \cdot r^{2\nu} \; ; && \text{(A.19b)} \\ A_s(x,z,s) &= 0 && \text{(A.19c)} \end{aligned}$$

with

$$r^2 = x^2 + z^2 \; .$$

Thus we can write :

$$\begin{aligned} \frac{\epsilon}{E_0} A_x &= -H(s) \cdot z + \frac{1}{8} H''(s) \cdot (x^2 + z^2) \cdot z + \cdots \; ; && \text{(A.20a)} \\ \frac{\epsilon}{E_0} A_z &= +H(s) \cdot x - \frac{1}{8} H''(s) \cdot (x^2 + z^2) \cdot x + \cdots && \text{(A.20b)} \end{aligned}$$

with

$$\begin{aligned} H(s) &= \frac{1}{2} \cdot \frac{\epsilon}{E_0} \cdot b_0(s) && \text{(A.21)} \\ &\equiv \frac{1}{2} \cdot \frac{\epsilon}{E_0} \cdot B_s(0,0,s) \; . \end{aligned}$$

Note that the cyclotron radius for the longitudinal field (A.18) is given by

$$R = \frac{1}{2 \cdot H} \; .$$

A.2.5 Dipole

$$\left\{ \begin{aligned} B_x &= \Delta \hat{B}_x \cdot \delta(s - s_0) \; ; \\ B_z &= \Delta \hat{B}_z \cdot \delta(s - s_0) \end{aligned} \right. \tag{A.22}$$

so that

$$\frac{\epsilon}{E_0} A_s = \frac{\epsilon}{E_0} \cdot \delta(s - s_0) \cdot \left[\Delta \hat{B}_x \cdot z - \Delta \hat{B}_z \cdot x \right] \; . \tag{A.23}$$

A.2.6 Cavity Field

For a longitudinal electric field

$$\begin{aligned}
\varepsilon_x &= 0 \; ; \\
\varepsilon_z &= 0 \; ; \\
\varepsilon_s &= \epsilon(s,\sigma)
\end{aligned} \tag{A.24}$$

we write:

$$\begin{aligned}
A_x &= 0 \; ; \\
A_z &= 0 \; ; \\
A_s &= \int_{\sigma_0}^{\sigma} d\tilde{\sigma} \cdot \varepsilon(s,\tilde{\sigma}) \; ,
\end{aligned} \tag{A.25}$$

which by (A.3) immediately gives ε_s.

Now the cavity field may be represented by

$$\varepsilon(s,\sigma) = V(s) \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \tag{A.26}$$

and we obtain using (A.25):

$$A_s = -\frac{L}{2\pi \cdot h} \cdot V(s) \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \; , \tag{A.27}$$

in which the phase φ is defined so that the average energy radiated away in the bending magnets is replaced by the cavities and h is the harmonic number.

A.3 Series Expansion of the Hamiltonian

The eqns. (A.9), (A.12), (A.15), (A.23) and (A.27) can now be combined as

$$\begin{aligned}
\frac{\epsilon}{E_0} A_s &= -\frac{1}{2}(1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz \\
&\quad -\frac{L}{2\pi \cdot h} \cdot \frac{\epsilon V(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] \\
&\quad +\frac{\epsilon}{E_0} [\Delta B_x \cdot z - \Delta B_z \cdot x]
\end{aligned} \tag{A.28}$$

with

$$\Delta B_x = \; = \sum_{\mu} \Delta \hat{B}_x^{(\mu)} \cdot \delta(s - s_\mu) \; ; \tag{A.29a}$$

$$\Delta B_z = \; = \sum_{\mu} \Delta \hat{B}_z^{(\mu)} \cdot \delta(s - s_\mu) \; . \tag{A.29b}$$

Together with eqns. (A.20a, b) all the components of the vector potential $\vec{A}$ appearing in the Hamiltonian (A.1) are now known.

Furthermore, since

$$
\begin{aligned}
|p_x - \frac{e}{E_0}A_x| &\ll 1 ;\\
|p_z - \frac{e}{E_0}A_z| &\ll 1
\end{aligned}
$$

the square root

$$
\left[1 - \frac{(p_x - \frac{e}{E_0}A_x)^2}{(1+p_\sigma)^2} - \frac{(p_z - \frac{e}{E_0}A_z)^2}{(1+p_\sigma)^2}\right]^{1/2}
$$

in (A.1) may be expanded in a series :

$$
\begin{aligned}
&\left[1 - \frac{(p_x - \frac{e}{E_0}A_x)^2}{(1+p_\sigma)^2} - \frac{(p_z - \frac{e}{E_0}A_z)^2}{(1+p_\sigma)^2}\right]^{1/2} = \\
&\qquad 1 - \frac{1}{2}\cdot\frac{(p_x - \frac{e}{E_0}A_x)^2}{(1+p_\sigma)^2} - \frac{1}{2}\cdot\frac{(p_z - \frac{e}{E_0}A_z)^2}{(1+p_\sigma)^2} + \cdots
\end{aligned}
\tag{A.30}
$$

so that in practice the particle motion can be conveniently calculated to various orders of approximation.

In the following we shall use (within the framework of a linear symplectic treatment of the synchro-betatron oscillations) a series expansion of the Hamiltonian up to second order in the variables $x,\ p_x,\ z,\ p_z, \sigma,\ p_\sigma$. Then we obtain, using eqns. (A.20) and (A.28):

$$
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{A.31}
$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by:

$$
\begin{aligned}
\mathcal{H}_0 &= \frac{1}{2}\cdot\left\{[p_x + H\cdot z]^2 + [p_z - H\cdot x]^2 + G_1\cdot x^2 + G_2\cdot z^2 - 2N\cdot xz\right\}\\
&\quad -\frac{1}{2}\sigma^2\cdot\frac{eV}{E_0}\cdot h\cdot\frac{2\pi}{L}\cos\varphi - [K_x\cdot x + K_z\cdot z]\cdot p_\sigma ; &\text{(A.32a)}\\
\mathcal{H}_1 &= -\sigma\cdot\frac{eV}{E_0}\sin\varphi - \frac{e}{E_0}\Delta B_x\cdot z + \frac{e}{E_0}\Delta B_z\cdot x &\text{(A.32b)}
\end{aligned}
$$

with

$$
G_1 = K_x^2 + g : \quad G_2 = K_z^2 - g
$$

(a constant term, $(L/2\pi h)\cdot(eV/E_0)\cdot\cos\varphi$, in the Hamiltonian, which has no influence on the motion has been dropped) and for the corresponding canonical equations we get:

$$
\begin{aligned}
x' &= p_x + H\cdot z ;\\
p_x' &= K_x\cdot p_\sigma + [p_z - H\cdot x]\cdot H - G_1\cdot x + N\cdot z - \frac{e}{E_0}\Delta B_z ;\\
z' &= p_z - H\cdot x ;\\
p_z' &= K_z\cdot p_\sigma - [p_x + H\cdot z]\cdot H - G_2\cdot z - N\cdot x - \frac{e}{E_0}\Delta B_x ;\\
\sigma' &= -[K_x\cdot x + K_z\cdot z] ;\\
p_\sigma' &= \frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\sigma + \frac{eV}{E_0}\cdot\sin\varphi .
\end{aligned}
\tag{A.33}
$$

Remark:

By taking into account the denominator $(1 + p_\sigma)^2$ in eqn. (A.30) we may write the Hamiltonian of a cavity field (see eqns. (A.1) and (A.27)) to third order as:

$$\mathcal{H}_{Cav} = \frac{1}{2} \cdot \frac{p_x^2}{1+p_\sigma} + \frac{1}{2} \cdot \frac{p_z^2}{1-p_\sigma} + \frac{L}{2\pi \cdot h} \cdot \frac{eV}{E_0} \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] . \tag{A.34}$$

This leads to the canonical equations [42]:

$$\begin{aligned}
x' &= +\frac{\partial}{\partial p_x} \mathcal{H}_{Cav} = \frac{p_x}{1+p_\sigma} ; \\
p_x' &= -\frac{\partial}{\partial x} \mathcal{H}_{Cav} = 0 ; \\
z' &= +\frac{\partial}{\partial p_z} \mathcal{H}_{Cav} = \frac{p_x}{1+p_\sigma} ; \\
p_z' &= -\frac{\partial}{\partial z} \mathcal{H}_{Cav} = 0 ; \\
\sigma' &= +\frac{\partial}{\partial p_\sigma} \mathcal{H}_{Cav} = -\frac{1}{2}\left[(x')^2 + (z')^2\right] ; \\
p_\sigma' &= -\frac{\partial}{\partial \sigma} \mathcal{H}_{Cav} = \frac{eV}{E_0} \cdot \sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] .
\end{aligned} \tag{A.35}$$

By eliminating the quantities p_x' and p_z' from (A.35) we get:

$$\begin{aligned}
x'' &= \frac{p_x'}{1+p_\sigma} - \frac{p_x}{(1+p_\sigma)^2} \cdot p_\sigma' \\
&= -x' \cdot \frac{p_\sigma'}{(1+p_\sigma)} \quad \text{since } p_x' = 0 \text{ and } \frac{p_x}{1+p_\sigma} = x' \\
&\approx -x' \cdot \frac{eV}{E_0} \sin\varphi ; \qquad \text{(A.36a)} \\
z'' &\approx -z' \cdot \frac{eV}{E_0} \sin\varphi . \qquad \text{(A.36b)}
\end{aligned}$$

Thus by deeper inspection we find in eqn. (A.36) the linear terms

$$-x' \cdot \frac{eV}{E_0} \sin\varphi \quad \text{and} \quad -z' \cdot \frac{eV}{E_0} \sin\varphi$$

additional to those appearing at second order on the r.h.s. of eqn. (A.33) and which would otherwise have been lost by a naive linearization. These terms are not symplectic and produce a damping of the orbital oscillations by the cavity fields [42].

Appendix B: The BMT-Equation in Natural Coordinates

According to eqns. (4.15), (4.16) and (2.15b) the precession vector $\vec{\Omega}$ in the BMT-equation (4.16a) may be written in the form:

$$
\begin{aligned}
\vec{\Omega} &= \frac{l'}{c} \cdot \vec{\Omega}_0 - K_z \cdot \vec{e}_x + K_x \cdot \vec{e}_z \\
&= \frac{l'}{c} \cdot \frac{e}{m_0 \gamma c} \cdot \left\{ -(1+\gamma a) \cdot \vec{B} + \frac{a\gamma^2}{1+\gamma} \cdot \frac{1}{c^2} \cdot (\dot{\vec{r}} \vec{B}) \cdot \dot{\vec{r}} \right. \\
&\quad \left. + \left[a\gamma + \frac{\gamma}{1+\gamma} \right] \dot{\vec{r}} \times \frac{\vec{\epsilon}}{c} \right\} - K_z \cdot \vec{e}_x + K_x \cdot \vec{e}_z
\end{aligned}
\tag{B.1}
$$

with (see eqn.(4.12))

$$
\begin{aligned}
l' &= +\sqrt{(x')^2 + (z')^2 + (1 + K_x \cdot x + K_z \cdot z)^2} \\
&= 1 + K_x \cdot x + K_z \cdot z + \cdots .
\end{aligned}
\tag{B.2}
$$

In order to express $\vec{\Omega}$ as a function of $x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta = p_\sigma$ we remark that one can write:

$$
\frac{1}{c} \cdot \dot{\vec{r}} = \frac{1}{l'} \cdot [\vec{e}_s \cdot (1 + K_x \cdot x + K_z \cdot z) + \vec{e}_x \cdot x' + \vec{e}_z \cdot z'] \tag{B.3}
$$

and

$$
\begin{aligned}
m_0 \gamma c^2 &\equiv E = E_0 \cdot (1 + \eta) ; && \text{(B.4a)} \\
\gamma &= \gamma_0 \cdot (1 + \eta) ; && \text{(B.4b)} \\
\frac{1}{1+\gamma} &= \frac{1}{(1+\gamma_0) + \gamma_0 \cdot \eta} \\
&= \frac{1}{1+\gamma_0} \cdot \left[1 - \frac{\gamma_0}{1+\gamma_0} \cdot \eta \right] + \cdots . && \text{(B.4c)}
\end{aligned}
$$

Furthermore, for the magnetic field $\vec{B}$ we have (see Appendix A):

$$
\begin{aligned}
\frac{e}{E_0} B_x &= -K_z + \frac{e}{E_0} \Delta B_x + (N - H') \cdot x + g \cdot z ; && \text{(B.5a)} \\
\frac{e}{E_0} B_z &= +K_x + \frac{e}{E_0} \Delta B_z - (N + H') \cdot z + g \cdot x ; && \text{(B.5b)} \\
\frac{e}{E_0} B_s &= 2 \cdot H && \text{(B.5c)}
\end{aligned}
$$

and for the electric field $\vec{\epsilon}$ we have:

$$
\vec{\epsilon} = \vec{e}_s \cdot V(s) \left[\sin\varphi + \sigma(s) \cdot h \cdot \frac{2\pi}{L} \cos\varphi \right] . \tag{B.6}
$$

Then, putting (B.2 - 6) into (B.1), we obtain in linear order the eqns. (4.19).

Appendix C: Derivation of the Linearized Equations of Spin Motion

C.1 Perturbation Theory

In order to derive the equations of spin motion the component $\vec{\omega}$ in (5.8) will be considered as a small perturbation. Making the ansatz

$$\vec{\xi} = \vec{\xi}^{\,(0)} - \vec{\xi}^{\,(1)} \tag{C.1}$$

with

$$\vec{\xi}^{\,(0)} = \xi_s^{\,(0)} \cdot \vec{e}_s + \xi_x^{\,(0)} \cdot \vec{e}_x + \xi_z^{\,(0)} \cdot \vec{e}_z \tag{C.1a}$$

and

$$\vec{\xi}^{\,(1)} = \xi_s^{\,(1)} \cdot \vec{e}_s + \xi_x^{\,(1)} \cdot \vec{e}_x + \xi_z^{\,(1)} \cdot \vec{e}_z \tag{C.1b}$$

(where $\vec{\xi}^{\,(0)}$ denotes the spin vector on the closed orbit) and using eqn. (4.16a) on the closed orbit:

$$\vec{e}_s \cdot \frac{d}{ds}\, \xi_s^{\,(0)} + \vec{e}_x \cdot \frac{d}{ds}\, \xi_x^{\,(0)} + \vec{e}_z \cdot \frac{d}{ds}\, \xi_z^{\,(0)} = \vec{\Omega}^{(0)} \times \vec{\xi}^{\,(0)} \tag{C.2}$$

we obtain the following expression for $\vec{\xi}^{\,(1)}$ in linear order of perturbation theory:

$$\vec{e}_s \cdot \frac{d}{ds}\, \xi_s^{\,(1)} + \vec{e}_x \cdot \frac{d}{ds}\, \xi_x^{\,(1)} + \vec{e}_z \cdot \frac{d}{ds}\, \xi_z^{\,(1)} = \vec{\Omega}^{(0)} \times \vec{\xi}^{\,(1)} + \vec{\omega} \times \vec{\xi}^{\,(0)} \,. \tag{C.3}$$

As A. Chao has shown [11] eqn. (C.2) now can be used to define a new system of orthogonal unit vectors which considerably simplify the spin motion determined by (C.3).

C.2 The Periodic Spin Frame $(\vec{n}_0, \vec{m}, \vec{l})$ along the Closed Orbit

In the following we shall introduce a compact matrix notation. Rewriting an arbitrary vector

$$\vec{A} = A_s \cdot \vec{e}_s + A_x \cdot \vec{e}_x + A_z \cdot \vec{e}_z$$

as a column vector with components A_s, A_x, A_z:

$$A_s \cdot \vec{e}_s + A_x \cdot \vec{e}_x + A_z \cdot \vec{e}_z = \begin{pmatrix} A_s \\ A_x \\ A_z \end{pmatrix}$$

and defining the derivative of a column vector with respect to the arc length s as the derivative of the corresponding components A_i but not of the unit vectors:

$$\frac{d}{ds} \begin{pmatrix} A_s \\ A_x \\ A_z \end{pmatrix} = \vec{e}_s \cdot \frac{d}{ds}\, A_s + \vec{e}_x \cdot \frac{d}{ds}\, A_x + \vec{e}_z \cdot \frac{d}{ds}\, A_z$$

we get from (C.2):

$$\frac{d}{ds}\ \vec{\xi}^{\,(0)}(s) = \underline{\Omega}^{(0)}(s)\cdot\vec{\xi}^{\,(0)}(s) \tag{C.4}$$

where we have set

$$\vec{\xi}^{\,(0)} = \begin{pmatrix} \xi_s^{(0)} \\ \xi_x^{(0)} \\ \xi_z^{(0)} \end{pmatrix} \tag{C.5a}$$

and

$$\underline{\Omega}^{(0)}(s) = \begin{pmatrix} 0 & -\Omega_z^{(0)} & \Omega_x^{(0)} \\ \Omega_z^{(0)} & 0 & -\Omega_s^{(0)} \\ -\Omega_x^0 & \Omega_s^{(0)} & 0 \end{pmatrix}\ . \tag{C.5b}$$

The transfer matrix $\underline{M}_{(spin)}(s, s_0)$ for the spin motion defined by

$$\vec{\xi}^{\,(0)}(s) = \underline{M}_{(spin)}(s, s_0)\cdot\vec{\xi}^{\,(0)}(s_0)$$

satisfies the relationships:

$$\underline{M}^T_{(spin)}(s, s_0)\cdot\underline{M}_{(spin)}(s, s_0) \ = \ \underline{1}\ ; \tag{C.6a}$$

$$\det\,[\underline{M}_{(spin)}(s, s_0)] = 1 \tag{C.6b}$$

since (using eqn. (C.4))

$$\begin{aligned} \frac{d}{ds}\ \underline{M}_{(spin)}(s, s_0) &= \underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s, s_0)\ ; \\ \underline{M}_{(spin)}(s_0, s_0) &= \underline{1} \end{aligned}$$

and therefore (with $[\underline{\Omega}^{(0)}]^T = -\underline{\Omega}^{(0)}$)

$$\begin{aligned} \frac{d}{ds}\ \left[\underline{M}^T_{(spin)}(s, s_0)\cdot\underline{M}_{(spin)}(s, s_0)\right] &= \left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s, s_0)\right]^T\cdot\underline{M}_{(spin)}(s, s_0) \\ &\quad +\underline{M}^T_{(spin)}(s, s_0)\cdot\left[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s, s_0)\right] \\ &= -\underline{M}_{(spin)}(s, s_0)^T\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s, s_0) \\ &\quad +\underline{M}^T_{(spin)}(s, s_0)\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(spin)}(s, s_0) \\ &= \underline{0}\ ; \end{aligned}$$

$$\det\,M_{(spin)}(s, s_0) \ = \ \det\,M_{(spin)}(s_0, s_0) = 1\ ,$$

i.e. $\underline{M}_{(spin)}(s, s_0)$ is an orthogonal matrix with determinant 1.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{M}(s_0+L,s_0)$ with the eigenvalues α_μ and eigenvectors $\vec{r}_\mu(s_0)$:

$$\underline{M}(s_0+L,s_0)\ \vec{r}_\mu(s_0) = \alpha_\mu \cdot \vec{r}_\mu(s_0)\ ; \qquad (C.7)$$

$$(\mu = 1,\ 2,\ 3).$$

Because of (C.6a,b) we can write [13]:

$$\begin{aligned} \alpha_1 &= 1\ ; \\ \alpha_2 &= e^{i \cdot 2\pi \cdot Q_{spin}}\ ; \\ \alpha_3 &= e^{-i \cdot 2\pi \cdot Q_{spin}}\ ; \end{aligned} \qquad (C.8)$$

$$(Q_{spin} = \text{real number})$$

and

$$\begin{aligned} \vec{r}_1(s_0) &= \vec{n}_0(s_0)\ ; & (C.9a) \\ \vec{r}_2(s_0) &= \vec{m}_0(s_0) + i \cdot \vec{l}_0(s_0)\ ; & (C.9b) \\ \vec{r}_3(s_0) &= \vec{m}_0(s_0) - i \cdot \vec{l}_0(s_0)\ ; & (C.9c) \end{aligned}$$

$$(\vec{n}_0, \vec{m}_0, \vec{l}_0 = \text{real vectors})\ .$$

If we require that

$$\begin{aligned} \vec{r}_1^{\,+} \cdot \vec{r}_1 &= 1\ ; & (C.10a) \\ \vec{r}_2^{\,+} \cdot \vec{r}_2 &\equiv \vec{r}_3^{\,+} \cdot \vec{r}_3 = 2\ ; & (C.10b) \end{aligned}$$

$$(\text{normalizing conditions})$$

we find, using eqn. (C.6a) [13]:

$$|\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{l}_0(s_0)| = 1\ ; \qquad (C.11a)$$

$$\vec{n}_0(s_0)\ \perp\ \vec{m}_0(s_0)\ \perp\ \vec{l}_0(s_0)\ . \qquad (C.11b)$$

Thus the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$ form an orthogonal system of unit vectors. Choosing the direction of $\vec{n}_0(s_0)$ such that

$$\vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{l}_0(s_0) \qquad (C.11c)$$

these vectors form a right-handed coordinate system.

In this way we have found a coordinate frame for the position $s = s_0$.

An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_{(spin)}(s,s_0)$ to the vectors $\vec{n}_0(s_0)$, $\vec{m}_0(s_0)$ and $\vec{l}_0(s_0)$:

$$\begin{aligned} \vec{n}_0(s) &= \underline{M}_{(spin)}(s,s_0)\ \vec{n}_0(s_0)\ ; & (C.12a) \\ \vec{m}_0(s) &= \underline{M}_{(spin)}(s,s_0)\ \vec{m}_0(s_0)\ ; & (C.12b) \\ \vec{l}_0(s) &= \underline{M}_{(spin)}(s,s_0)\ \vec{l}_0(s_0)\ . & (C.12b) \end{aligned}$$

Because of eqn. (C.6a,b) the orthogonality relations remain unchanged:

$$\vec{n}_0(s) = \vec{m}_0(s) \times \vec{l}_0(s) \tag{C.13a}$$
$$\vec{m}_0(s) \perp \vec{l}_0(s) ; \tag{C.13b}$$
$$|\vec{n}_0(s)| = |\vec{m}_0(s)| = |\vec{l}_0(s)| = 1 . \tag{C.13c}$$

The coordinate frame defined by $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particles:

$$\begin{aligned} \vec{m}_0(s_0+L) + i\vec{l}_0(s_0+L) &= \underline{M}_{(spin)}(s_0+L, s_0)\left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &= e^{i \cdot 2\pi \cdot Q_{spin}} \cdot \left[\vec{m}_0(s_0) + i\vec{l}_0(s_0)\right] \\ &\neq \vec{m}_0(s_0) + i\vec{l}_0(s_0) \end{aligned}$$

$$(\text{if } Q_{spin} \neq \text{integer}).$$

i.e. although $\vec{n}_0(s)$ is periodic by eqns. (C.8), (C.9a), $\vec{m}_0(s)$ and $\vec{l}_0(s)$ are not periodic.

But by introducing a phase function $\psi(s)$ and using another orthogonal matrix $\underline{D}(s, s_0)$:

$$\underline{D}(s, s_0) = \begin{pmatrix} \cos[\psi(s) - \psi(s_0)] & \sin[\psi(s) - \psi(s_0)] \\ -\sin[\psi(s) - \psi(s_0)] & \cos[\psi(s) - \psi(s_0)] \end{pmatrix} \tag{C.14}$$

with

$$\underline{D}^T(s, s_0) \cdot \underline{D}(s, s_0) = \underline{1} ; \tag{C.15a}$$

$$\det\left[\underline{D}(s, s_0)\right] = 1 \tag{C.15b}$$

we can construct a periodic orthogonal system of unit vectors from $\vec{n}_0(s)$, $\vec{m}_0(s)$ and $\vec{l}_0(s)$. Namely, if we put [44]:

$$\begin{pmatrix} \vec{m}(s) \\ \vec{l}(s) \end{pmatrix} = \underline{D}(s, s_0) \begin{pmatrix} \vec{m}(s_0) \\ \vec{l}(s_0) \end{pmatrix}$$

$$\Longrightarrow \quad \begin{aligned} \vec{m}(s) + i\vec{l}(s) &= e^{-i \cdot [\psi(s) - \psi(s_0)]} \cdot \left[\vec{m}_0(s) + i\vec{l}_0(s)\right] \\ &\neq \vec{m}_0(s_0) + i\vec{l}_0(s_0) \end{aligned} \tag{C.16}$$

we find, using eqns. (C.15a, b) :

$$\vec{n}_0(s) = \vec{m}(s) \times \vec{l}(s) ; \tag{C.17a}$$
$$\vec{m}(s) \perp \vec{l}(s) ; \tag{C.17b}$$
$$|\vec{n}_0(s)| = |\vec{m}(s)| = |\vec{l}(s)| = 1 . \tag{C.17c}$$

Since

$$\vec{m}(s_0+L) + i \cdot \vec{l}(s_0+L) = e^{-i \cdot [\psi(s_0 - L) - \psi(s_0)]} \cdot \left[\vec{m}(s_0) + i \cdot \vec{l}(s_0)\right]$$

it follows, that the condition of periodicity for $\vec{n}_0$, $\vec{m}$ and $\vec{l}$:

$$\left(\vec{n}_0,\ \vec{m},\ \vec{l}\right)_{s=s_0+L} = \left(\vec{n}_0,\ \vec{m},\ \vec{l}\right)_{s=s_0} \tag{C.18}$$

can indeed be fulfilled if the phase function $\psi(s)$ satisfies the following relationship:

$$\psi(s_0 + L) - \psi(s_0) = 2\pi \cdot Q_{spin}\ ; \tag{C.19a}$$

$$(Q_{spin} = \text{spin tune}).$$

For instance we can choose:

$$\psi(s) = 2\pi \cdot Q_{spin} \cdot \frac{s}{L}\ . \tag{C.19b}$$

In this frame, spins on the closed orbit precess uniformly with respect to $\vec{m}$ and $\vec{l}$.

Taking the derivatives of $\vec{m}(s)$ and $\vec{l}(s)$ with respect to s, and taking into account eqns. (C.16), (C.12), and (C.4) we get

$$\frac{d}{ds}\,\vec{m}(s) \;=\; \underline{\Omega}^{(0)}(s)\ \vec{m}(s) + \psi\,'(s) \cdot \vec{l}(s)\ ; \tag{C.20a}$$

$$\frac{d}{ds}\,\vec{l}(s) \;=\; \underline{\Omega}^{(0)}(s)\ \vec{l}(s) - \psi\,'(s) \cdot \vec{m}(s) \tag{C.20b}$$

and $\vec{n}_0(s)$ satisfies (see (C.12a))

$$\frac{d}{ds}\,\vec{n}_0(s) \;=\; \underline{\Omega}^{(0)}(s)\ \vec{n}_0(s)\ . \tag{C.20c}$$

Finally, the vectors

$$\vec{r}_1(s) \;=\; \vec{n}_0(s) \;\equiv\; \underline{M}_{(spin)}(s, s_0)\ \vec{r}_1(s_0)\ ; \tag{C.21a}$$

$$\vec{r}_2(s) \;=\; \vec{m}_0(s) + i \cdot \vec{l}_0(s) \;\equiv\; \underline{M}_{(spin)}(s, s_0)\ \vec{r}_2(s_0)\ ; \tag{C.21b}$$

$$\vec{r}_3(s) \;=\; \vec{m}_0(s) - i \cdot \vec{l}_0(s) \;\equiv\; \underline{M}_{(spin)}(s, s_0)\ \vec{r}_3(s_0) \tag{C.21c}$$

are eigenvectors of the revolution matrix $\underline{M}_{(spin)}$ with the same eigenvalues as in (C.8):

$$\underline{M}(s + L, s)\ \vec{r}_\mu(s) = \alpha_\mu \cdot \vec{r}_\mu(s)\ . \tag{C.22}$$

Thus, the eigenvalues α_μ and the quantity Q_{spin} defined by eqn. (C.8) are independent of the chosen initial position s_0.

C.3 The Linearized Equations of Spin Motion in the $(\vec{n}_0,\ \vec{m},\ \vec{l})$ Spin Frame

Following A. Chao [11] we make the following ansatz:

$$\vec{\xi}^{\,(0)}(s) \;=\; \vec{n}_0(s)\ ; \tag{C.23a}$$

$$\vec{\xi}^{\,(1)}(s) \;=\; \alpha(s) \cdot \vec{m}(s) + \beta(s) \cdot \vec{l}(s)\ ; \tag{C.23b}$$

$$(|\alpha|^2| + |\beta|^2| \ll 1)$$

to solve the equations of motion (C.2) and (C.3). Because of (C.20c) the expression (C.23a) is a solution of (C.2) and putting (C.23a,b) into eqn. (C.3) we get:

$$\alpha' = (l_s,\ l_x,\ l_z) \cdot \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} + \beta \cdot \psi' \tag{C.24a}$$

$$\beta' = -(m_s,\ m_x,\ m_z) \cdot \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} - \alpha \cdot \psi' \tag{C.24b}$$

where we have taken into account (C.20a, b).

Using the relations

$$\begin{aligned} \tilde{x}' &= \tilde{p}_x + H \cdot \tilde{z} ; \\ \tilde{z}' &= \tilde{p}_z - H \cdot \tilde{x} \end{aligned}$$

which result from (5.3) and (4.6 - 8) and taking into account (4.19) and (5.10), eqn. (5.9b) can be rewritten in the form:

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3\times 6)} \cdot \vec{\tilde{y}} \ . \tag{C.25}$$

The matrix elements of

$$\underline{F}_{(3\times 6)} \equiv \underline{F} = ((F_{ik}))$$

are given by (see (4.19), (5.9) and (5.10)):

$$\begin{aligned}
F_{12} &= -a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot K_z ; \\
F_{14} &= +a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot K_x ; \\
F_{16} &= 2H \cdot \left[1 + a\frac{\gamma_0^2}{(1+\gamma_0)^2}\right] ; \\
F_{21} &= -(1+a\gamma_0) \cdot (N - H') ; \\
F_{22} &= +a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot 2H ; \\
F_{23} &= +(1+a\gamma_0) \cdot G_2 + a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot 2H^2 ; \\
F_{24} &= \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{\epsilon}{E_0} V(s) \sin\varphi ; \\
F_{26} &= -K_z ; \\
F_{31} &= -(1+a\gamma_0) \cdot G_1 - a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot 2H^2 ; \\
F_{32} &= -\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{\epsilon}{E_0} V(s) \sin\varphi ; \\
F_{33} &= +(1+a\gamma_0) \cdot (N + H') ; \\
F_{34} &= +a\gamma_0 \cdot \frac{\gamma_0}{1+\gamma_0} \cdot 2H ; \\
F_{36} &= +K_x ; \\
F_{ik} &= 0 \quad \text{otherwise} .
\end{aligned} \tag{C.26}$$

Finally, taking into account (C.25) the spin equation (C.24) can be rewritten in a form given by eqn. (5.12).

Appendix D: Perturbation Theory and Robinson-Theorem

D.1 Introductory Remark: Two Methods for Calculating the Damping Constants

In chapter 9 we have derived an analytic expression for the damping constants α_k of the (coupled) synchro-betatron oscillations (see eqn. (9.18)). This expression allows the calculation of α_k if one knows the eigenvectors $\vec{v}_k(s)$ of the unperturbed problem (6.2) and the matrix elements δA_{ik} of eqn. (4.8).

On the other hand, A. Chao [11] calculates the damping constants by using the eigenvalue spectrum of the revolution matrix

$$\underline{M}(s_0 + L, s_0) + \delta\underline{M}(s_0 + L, s_0)$$

of the perturbed problem

$$\frac{d}{ds}\vec{\tilde{y}} = (\underline{A} + \delta\underline{A}) \cdot \vec{\tilde{y}} \,. \tag{D.1}$$

This matrix with the perturbation part $\delta\underline{M}(s_0 + L, s_0)$ is not symplectic in contrast to

$$\underline{M}(s_0 + L, s_0) \,.$$

Therefore, writing the perturbed eigenvalues $(\lambda_k + \delta\lambda_k)$ in the form (see eqn. (6.16)):

$$\lambda_k + \delta\lambda_k = e^{-i \cdot 2\pi(Q_k + \delta Q_k)}$$

one will generally obtain complex values for the Q-shift δQ_k caused by the perturbation $\delta\underline{A}$. According to A. Chao [11] we put

$$\begin{aligned} \alpha_k &= -2\pi \cdot \Im m\{Q_k + \delta Q_k\} \\ &= -2\pi \cdot \Im m\{\delta Q_k\} \,. \end{aligned} \tag{D.2}$$

The purpose of this appendix is to show the equivalence of (9.18) and (D.2). As a by-product of this calculation the well-known Robinson theorem will be rederived.

D.2 Equivalence of the two Methods

D.2.1 Calculation for the Perturbed Part of the Revolution Matrix

In order to prove the equivalence of the two methods mentioned above we determine the perturbation part $\delta\underline{M}(s_0 + L, s_0)$ of the revolution matrix (of the perturbed problem).

According to eqn. (D.1) the transfer matrix

$$\underline{M}(s, s_0) + \delta\underline{M}(s, s_0)$$

obeys the equation:

$$\frac{d}{ds}[\underline{M}(s,s_0)+\delta\underline{M}(s,s_0)] = [\underline{A}(s)+\delta\underline{A}(s)]\cdot[\underline{M}(s,s_0)+\delta\underline{M}(s,s_0)] \; ; \qquad \text{(D.3a)}$$

$$\underline{M}(s_0,s_0)+\delta\underline{M}(s_0,s_0) = \underline{1} \; . \qquad \text{(D.3b)}$$

Taking into account the corresponding equations for the unperturbed transfer matrix $\underline{M}(s,s_0)$:

$$\frac{d}{ds}\underline{M}(s,s_0) = \underline{A}(s)\cdot\underline{M}(s,s_0) \; ;$$

$$\underline{M}(s_0,s_0) = \underline{1}$$

we obtain from (D.3), to first order, the differential equation for $\delta\underline{M}(s,s_0)$:

$$\frac{d}{ds}\delta\underline{M}(s,s_0)] = \underline{A}(s)\cdot\delta\underline{M}(s,s_0)+\delta\underline{A}(s)\cdot\underline{M}(s,s_0)$$

with the initial condition:

$$\delta\underline{M}(s_0,s_0) = \underline{0} \; .$$

The solution of this equation (and thus the first order solution of eqn. (D.3)) reads as:

$$\begin{aligned}\delta\underline{M}(s,s_0) &= \int_{s_0}^{s} d\tilde{s}\cdot\underline{M}(s.\tilde{s})\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0)\\ &= \underline{M}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s_0)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0) \; .\end{aligned}$$

For the perturbative part $\delta\underline{M}(s_0+L,s_0)$ of the revolution matrix one therefore gets, to first order, the expression:

$$\begin{aligned}\delta\underline{M}(s_0+L,s_0) &= \int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{M}(s_0+L,\tilde{s})\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0)\\ &= \underline{M}(s_0+L,s_0)\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s_0)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s_0) \qquad \text{(D.4a)}\end{aligned}$$

and for $\delta\underline{M}(s+L,s)$ one thus may write:

$$\delta\underline{M}(s+L,s) = \underline{M}(s+L,s)\cdot\int_{s}^{s+L} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s) \; . \qquad \text{(D.4b)}$$

D.2.2 Perturbation Theory

Eqn. (D.4b) determines the perturbed part $\delta\underline{M}(s+L.s)$ of the revolution matrix if the (unperturbed) transfer matrix $\underline{M}(\tilde{s},s)$ and the perturbation $\delta\underline{A}(\tilde{s})$ are known. Using the eigenvalue equation

$$(\underline{M}+\delta\underline{M})\cdot(\vec{v}_\mu+\delta\vec{v}_\mu) = (\lambda_\mu+\delta\lambda_\mu)\cdot(\vec{v}_\mu-\delta\vec{v}_\mu) \; ;$$
$$(\mu = \pm I, \; \pm II, \; \pm III)$$

or (since $\underline{M}\vec{v}_\mu = \lambda_\mu\vec{v}_\mu$)

$$\underline{M}\cdot\delta\vec{v}_\mu+\delta\underline{M}\cdot\vec{v}_\mu = \lambda_\mu\cdot\delta\vec{v}_\mu+\delta\lambda_\mu\cdot\vec{v}_\mu \qquad \text{(D.5)}$$

we can calculate the Q-shift

$$\delta Q_\kappa = \frac{i}{2\pi \cdot \lambda_\kappa} \cdot \delta\lambda_\kappa \tag{D.6}$$

caused by $\delta\underline{M}$ [24,30].

For that purpose we expand $\delta\vec{v}_\mu$ in terms of the eigenvectors $\vec{v}_\mu$ of the unperturbed problem:

$$\delta\vec{v}_\mu = \sum_\nu a_{\mu\nu} \cdot \vec{v}_\nu \tag{D.7}$$

and by inserting (D.7) into (D.5) we get:

$$\sum_\nu a_{\mu\nu} \cdot \lambda_\nu \vec{v}_\nu + \delta\underline{M} \cdot \vec{v}_\mu = \lambda_\mu \cdot \sum_\nu a_{\mu\nu}\vec{v}_\nu + \delta\lambda_\mu \cdot \vec{v}_\mu \ . \tag{D.8}$$

Multiplying this equation from the left hand side with

$$\frac{1}{i} \cdot \vec{v}_\kappa^{\,+}\underline{S}$$

and taking into account eqn. (6.28) we obtain

$$a_{\mu\kappa} \cdot \lambda_\kappa \cdot \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa + \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S} \cdot \delta\underline{M} \cdot \vec{v}_\mu \;=\; \lambda_\mu \cdot a_{\mu\kappa} \cdot \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa + \delta\lambda_\kappa \cdot \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa \cdot \delta_{\mu\kappa} \tag{D.9}$$

with

$$\frac{1}{i} \cdot \vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa = \begin{cases} +1 & \text{for } \kappa = I,\ II,\ III; \\ -1 & \text{for } \kappa = -I, -II, -III. \end{cases} \tag{D.10}$$

For $\kappa \neq \mu$ the expansion coefficients are given by (see eqn. (D.4))

$$\begin{aligned} a_{\mu\kappa} &= \left(\frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \cdot \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S} \cdot \delta\underline{M} \cdot \vec{v}_\mu(s) \\ &= \left(\frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S}\vec{v}_\kappa\right) \frac{1}{\lambda_\mu - \lambda_\kappa} \cdot \frac{1}{i}\,\vec{v}_\kappa^{\,+}\underline{S} \cdot \underline{M}(s+L,s) \\ &\quad \times \int_s^{s+L} d\tilde{s} \cdot \underline{M}^{-1}(\tilde{s},s) \cdot \delta\underline{A}(\tilde{s}) \cdot \underline{M}(\tilde{s},s) \cdot \vec{v}_\mu(s) \ . \end{aligned}$$

Using the symplectic condition of the transfer matrix $\underline{M}(s_1, s_2)$:

$$\underline{M}^T(s_1,s_2) \cdot \underline{S} \cdot \underline{M}(s_1,s_2) = \underline{S}$$

and the equation

$$\begin{aligned} \vec{v}_\kappa^{\,+}(s) \cdot \underline{S} \cdot \underline{M}(s+L,s) &= \vec{v}_\kappa^{\,+}(s) \cdot \left[\underline{M}^{-1}(s+L,s)\right]^T \cdot \underline{S} \\ &= \left[\underline{M}^{-1}(s+L,s) \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\ &= \left[\lambda_\kappa^{-1} \cdot \vec{v}_\kappa(s)\right]^+ \cdot \underline{S} \\ &= \lambda_\kappa \cdot \vec{v}_\kappa^{\,+}(s) \cdot \underline{M}^T(\tilde{s},s) \cdot \underline{S} \cdot \underline{M}(\tilde{s},s) \\ &\quad \left(\text{since } \ (\lambda_\kappa^{-1})^* = \lambda_\kappa \ \text{ and } \ \underline{S} = \underline{M}^T \cdot \underline{S} \cdot \underline{M}\right) \\ &= \lambda_\kappa \cdot [\underline{M}(\tilde{s},s) \cdot \vec{v}_\kappa(s)]^+ \cdot \underline{S} \cdot \underline{M}(\tilde{s},s) \\ &= \lambda_\kappa \cdot \vec{v}_\kappa^{\,+}(\tilde{s}) \cdot \underline{S} \cdot \underline{M}(\tilde{s},s) \end{aligned} \tag{D.11}$$

$a_{\mu\kappa}$ can be rewritten as

$$\begin{aligned} a_{\mu\kappa} &= \left(\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\frac{\lambda_\kappa}{\lambda_\mu-\lambda_\kappa} \\ &\quad\times\frac{1}{i}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\mu(\tilde{s}) \end{aligned} \tag{D.12}$$

so that the perturbation $\delta\vec{v}_\mu$ of $\vec{v}_\mu$ is given by (see eqns. (D.7) and (D.12)):

$$\begin{aligned} \delta\vec{v}_\mu(s) &= \sum_{\kappa\neq\mu}\left(\frac{1}{i}\,\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\frac{\lambda_\kappa}{\lambda_\mu-\lambda_\kappa} \\ &\quad\times\frac{1}{i}\cdot\left[\int_s^{s+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\mu(\tilde{s})\right]\cdot\vec{v}_\kappa(s) \\ &\quad+a_{\mu\mu}\cdot\vec{v}_\mu(s)\ . \end{aligned} \tag{D.13}$$

Here the coefficient $a_{\mu\mu}$ remains undetermined but can be determined to first order by using the normalization condition (6.28) applied to the perturbed eigenvector $\vec{v}_\mu + \delta\vec{v}_\mu$:

$$[\vec{v}_\mu(s)+\delta\vec{v}_\mu(s)]^{+}\cdot\underline{S}\cdot[\vec{v}_\mu(s)+\delta\vec{v}_\mu(s)] = \vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\mu(s)$$

with $\delta\vec{v}_\mu$ given by (D.13) which leads to:

$$\begin{aligned} 0 &= \delta\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\mu(s)+\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\delta\vec{v}_\mu(s) \\ &= (a_{\mu\mu}+a_{\mu\mu}^{*})\cdot\left[\vec{v}_\mu^{+}(s)\cdot\underline{S}\cdot\vec{v}_\mu(s)\right] \end{aligned}$$

$$\Longrightarrow\quad a_{\mu\mu} = i\cdot\varphi_\mu$$

where φ_μ is an arbitrary real number. This is consistent with the fact that one can multiply an eigenvector $\vec{v}_\mu$ with an arbitrary phase factor $e^{i\varphi_\mu}$ without disturbing the normalization. Without loss of generality we may set:

$$\varphi_\mu = 0 \quad\Longrightarrow\quad a_{\mu\mu} = 0\ .$$

For $\mu=\kappa$ the first terms on both sides of eqn. (D.9) cancel and one obtains with (D.4), (D.6) and (D.11) the following approximate expression for the Q-shift δQ_κ in linear order:

$$\begin{aligned} \delta Q_\kappa &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\,\underline{S}\cdot\delta\underline{M}(s+L,s)\cdot\vec{v}_\kappa(s) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi\cdot\lambda_\kappa}\cdot\vec{v}_\kappa^{+}\,\underline{S}\cdot\underline{M}(s-L,s) \\ &\quad\times\int_s^{s+L} d\tilde{s}\cdot\underline{M}^{-1}(\tilde{s},s)\cdot\delta\underline{A}(\tilde{s})\cdot\underline{M}(\tilde{s},s)\vec{v}_\kappa(s) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_s^{s+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s}) \\ &= \left(\frac{1}{i}\cdot\vec{v}_\kappa^{+}\underline{S}\vec{v}_\kappa\right)\cdot\frac{1}{2\pi}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\vec{v}_\kappa^{+}(\tilde{s})\cdot\underline{S}\cdot\delta\underline{A}(\tilde{s})\cdot\vec{v}_\kappa(\tilde{s}) \end{aligned}$$

(in the last step we have used the fact that the integrand is a periodic function of period L; see eqn. (6.22))
or for $\kappa = k$ and $\kappa = -k$ $\quad (k = \pm I, \pm II, \pm III)$:

$$\delta Q_k = \frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_k(\tilde{s}) \; ; \tag{D.14a}$$

$$\delta Q_{-k} = -\frac{1}{2\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_{-k}^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_{-k}(\tilde{s}) \; . \tag{D.14b}$$

Using the facts that:

$$\begin{aligned} \delta Q_\kappa^* &= \left(\frac{1}{i} \cdot \vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right)^+ \cdot \frac{1}{2\pi} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \left[\vec{v}_\kappa^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\kappa(\tilde{s})\right]^+ \\ &= \left(\frac{1}{i} \cdot \vec{v}_\kappa^+ \underline{S} \vec{v}_\kappa\right) \cdot \frac{1}{2\pi} \int_{s_0}^{s_0+L} d\tilde{s} \cdot \left[-\vec{v}_\kappa^+(\tilde{s}) \cdot \delta\underline{A}^T(\tilde{s}) \cdot \underline{S} \cdot \vec{v}_\kappa(\tilde{s})\right] \end{aligned}$$

as well as

$$\vec{v}_{-\kappa} = (\vec{v}_\kappa)^*$$

the following relations can be derived from (D.14a,b):

$$\begin{aligned} \Re e\{\delta Q_k\} &= \frac{1}{4\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^+(\tilde{s}) \cdot \left[\underline{S} \cdot \delta\underline{A}(\tilde{s}) - \delta\underline{A}^T(\tilde{s}) \cdot \underline{S}\right] \cdot \vec{v}_k(\tilde{s}) \\ &= -\Re e\{\delta Q_{-k}\} \; ; \end{aligned} \tag{D.15a}$$

$$\begin{aligned} \Im m\{\delta Q_k\} &= -\frac{i}{4\pi} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \vec{v}_k^+(\tilde{s}) \cdot \left[\underline{S} \cdot \delta\underline{A}(\tilde{s}) + \delta\underline{A}^T(\tilde{s}) \cdot \underline{S}\right] \cdot \vec{v}_k(\tilde{s}) \\ &= +\Im m\{\delta Q_{-k}\} \; . \end{aligned} \tag{D.15b}$$

This means that in addition to a real Q-shift, there is also a complex Q-shift, and comparing (9.18) with (D.15) we find the desired result that the two methods mentioned at the beginning of the appendix are equivalent. Equation (9.18) allows the calculation of the damping constants simply by a numerical integration if one knows the eigenvectors of the unperturbed problem instead of calculating the eigenvalue spectrum of the perturbed revolution matrix necessary for evaluating (D.2).

Finally it is worth mentioning that the applied perturbation theory is only valid if $\delta\lambda_\mu$ and $\delta\vec{v}_\mu$ are small compared with the unperturbed quantities λ_μ and $\vec{v}_\mu$:

$$\begin{aligned} |\delta\lambda_\mu| &\ll |\lambda_\mu| \; ; \\ ||\delta\vec{v}_\mu| &\ll ||\vec{v}_\mu|| \; . \end{aligned}$$

Therefore, in order to apply this kind of perturbation theory the following condition must hold (see eqn. (D.13)):

$$\left|\int_s^{s+L} d\tilde{s} \cdot \left[\vec{v}_\kappa^+(\tilde{s}) \cdot \underline{S} \cdot \delta\underline{A}(\tilde{s}) \cdot \vec{v}_\mu(\tilde{s})\right]\right| \ll |\lambda_\mu - \lambda_\kappa| \; .$$

This condition is well satisfied if the values for different $\lambda_\mu, \lambda_\kappa$ are far apart. However the calculation breaks down if two eigenvalues coincide:

$$\lambda_k = e^{-i \cdot 2\pi Q_k} \approx \lambda_{k'} = e^{-i \cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k - Q_{k'} = n$$

or

$$\lambda_k = e^{-i \cdot 2\pi Q_k} \approx \lambda_{-k'} = \epsilon^{+i \cdot 2\pi Q_{k'}} \Longleftrightarrow Q_k + Q_{k'} = n$$

(n=integer).

Since these Q-resonances can lead to instabilities of the particle motion we do not investigate these effects in this report.

D.3 Robinson's Theorem

As a by-product of the calculations of the last sections one can demonstrate the well-known Robinson theorem [45]. For that purpose we use the mathematical theorem that the determinant of the perturbed matrix can be represented as the product of the corresponding eigenvalues

$$\begin{aligned}
\lambda_k + \delta\lambda_k &= \epsilon^{-i \cdot 2\pi \cdot [Q_k + \Re e\{\delta Q_k\} + i \cdot \Im m\{\delta Q_k\}]} \\
&= \epsilon^{-\alpha_k - i \cdot 2\pi \cdot [Q_k + \Re e\{\delta Q_k\}]} ;
\end{aligned}$$

$$\begin{aligned}
\lambda_{-k} + \delta\lambda_{-k} &= \epsilon^{-i \cdot 2\pi \cdot [Q_{-k} + \Re e\{\delta Q_{-k}\} + i \cdot \Im m\{\delta Q_{-k}\}]} \\
&= \epsilon^{-\alpha_k + i \cdot 2\pi \cdot [Q_k + \Re e\{\delta Q_k\}]}
\end{aligned}$$

to give

$$\begin{aligned}
\det\ \{\underline{M}(s_0 + L, s_0) + \delta\underline{M}(s_0 + L, s_0)\} &= (\lambda_I - \delta\lambda_I) \cdot (\lambda_{-I} + \delta\lambda_{-I}) \times \\
&\quad (\lambda_{II} + \delta\lambda_{II}) \cdot (\lambda_{-II} + \delta\lambda_{-II}) \times \\
&\quad (\lambda_{III} + \delta\lambda_{III}) \cdot (\lambda_{-III} + \delta\lambda_{-III}) \\
&= \epsilon^{-2 \cdot [\alpha_I + \alpha_{II} + \alpha_{III}]} . \qquad \text{(D.16)}
\end{aligned}$$

On the other hand we obtain from eqns. (D.3a, b) [46] :

$$\begin{aligned}
&\det\ \{\underline{M}(s_0 + L, s_0) + \delta\underline{M}(s_0 + L, s_0)\} \\
&= \exp\left\{\int_{s_0}^{s_0+L} d\tilde{s} \cdot \mathrm{Sp}\ [\underline{A}(\tilde{s}) + \delta\underline{A}(\tilde{s})]\right\} . \qquad \text{(D.17)}
\end{aligned}$$

Comparing eqns. (D.16) and (D.17) we obtain :

$$\alpha_I + \alpha_{II} + \alpha_{III} = -\frac{1}{2} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \mathrm{Sp}\ [\delta\underline{A}(\tilde{s})] \qquad \text{(D.18)}$$

(see also Ref. [28]) where we have used:

$$\mathrm{Sp}\ [\underline{A}(\tilde{s})] = 0$$

(see eqn. (4.7)). Furthermore, taking into account (4.8) we find:

$$-\frac{1}{2} \cdot \mathrm{Sp}\ [\delta \underline{A}(\tilde{s})] = \frac{e\hat{V}}{E_0} \sin\varphi + C_1 \cdot (K_x^2 + K_z^2). \tag{D.19}$$

Inserting (D.19) into (D.18) and using the fact that

$$E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \frac{e\hat{V}}{E_0} \sin\varphi = U_0$$

(= average energy gained by a particle in one turn)

and

$$E_0 \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot C_1 \left[K_x^2 + K_z^2\right] = U_0$$

(= average energy radiated by a particle in one turn)

(see eqn. (4.10)) we get Robinson' theorem

$$\alpha_I + \alpha_{II} + \alpha_{III} = 2 \cdot \frac{U_0}{E_0}\ . \tag{D.20}$$

If we know for example two damping constants the third constant is then automatically fixed by this relation.

Note that our derivation is valid for an arbitrarily coupled system. J.M. Jowett has given a local version of the Robinson-Theorem [2].

Appendix E: The Essential Uniqueness of the Fokker-Planck Solutions for Large Times

The following considerations are based mainly on a method outlined in Ref. [8]. The difference between our treatment and that of Ref. [8] lies in the use of different boundary conditions.

In order to investigate the asymptotic time behaviour of the Fokker-Planck solutions we first introduce the abbreviations

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \equiv (J_I, J_{II}, J_{III}, J_{IV}, \Phi_I, \Phi_{II}, \Phi_{III}. \Phi_{IV}) \tag{E.1}$$

so that the Fokker-Planck equation (9.15) can be written in the form:

$$\frac{\partial W}{\partial s} = \left\{ -\sum_{i=1}^{8} \frac{\partial}{\partial x_i} D_i + \sum_{i,j=1}^{8} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} \right\} W \tag{E.2}$$

where

$$\begin{aligned}(D_1,\ D_2,\ D_3,\ D_4,\ D_5,\ D_6,\ D_7,\ D_8) \\ = (-2a_I J_I + M_I,\ -2a_{II} J_{II} + M_{II},\ -2a_{III} J_{III} + M_{III}, \\ -2a_{IV} J_{IV} + M_{IV},\ b_I,\ b_{II},\ b_{III},\ b_{IV})\end{aligned} \tag{E.3}$$

and where

$$D_{ij} = \delta_{ij} \cdot \tilde{D}_i \tag{E.4a}$$

with

$$\begin{aligned} &(\tilde{D}_1,\ \tilde{D}_2,\ \tilde{D}_3,\ \tilde{D}_4,\ \tilde{D}_5,\ \tilde{D}_6,\ \tilde{D}_7,\ \tilde{D}_8) \\ &\quad \equiv (J_I \cdot M_I,\ J_{II} \cdot M_{II},\ J_{III} \cdot M_{III},\ J_{IV} \cdot M_{IV},\ \frac{M_I}{4J_I},\ \frac{M_{II}}{4J_{II}},\ \frac{M_{III}}{4J_{III}},\ \frac{M_{IV}}{4J_{IV}})\ . \end{aligned} \tag{E.4b}$$

From eqns. (4.3b), (7.7), and (9.13c,f) it is clear that the diffusion matrix $((D_{ij}))$ is positive definite which is connected with the positivity of the radiative energy loss of the electron. We will need this property later.

We now introduce the Lyapunov functional of W with respect to another special physical solution W_0 [12] :

$$\begin{aligned} \hat{H}(s) &= \int_V d^8x \cdot W \cdot \ln\left(\frac{W}{W_0}\right) \\ &= \int_V d^8x \cdot W \cdot [\ln W - \ln W_0] \end{aligned} \tag{E.5}$$

(V denotes the action-angle phase space).

Defining the quantity

$$R = \frac{W}{W_0} \tag{E.6}$$

and using the relation ($R \geq 0$)

$$R \ln R - R + 1 = \int_0^R dx \cdot \ln x \geq 0 \tag{E.7}$$

as well as the normalization condition for W and W_0:

$$\int_V d^8x \cdot W = 1\ ; \tag{E.8a}$$

$$\int_V d^8x \cdot W_0 = 1 \tag{E.8b}$$

we obtain the inequality:

$$\begin{aligned} \hat{H}(s) &= \int_V d^8x \cdot W \cdot \ln R \\ &= \int_V d^8x \cdot [W \cdot \ln R - W + W_0] \\ &= \int_V d^8x \cdot W_0 \cdot [R \cdot \ln R - R + 1] \geq 0\ . \end{aligned} \tag{E.9}$$

i.e. $H(s)$ cannot have negative values.

[12] A solution W of the Fokker-Planck equation (9.15) is called physical if it is normalized and nonnegative and if the moments of $J_I, J_{II}, J_{III}, J_{IV}$ with respect to W are finite.

For the derivative

$$\frac{\partial}{\partial s}\,\hat{H}(s)$$

of the Lyapunov functional we get:

$$\begin{aligned}
\frac{\partial}{\partial s}\,\hat{H}(s) &= \int_V d^8x\cdot\left\{\left(\frac{\partial}{\partial s}\,W\right)\cdot\ln R + W\cdot\left[\frac{1}{W}\cdot\frac{\partial}{\partial s}\,W - \frac{1}{W_0}\cdot\frac{\partial}{\partial s}\,W_0\right]\right\} \\
&= \int_V d^8x\cdot\left\{\left(\frac{\partial}{\partial s}\,W\right)\cdot\ln R - R\cdot\left(\frac{\partial}{\partial s}\,W_0\right)\right\} + \frac{\partial}{\partial s}\int_V d^6x\cdot W \\
&= \int_V d^8x\cdot\left\{\left(\frac{\partial}{\partial s}\,W\right)\cdot\ln R - R\cdot\left(\frac{\partial}{\partial s}\,W_0\right)\right\}\,. \qquad \text{(E.10)}
\end{aligned}$$

Using eqn. (E.2), the first term on the r.h.s. of eqn. (E.10) can be written as:

$$\begin{aligned}
\int_V d^8x\cdot\left(\frac{\partial}{\partial s}\,W\right)\cdot\ln R &= \int_V d^8x\cdot\ln R\cdot\left\{-\sum_{i=1}^{8}\frac{\partial}{\partial x_i}D_i + \sum_{i,j=1}^{8}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}\right\}W \\
&= \int_V d^8x\cdot W\cdot\left\{\sum_{i=1}^{8}D_i + \sum_{i,j=1}^{8}D_{ij}\frac{\partial}{\partial x_j}\right\}\frac{\partial}{\partial x_i}\,\ln R \\
&= \int_V d^8x\cdot W\cdot\left\{\sum_{i=1}^{8}D_i + \sum_{i,j=1}^{8}D_{ij}\frac{\partial}{\partial x_j}\right\}\left(\frac{1}{R}\frac{\partial R}{\partial x_i}\right) \\
&= \int_V d^8x\cdot\frac{W}{R}\cdot\left\{\sum_{i=1}^{8}D_i + \sum_{i,j=1}^{8}D_{ij}\frac{\partial}{\partial x_j}\right\}\frac{\partial R}{\partial x_i} \\
&\quad - \int_V d^8x\cdot W\cdot\sum_{i,j=1}^{8}D_{ij}\cdot\frac{1}{R^2}\cdot\frac{\partial R}{\partial x_j}\cdot\frac{\partial R}{\partial x_i} \\
&= \int_V d^8x\cdot R\cdot\left\{-\sum_{i=1}^{8}\frac{\partial}{\partial x_i}D_i + \sum_{i,j=1}^{8}\frac{\partial^2}{\partial x_i\partial x_j}D_{ij}\right\}W_0 \\
&\quad - \int_V d^8x\cdot W\cdot\sum_{i,j=1}^{8}D_{ij}\cdot\frac{1}{R}\frac{\partial R}{\partial x_i}\cdot\frac{1}{R}\frac{\partial R}{\partial x_j} \\
&= \int_V d^8x\cdot R\cdot\left(\frac{\partial}{\partial s}\,W_0\right) \\
&\quad - \int_V d^8x\cdot W\cdot\sum_{i,j=1}^{8}D_{ij}\cdot\frac{1}{R}\frac{\partial R}{\partial x_i}\cdot\frac{1}{R}\frac{\partial R}{\partial x_j}\,. \qquad \text{(E.11)}
\end{aligned}$$

To carry out the partial integration we have used the fact that W and W_0 are periodic functions in Φ_k ($k = I,\ II,\ III,\ IV$) with period 2π and that the probability currents (see eqn. (10.3b)) of W, i.e.

$$\Im_i \equiv D_iW - \sum_{i,j=1}^{8}\frac{\partial}{\partial x_j}\left[D_{ij}W\right]\,; \quad (i = 1,2,3,4) \qquad \text{(E.12)}$$

and of W_0 associated with the action variables J_k ($k = I,\ II,\ III,\ IV$) vanish for $J_k = 0$ and $J_k \to \infty$.

Equations (E.10) and (E.11) lead to

$$\frac{\partial}{\partial s}\,\hat{H}(s) \;=\; -\int_V d^8x\cdot W\cdot\sum_{i,j=1}^{8} D_{ij}\cdot\frac{\partial}{\partial x_i}\left[\ln\left(\frac{W}{W_0}\right)\right]\cdot\frac{\partial}{\partial x_j}\left[\ln\left(\frac{W}{W_0}\right)\right]\,. \tag{E.13}$$

Now, since the diffusion matrix $((D_{ij}))$ is positive definite we have

$$\frac{\partial}{\partial s}\,\hat{H}(s) < 0 \tag{E.14a}$$

if

$$\sum_{i=1}^{8}\left\{\frac{\partial}{\partial x_i}\left[\frac{W}{W_0}\right]\right\}^2 \neq 0\,. \tag{E.14b}$$

Hence from eqns. (E.9) and (E.14) we have:

$$\lim_{s\to\infty}\left[\frac{\partial}{\partial x_i}\left(\frac{W}{W_0}\right)\right] = 0 \tag{E.15}$$

(otherwise $\hat{H}(s)$ would decrease below 0.)

From (E.15) and using (E.9) we can finally see that the two solutions W and W_0 must coincide for long times.

Choosing for the orbital part of W_0, the stationary solution (10.7), and for the spin part of W_0, a solution of eqn. (10.49) which is independent of the phase Φ_{IV}, we obtain the result that for the orbital motion the stationary distribution (10.7) is unique and that for the spin motion, an arbitrary phase distribution will converge to a uniform distribution in Φ_{IV}.

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