Z. Phys. C 64, 117–142 (1994)

ZEITSCHRIFT
FÜR PHYSIK C

A canonical 8-dimensional formalism for classical spin-orbit motion in storage rings

I. A new pair of canonical spin variables

D.P. Barber, K. Heinemann, G. Ripken

Deutsches Elektronen-Synchrotron DESY, D-22603 Hamburg, Germany

Received: 30 November 1993

Abstract. In this paper we present a classical symplectic treatment of linear and nonlinear spin-orbit motion for storage rings using a fully coupled eight-dimensional formalism which generalizes earlier investigations of coupled synchro-betatron oscillations [1, 2] by introducing two new real canonical spin variables which behave, in a small-angle limit, like those already used in linearised spin theory. Thus in addition to the usual $x-z-s$ couplings, both the spin to orbit and orbit to spin coupling are described canonically. Since the spin Hamiltonian can be expanded in a Taylor series in canonical variables, the formalism is convenient for use in 8-dimensional symplectic tracking calculations with the help, for example, of Lie algebra or differential algebra [3, 4], for the study of chaotic spin motion, for construction of spin normal forms and for studying the effect of Stern-Gerlach forces [5].

1 Introduction

The most elegant route to the equations of classical spin-orbit motion in accelerators and storage rings is by way of the semiclassical spin-orbit Hamiltonian of Derbenev and Kondratenko [6]. This not only leads to the Thomas-Bargmann-Michel-Telegdi (T-BMT) [7, 8] equation for the spin precession but also delivers the equations of orbital motion including the correct relativistic generalization of the Stern-Gerlach (SG) forces.

The standard tool for describing particle motion in accelerators and storage rings is canonical perturbation theory. However, a direct extension of these methods to include classical spin motion is not straightforward since the appropriate canonical spin variables, J (the spin projection onto some axis) and Ψ (the phase angle) are not normally "small" and cannot be handled perturbatively. On the other hand, it is standard practice to describe the spin motion with respect to a special comoving coordinate system [9, 10] which allows the spin coordinates to be linearized. But these latter are not canonical.

In this paper we introduce two new real canonical spin variables which allow the usual six dimensional canonical perturbation methods for orbit motion to be extended naturally into a canonical eight dimensional theory which reduces to the SLIM [9, 10] formalism when the spin variables are so small that the theory can be linearized. The new variables uniquely parametrize the spin motion over (almost) the whole "spin sphere".

This formalism in fact represents a natural generalization of the earlier work of [1, 2] where we presented an analytical technique for investigating linear and nonlinear coupled synchro - betatron oscillations which handles the combined external magnetic and electric forces in a consistent canonical manner and which includes consistently and canonically the synchrotron oscillations in the electric fields of the accelerating cavities. The motion was described by using the canonical variables $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{p}_\sigma$ of the fully six-dimensional formalism. The equations derived in those papers provide the basis of a symplectic, nonlinear, 6-dimensional tracking program.

With the two new spin variables $(\hat{\alpha}, \hat{\beta})$ the spin part of the Hamiltonian takes a form which can be expanded into a power series in an economic way, leading to various orders of approximation of the canonical spin equations. It is this property which distinguishes our canonical coordinates $\hat{\alpha}$ and $\hat{\beta}$ from others occuring in the literature [11, 12].

Armed with the complete set $\hat{x}$, $\hat{p}_x$, $\hat{z}$, $\hat{p}_z$, $\hat{\sigma}$, $\hat{p}_\sigma$, $\hat{\alpha}$, $\hat{\beta}$ we are then in a position to develop, in the framework of this 8-dimensional formalism, a symplectic treatment of the combined orbital and spin motion in storage rings.

The equations so derived can serve to develop a nonlinear, 8-dimensional (symplectic) tracking program and modern methods such as Lie algebra, normal forms and differential algebra which are commonly used for orbital motion could also be used. Such a program could be used to study (in addition to orbital problems) chaotic behaviour of spin motion when spin-orbit resonances are wide and overlap as, for example, for protons in the TeV range and to investigate the influence of Stern-Gerlac forces. Furthermore, our formalism automatically includes provision for describing the effects of skew quadrupoles and solenoids.

The requirement that the theory be based on a Hamiltonian and includes the effects of skew quadrupoles and solenoids is not just academic but could also have a practical use. For example, in recent years there have been a number of proposals for obtaining spin polarized antiprotons [13–16]. One of these [15, 17] involves using the Stern-Gerlach forces experienced by particles in strong quadrupoles in a storage ring to drive the orbital motion in resonance and thereby spatially separate the opposite spin component sub-ensembles of an initially unpolarized ensemble in a storage ring into two separate ensembles oscillating in antiphase. On the other hand, Derbenev [5] has pointed out, essentially on the basis of conservation laws applied to the coupled spin-orbit oscillator system, that the betatron amplitude achievable in the original proposal could be limited. But his treatment dealt with just one mode of uncoupled betatron motion. Clearly, since the SG forces are so small the safest way to untangle the pros and cons is to begin with a fully symplectic description in which the action of the spin motion on the orbit and the action of the orbit motion of the spin are considered simultaneously so that the exchange of energy between all four oscillation modes (three for the orbit and one for the spin) is handled automatically. This is could be especially relevant when the spin and orbital motion are in resonance. In this way it should be possible to separate spurious orbit amplitude growth, which might occur simply if the formalism were not symplectic, from growth due to SG effects.

The formalism set up here is, in the spirit of the semiclassical origin of the D-K Hamiltonian, correct at semiclassical order i.e. following some notions of Yokoya [11] it contains no terms higher than first order in $\hbar$ [11].

This paper is the first of a series in which we study the interaction of the spin and orbit motion using methods familiar from other parts of storage ring physics.

After basing the calculation of the spin motion on a combined spin-orbit Hamiltonian it is natural to try to find the action angle variables for the combined motion. This topic has already been treated in [6, 11, 18] and is the subject of paper II [19]. Finding the action angle variables for spin involves finding the semiclassical spin quantization axis along which the spin component is an integral of motion. This axis, the so called $\mathbf{n}$ axis, is a single valued function of the canonical orbit coordinates and the azimuth [19, 20]. Away from tunes where the spin and orbit motion are in resonance, $\mathbf{n}$ is almost parallel to the periodic spin axis $\mathbf{n}_0$ defined on the closed orbit [9,10]. But near resonance $\mathbf{n}$ can deviate strongly from $\mathbf{n}_0$ and can depend strongly on the position of the particle in the orbital phase space. On resonance $\mathbf{n}$ is not even uniquely defined. In any case it is clear [6, 11, 18, 20–22] that for *motion* in the spatially and temporally varying fields of a storage ring it is incorrect to assume that the spin quantization axis along which the spin component is constant is parallel to the local magnetic field. Clearly, this is a key point if arguments based on conservation of laws are used to discuss Spin - Splitter systems where there are exotic field configurations.

In detail, our considerations are organized as follows:

Starting in Sect. 2 from the Hamiltonian of a charged particle for spin-orbit motion in an electromagnetic field, described in a fixed Cartesian coordinate system, in Sect. 3 we use a canonical transformation to arrive at the symplectic formalism for spin-orbit motion expressed in machine coordinates, taking into account all kinds of coupling induced by skew quadrupoles and solenoids (coupling of betatron motion), by a non-vanishing dispersion in the cavities (synchro-betatron coupling) and by Stern-Gerlach forces.

The vector potentials we need to describe the electromagnetic field are calculated in Appendix A.

In Sect. 4 the arc length of the design orbit as independent variable (instead of the time t) is introduced and new (small and oscillating) variables σ, p_σ are defined which describe the longitudinal oscillations.

Spin motion with respect to the dreibein $(\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_z)$ is investigated in Sect. 5 and the corresponding Hamiltonian is derived by applying a transformation similar to that used by Yokoya.

Then in Sect. 6 and, with the help of Appendix B, we define an 8-dimensional closed orbit which we introduce as a new reference orbit for spin-orbit motion. The Hamiltonian with respect to the closed orbit is again obtained by using canonical transformations, whereby the canonical variables $\hat{\alpha}$ and $\hat{\beta}$ are introduced to describe the spin motion.

A summary is finally presented in Sect. 7.

2 Spin-orbit motion in a fixed coordinate system

2.1 The starting Hamiltonian

The starting point of our description of classical spin-orbit motion will be the classical Hamiltonian, $\mathscr{H}$:

$$\mathscr{H}(\mathbf{r}, \psi; \mathbf{P}, J; t) = \mathscr{H}_{\text{orb}}(\mathbf{r}, \mathbf{P}, t) + \mathbf{\Omega}_0(\mathbf{r}, \mathbf{P}, t) \cdot \boldsymbol{\xi}, \tag{2.1}$$

with

$$\mathscr{H}_{\text{orb}}(\mathbf{r}, \mathbf{P}, t) = c \cdot \{\boldsymbol{\pi}^2 + m_0^2 c^2\}^{1/2} + e\phi, \tag{2.2}$$

and

$$\mathbf{\Omega}_0 = -\frac{e}{m_0 c}\left[\left(\frac{1}{\gamma} + a\right) \cdot \mathscr{B} - \frac{a(\boldsymbol{\pi} \cdot \mathscr{B})}{\gamma(\gamma+1) m_0^2 c^2} \cdot \boldsymbol{\pi} - \frac{1}{m_0 c \gamma}\left(a + \frac{1}{1+\gamma}\right) \boldsymbol{\pi} \times \mathscr{E}\right], \tag{2.3}$$

where $\mathbf{r}$ and $\mathbf{P}$ are canonical orbital position and momentum variables, $\boldsymbol{\xi}$ is a classical spin vector of length $\hbar/2$ and where $\boldsymbol{\pi}$ and γ are given by:

$$\boldsymbol{\pi} = \mathbf{P} - \frac{e}{c}\mathbf{A} \quad \text{(kinetic momentum vector)}, \tag{2.4}$$

$$\gamma = \frac{1}{m_0 c} \cdot \sqrt{m_0^2 c^2 + \boldsymbol{\pi}^2} \quad \text{(Lorentz factor)}. \tag{2.5}$$

The following abbreviations have been used:

- e = charge of the particle;
- m_0 = rest mass of the particle;
- c = velocity of light;
- $\mathscr{E}$ = electric field;
- $\mathscr{B}$ = magnetic field;
- $\mathbf{r}$ = radius vector of the particle;
- $\boldsymbol{\xi}$ = classical spin angular momentum vector in the rest frame of the particle of length $\hbar/2$;
- $a=(g-2)/2$ (0.00116 for electrons, 1.793 for protons) and quantifies the anomalous spin g factor;
- $2\pi\hbar$ = Planck's constant.

The quantities $\mathbf{A}$ and ϕ appearing in (2.2) and (2.4) are the vector and scalar potentials from which the electric field $\mathscr{E}$ and the magnetic field $\mathscr{B}$ are derived as:

$$\mathscr{E} = -\text{grad }\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{2.6a}$$

$$\mathscr{B} = \text{curl }\mathbf{A}\,. \tag{2.6b}$$

Our starting Hamiltonian (2.1) is that which is often used for describing the spin-orbit dynamics in accelerators [5, 6, 11, 23, 24] and is the classical reinterpretation of the effective quantum mechanical Hamiltonian derived by a unitary transformation of the Dirac Hamiltonian and by working in the semiclassical limit. This latter is valid when the external electromagnetic field is weak and it neglects bremsstrahlung effects [25].

In terms of the three unit cartesian coordinate vectors in the fixed laboratory frame, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we can write $\mathbf{r}$, $\mathbf{P}$ and $\boldsymbol{\xi}$ as:

$$\mathbf{r} = X_1\cdot\mathbf{e}_1 + X_2\cdot\mathbf{e}_2 + X_3\cdot\mathbf{e}_3, \tag{2.7a}$$

$$\mathbf{P} = P_1\cdot\mathbf{e}_1 + P_2\cdot\mathbf{e}_2 + P_3\cdot\mathbf{e}_3, \tag{2.7b}$$

$$\boldsymbol{\xi} = \xi_1\cdot\mathbf{e}_1 + \xi_2\cdot\mathbf{e}_2 + \xi_3\cdot\mathbf{e}_3. \tag{2.7c}$$

Furthermore, we write the components of $\boldsymbol{\xi}$ in the form:

$$\begin{cases} \xi_1 = \sqrt{\xi^2 - J^2}\cdot\cos\psi, \\ \xi_2 = \sqrt{\xi^2 - J^2}\cdot\sin\psi, \\ \xi_3 = J, \end{cases} \tag{2.8}$$

with

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{\hbar^2}{4}.$$

We will treat ψ and J as canonical spin variables [11, 23] to be used on an equal basis with $\mathbf{r}$ and $\mathbf{P}$. The spin vector $\boldsymbol{\xi}$ is of constant length since it obeys a precession equation. See below.

With (2.1) and (2.8) we have the Hamiltonian for the canonical variables $\mathbf{r}, \mathbf{P}, \psi, J$.

One of the aims of this paper is to transform from the canonical variables $(\mathbf{r}, \mathbf{P}, \psi, J)$ to the new set of canonical variables $(\hat{x}, \hat{z}, \hat{\sigma}, \hat{\alpha};\ \hat{p}_x, \hat{p}_z, \hat{p}_\sigma, \hat{\beta})$ (see (6.39)).

2.2 The equations of motion

2.2.1 Orbital motion. With this Hamiltonian (2.1) the orbital equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}X_k = +\frac{\partial \mathscr{H}_{\text{orb}}}{\partial P_k} + \frac{\partial \mathbf{\Omega}_0}{\partial P_k}\cdot\boldsymbol{\xi}, \tag{2.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_k = -\frac{\partial \mathscr{H}_{\text{orb}}}{\partial X_k} - \frac{\partial \mathbf{\Omega}_0}{\partial X_k}\cdot\boldsymbol{\xi}, \quad (k=1, 2, 3). \tag{2.9b}$$

The first terms on the rhs of (2.9) are the Lorentz terms and the second terms describe the Stern-Gerlach force [26]. Thus our Hamiltonian includes the SG force automatically. Note that here we deal with the relativistic generalization of the SG effect. It is clear (see (2.3)) that our SG terms reduce to the usual non-relativistic forms in the limit that γ becomes unity and that for $\gamma > 1$ the factor $g/2$ in the expression for the SG force in field gradients in [15] should be replaced by $(g-2)/2 + 1/\gamma$. Thus if γ is increased from 1 up to a large value, the SG force is reduced by the factor $(g-2)/g$. For protons ($g/2 = 2.793$) this gives a 36% reduction. But for electrons ($g/2 = 1.00116$) the reduction factor is large.

The discussion in this paper covers both relativistic and non-relativistic motion.

2.2.2 Spin motion. Using (2.8) and treating J, ψ as canonical variables, we can easily show that [11]:

$$\{\xi_1, \xi_2\}_{\psi, J} = \xi_3, \tag{2.10a}$$

$$\{\xi_2, \xi_3\}_{\psi, J} = \xi_1, \tag{2.10b}$$

$$\{\xi_3, \xi_1\}_{\psi, J} = \xi_2. \tag{2.10c}$$

These Poisson bracket relations for spin, which do not contain $\hbar$ on the rhs, are the classical analogues of the commutation relation among Pauli spin operators. Using these relations together with the canonical equations of the spin motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi = +\frac{\partial}{\partial J}\mathscr{H}_{\text{spin}}, \tag{2.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J = -\frac{\partial}{\partial \psi}\mathscr{H}_{\text{spin}}, \tag{2.11b}$$

where

$$\mathscr{H}_{\text{spin}} = \mathbf{\Omega}_0\cdot\boldsymbol{\xi}, \tag{2.12}$$

and

$$\mathbf{\Omega}_0 = \Omega_{01}\cdot\mathbf{e}_1 + \Omega_{02}\cdot\mathbf{e}_2 + \Omega_{03}\cdot\mathbf{e}_3, \tag{2.13}$$

so that

$$\begin{aligned}\mathbf{\Omega}_0\cdot\boldsymbol{\xi} &= \Omega_{01}\cdot\xi_1 + \Omega_{02}\cdot\xi_2 + \Omega_{03}\cdot\xi_3 \\ &= \sqrt{\xi^2 - J^2}\cdot[\Omega_{01}\cdot\cos\psi + \Omega_{02}\cdot\sin\psi] + \Omega_{03}\cdot J,\end{aligned} \tag{2.14}$$

we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi} = \mathbf{\Omega}_0\times\boldsymbol{\xi}. \tag{2.15}$$

Thus this Hamiltonian formalism reproduces the Thomas-BMT equation [7, 8].

The result (2.15) can also be obtained by using the equation of motion:

$$\frac{d}{dt}\xi=\{\xi,\mathcal{H}_{\text{spin}}\}_{\psi,J}\equiv\frac{\partial\xi}{\partial\psi}\cdot\frac{\partial\mathcal{H}_{\text{spin}}}{\partial J}-\frac{\partial\xi}{\partial J}\cdot\frac{\partial\mathcal{H}_{\text{spin}}}{\partial\psi}. \tag{2.16}$$

2.2.3 The combined form of the spin-orbit equations. The combined equations of spin-orbit motion can be written in the form:

$$\frac{d}{dt}X_k=+\frac{\partial\mathcal{H}}{\partial P_k}, \tag{2.17a}$$

$$\frac{d}{dt}P_k=-\frac{\partial\mathcal{H}}{\partial X_k}, \quad (k=1,2,3,4), \tag{2.17b}$$

with

$$X_4\equiv\psi, \tag{2.18a}$$

$$P_4\equiv J. \tag{2.18b}$$

3 Introduction of machine coordinates via a canonical transformation

3.1 Reference trajectory and coordinate frame

The position vector $\mathbf{r}$ in (2.1) refers to a fixed coordinate system with the coordinates X_1, X_2 and X_3. However, in accelerator physics, it is useful to introduce the natural coordinates x, z, s in a suitable curvilinear coordinate system. With this in mind we assume that an ideal closed design orbit exists which describes the path of a particle of constant energy E_0, i.e. we neglect energy variations due to cavities and to radiation loss. In addition we assume that there are no field errors or correction magnets. We also require that the design orbit comprises piecewise flat curves which lie either in the horizontal or vertical plane so that it has (piecewise) no torsion. The design orbit which will be used as the reference system will, in the following, be described by the vector $\mathbf{r}_0(s)$ where s is the length along the design orbit. An arbitrary particle orbit $\mathbf{r}(s)$ is then described by the deviation $\delta\mathbf{r}(s)$ of the particle orbit $\mathbf{r}(s)$ from the design orbit $\mathbf{r}_0(s)$:

$$\mathbf{r}(s)=\mathbf{r}_0(s)+\delta\mathbf{r}(s). \tag{3.1}$$

The vector $\delta\mathbf{r}$ can as usual [27] be described using an orthogonal coordinate system ("dreibein") accompanying the particle which travels along the design orbit and comprises

the unit tangent vector $\mathbf{e}_s(s)=\frac{d}{ds}\mathbf{r}_0(s)\equiv\mathbf{r}'_0(s)$,

a unit vector $\mathbf{e}_x(s)$,

which lies perpendicular to $\mathbf{e}_s$ in the horizontal plane [10]

and the unit vector $\mathbf{e}_z(s)=\mathbf{e}_s(s)\times\mathbf{e}_x(s)$.

In this natural coordinate system we may represent $\delta\mathbf{r}(s)$ as:

$$\delta\mathbf{r}(s)=(\delta\mathbf{r}\cdot\mathbf{e}_x)\cdot\mathbf{e}_x+(\delta\mathbf{r}\cdot\mathbf{e}_z)\cdot\mathbf{e}_z$$

(since the "dreibein" accompanies the particle, the $\mathbf{e}_s$-component of $\delta\mathbf{r}$ is always zero by definition).

Thus, the orbit-vector $\mathbf{r}(s)$ can be written in the form:

$$\mathbf{r}(x,z,s)=\mathbf{r}_0(s)+x(s)\cdot\mathbf{e}_x(s)+z(s)\cdot\mathbf{e}_z(s), \tag{3.2}$$

and the Serret-Frenet formulae for the dreibein $(\mathbf{e}_s,\mathbf{e}_x,\mathbf{e}_z)$ read as:

$$\frac{d}{ds}\mathbf{e}_x(s)=+K_x(s)\cdot\mathbf{e}_s(s), \tag{3.3a}$$

$$\frac{d}{ds}\mathbf{e}_z(s)=+K_z(s)\cdot\mathbf{e}_s(s), \tag{3.3b}$$

$$\frac{d}{ds}\mathbf{e}_s(s)=-K_x(s)\cdot\mathbf{e}_x(s)-K_z(s)\cdot\mathbf{e}_z(s), \tag{3.3c}$$

where we assume that

$$K_x(s)\cdot K_z(s)=0, \tag{3.4}$$

(piecewise no torsion) and where $K_x(s)$, $K_z(s)$ designate the curvatures in the x-direction and in the z-direction respectively.

Note that the sign of $K_x(s)$ and $K_z(s)$ is fixed by (3.3).

3.2 Introduction of the natural coordinates x, z, s via a canonical transformation

Writing:

$$\mathbf{r}=X_1\cdot\mathbf{e}_1+X_2\cdot\mathbf{e}_2+X_3\cdot\mathbf{e}_3=\mathbf{r}_0(s)+x\cdot\mathbf{e}_x(s)+z\cdot\mathbf{e}_z(s),$$

$$\mathbf{P}=P_1\cdot\mathbf{e}_1+P_2\cdot\mathbf{e}_2+P_3\cdot\mathbf{e}_3,$$

we can obtain the canonical transformation:

$$X_1,X_2,X_3,P_1,P_2,P_3\rightarrow x,z,s,p_x,p_z,p_s,$$

(ψ, J unchanged)

by introducing the generating function [28]:

$$F_3(x,z,s;P_1,P_2,P_3;t)=-[\mathbf{r}_0(s)+x\cdot\mathbf{e}_x(s)+z\cdot\mathbf{e}_z(s)]\cdot\mathbf{P}. \tag{3.5}$$

This leads to the transformation equations:

$$X_1=-\frac{\partial F_3}{\partial P_1}=[\mathbf{r}_0(s)+x(s)\cdot\mathbf{e}_x(s)+z(s)\cdot\mathbf{e}_z(s)]\cdot\mathbf{e}_1=\mathbf{r}\cdot\mathbf{e}_1, \tag{3.6a}$$

$$X_2=-\frac{\partial F_3}{\partial P_2}=[\mathbf{r}_0(s)+x(s)\cdot\mathbf{e}_x(s)+z(s)\cdot\mathbf{e}_z(s)]\cdot\mathbf{e}_2=\mathbf{r}\cdot\mathbf{e}_2, \tag{3.6b}$$

$$X_3=-\frac{\partial F_3}{\partial P_3}=[\mathbf{r}_0(s)+x(s)\cdot\mathbf{e}_x(s)+z(s)\cdot\mathbf{e}_z(s)]\cdot\mathbf{e}_3=\mathbf{r}\cdot\mathbf{e}_3, \tag{3.6c}$$

$$p_x=-\frac{\partial F_3}{\partial x}=\mathbf{e}_x(s)\cdot\mathbf{P}, \tag{3.6d}$$

$$p_z=-\frac{\partial F_3}{\partial z}=\mathbf{e}_z(s)\cdot\mathbf{P}, \tag{3.6e}$$

$$p_s=-\frac{\partial F_3}{\partial s}=[1+K_x\cdot x+K_z\cdot z]\cdot\mathbf{e}_s\cdot\mathbf{P}. \tag{3.6f}$$

Note, that (3.6a–c) reproduce the defining equation (2.1a) for the variables X_1, X_2 and X_3 and that (3.6d–f) determine the new momentum variables p_x, p_z and p_s. The spin variables ψ and J remain unchanged.

Because

$$\frac{\partial F_3}{\partial t}=0,$$

the Hamiltonian is transformed to:

$$\mathscr{H}\rightarrow\mathscr{H}+\frac{\partial F_3}{\partial t}=\mathscr{H}=\mathscr{H}_{\text{orb}}+\mathbf{\Omega}_0\cdot\boldsymbol{\xi}. \tag{3.7}$$

In order to obtain $\mathscr{H}$ in terms of the new variables x, p_x, z, p_z, s, p_s, we write:

$$\boldsymbol{\pi}=\pi_x\cdot\mathbf{e}_x+\pi_z\cdot\mathbf{e}_z+\pi_s\cdot\mathbf{e}_s, \tag{3.8}$$

with

$$\pi_x=\boldsymbol{\pi}\cdot\mathbf{e}_x\equiv\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)\cdot\mathbf{e}_x=p_x-\frac{e}{c}A_x, \tag{3.9a}$$

$$\pi_z=\boldsymbol{\pi}\cdot\mathbf{e}_z\equiv\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)\cdot\mathbf{e}_z=p_z-\frac{e}{c}A_z, \tag{3.9b}$$

$$\pi_s=\boldsymbol{\pi}\cdot\mathbf{e}_s\equiv\left(\mathbf{P}-\frac{e}{c}\mathbf{A}\right)\cdot\mathbf{e}_s=\frac{p_s}{[1+K_x\cdot x+K_z\cdot z]}-\frac{e}{c}A_s, \tag{3.9c}$$

whereby

$$\begin{aligned}\mathbf{A}&=A_x\cdot\mathbf{e}_x+A_z\cdot\mathbf{e}_z+A_s\cdot\mathbf{e}_s\\&=(\mathbf{A}\cdot\mathbf{e}_x)\cdot\mathbf{e}_x+(\mathbf{A}\cdot\mathbf{e}_z)\cdot\mathbf{e}_z+(\mathbf{A}\cdot\mathbf{e}_s)\cdot\mathbf{e}_s.\end{aligned} \tag{3.10}$$

Thus we obtain:

$$\begin{aligned}\mathscr{H}_{\text{orb}}=e\phi+c\cdot\Bigg\{&\left(p_x-\frac{e}{c}A_x\right)^2+\left(p_z-\frac{e}{c}A_z\right)^2\\&+\left(\frac{p_s}{[1+K_x\cdot x+K_z\cdot z]}-\frac{e}{c}A_s\right)^2+m_0^2c^2\Bigg\}^{1/2},\end{aligned} \tag{3.11}$$

and

$$\mathbf{\Omega}_0=\Omega_{0s}\cdot\mathbf{e}_s+\Omega_{0x}\cdot\mathbf{e}_x+\Omega_{0z}\cdot\mathbf{e}_z, \tag{3.12}$$

with

$$\begin{aligned}\Omega_{0s}=-\frac{e}{m_0c}\Bigg[&\left(\frac{1}{\gamma}+a\right)\cdot\mathscr{B}_s-\frac{a(\pi_s\cdot\mathscr{B}_s+\pi_x\cdot\mathscr{B}_x+\pi_z\cdot\mathscr{B}_z)}{\gamma(\gamma+1)\cdot m_0^2c^2}\cdot\pi_s\\&-\frac{1}{m_0c\gamma}\left(a+\frac{1}{1+\gamma}\right)(\pi_x\mathscr{E}_z-\pi_z\mathscr{E}_x)\Bigg],\end{aligned} \tag{3.13a}$$

$$\begin{aligned}\Omega_{0x}=-\frac{e}{m_0c}\Bigg[&\left(\frac{1}{\gamma}+a\right)\cdot\mathscr{B}_x-\frac{a(\pi_s\cdot\mathscr{B}_s+\pi_x\cdot\mathscr{B}_x+\pi_z\cdot\mathscr{B}_z)}{\gamma(\gamma+1)\cdot m_0^2c^2}\cdot\pi_x\\&-\frac{1}{m_0c\gamma}\left(a+\frac{1}{1+\gamma}\right)(\pi_z\mathscr{E}_s-\pi_s\mathscr{E}_z)\Bigg],\end{aligned} \tag{3.13b}$$

$$\begin{aligned}\Omega_{0z}=-\frac{e}{m_0c}\Bigg[&\left(\frac{1}{\gamma}+a\right)\cdot\mathscr{B}_z-\frac{a(\pi_s\cdot\mathscr{B}_s+\pi_x\cdot\mathscr{B}_x+\pi_z\cdot\mathscr{B}_z)}{\gamma(\gamma+1)\cdot m_0^2c^2}\cdot\pi_z\\&-\frac{1}{m_0c\gamma}\left(a+\frac{1}{1+\gamma}\right)(\pi_s\mathscr{E}_x-\pi_x\mathscr{E}_s)\Bigg],\end{aligned} \tag{3.13c}$$

whereby

$$\gamma=\frac{1}{m_0c}\sqrt{m_0^2c^2+\boldsymbol{\pi}^2}=\frac{\mathscr{H}_{\text{orb}}-e\phi}{m_0c^2}$$

and $\mathscr{B}$ and $\mathscr{E}$ have to be written as functions of s, x, z, t.

With (3.7), (3.11), and (3.13) we have the Hamiltonian for the canonical variables

$x, z, s, \psi; p_x, p_z, p_s, J$.

Remark: Equation (3.5) is an example of a point transformation

$$q_k\rightarrow q_k', \tag{3.14}$$

which may be written in the most general form as:

$$q_k=f_k(q_l', t). \tag{3.15}$$

This transformation can be obtained as a canonical transformation

$$q_k, p_k\rightarrow q_k', p_k' \tag{3.16}$$

by the generating function

$$F_3(q_l', p_l, t)=-\sum_n p_n\cdot f_n(q_l', t). \tag{3.17}$$

The corresponding transformation equations read as:

$$q_k=-\frac{\partial F_3}{\partial p_k}=f_k(q_l', t), \tag{3.18a}$$

$$p_k'=-\frac{\partial F_3}{\partial q_k'}=\sum_n p_n\cdot\frac{\partial}{\partial q_k'}f_n(q_l', t), \tag{3.18b}$$

$$\mathscr{H}'=\mathscr{H}+\frac{\partial F_3}{\partial t}. \tag{3.18c}$$

Here (3.18a) coincides with (3.15) defining the new variables q_k' and (3.18b) determines the new momenta p_k' corresponding to the variables q_k', whereas the new Hamiltonian $\mathscr{H}'$ is given by (3.18c) which has to be written in terms of q_k' and p_k'.

3.3 The equations of motion

3.3.1 Orbital motion. In the new orbital coordinates the equations of orbital motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}x=+\frac{\partial\mathscr{H}_{\text{orb}}}{\partial p_x}+\frac{\partial\mathbf{\Omega}_0}{\partial p_x}\cdot\boldsymbol{\xi},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_x=-\frac{\partial\mathscr{H}_{\text{orb}}}{\partial x}-\frac{\partial\mathbf{\Omega}_0}{\partial x}\cdot\boldsymbol{\xi}, \tag{3.19a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z=+\frac{\partial\mathscr{H}_{\text{orb}}}{\partial p_z}+\frac{\partial\mathbf{\Omega}_0}{\partial p_z}\cdot\boldsymbol{\xi},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_z=-\frac{\partial\mathscr{H}_{\text{orb}}}{\partial z}-\frac{\partial\mathbf{\Omega}_0}{\partial z}\cdot\boldsymbol{\xi}, \tag{3.19b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s=+\frac{\partial\mathscr{H}_{\mathrm{orb}}}{\partial p_s}+\frac{\partial\mathbf{\Omega}_0}{\partial p_s}\cdot\boldsymbol{\xi},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_s=-\frac{\partial\mathscr{H}_{\mathrm{orb}}}{\partial s}-\frac{\partial\mathbf{\Omega}_0}{\partial s}\cdot\boldsymbol{\xi}. \tag{3.19c}$$

3.3.2 Spin motion. Although we have not yet written the spin, $\boldsymbol{\xi}$, in terms of $\mathbf{e}_x$, $\mathbf{e}_z$, $\mathbf{e}_s$ the equations of spin motion are as before:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}=\mathbf{\Omega}_0\times\boldsymbol{\xi}, \tag{3.20}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi=+\frac{\partial}{\partial J}[\mathbf{\Omega}_0\cdot\boldsymbol{\xi}]=+\frac{\partial}{\partial J}\mathscr{H}, \tag{3.21a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J=-\frac{\partial}{\partial\psi}[\mathbf{\Omega}_0\cdot\boldsymbol{\xi}]=-\frac{\partial}{\partial\psi}\mathscr{H}. \tag{3.21b}$$

3.3.3 The combined form of the spin-orbit equations. The combined equations of spin-orbit motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}x=+\frac{\partial\mathscr{H}}{\partial p_x},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_x=-\frac{\partial\mathscr{H}}{\partial x}, \tag{3.22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}z=+\frac{\partial\mathscr{H}}{\partial p_z},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_z=-\frac{\partial\mathscr{H}}{\partial z}, \tag{3.22b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s=+\frac{\partial\mathscr{H}}{\partial p_s},\quad \frac{\mathrm{d}}{\mathrm{d}t}p_s=-\frac{\partial\mathscr{H}}{\partial s}, \tag{3.22c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi=+\frac{\partial\mathscr{H}}{\partial J},\quad \frac{\mathrm{d}}{\mathrm{d}t}J=-\frac{\partial\mathscr{H}}{\partial\psi}. \tag{3.22d}$$

4 The arc length of the design orbit as independent variable

In (3.22) the time t appeared as independent variable. In order, as usual in accelerator physics, to introduce the arc length s of the design orbit as independent variable we recall that (3.22) is equivalent to a version of Hamilton's principle [29]:

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\cdot\{\dot{x}\cdot p_x+\dot{z}\cdot p_z+\dot{s}\cdot p_s+\dot{\psi}\cdot J$$
$$-\mathscr{H}(x,z,s,\psi;p_x,p_z,p_s,J;t)\}=0, \tag{4.1a}$$

with

$$\begin{cases}\delta x(t_1)=\delta z(t_1)=\delta s(t_1)=\delta\psi(t_1)=0,\\ \delta p_x(t_1)=\delta p_z(t_1)=\delta p_s(t_1)=\delta J(t_1)=0,\\ \delta x(t_2)=\delta z(t_2)=\delta s(t_2)=\delta\psi(t_2)=0,\\ \delta p_x(t_2)=\delta p_z(t_2)=\delta p_s(t_2)=\delta J(t_2)=0,\\ \delta t_1=\delta t_2=0,\end{cases} \tag{4.1b}$$

where the variables $x, z, s, \psi, p_x, p_z, p_s, J, t$ are varied independently of each other and are held constant at the end points. (For the usual derivation of the Hamiltonian equations (3.22) from the variational principle (4.1) the variation of the time t is actually not needed. However, in order to be able to carry out the derivation of (4.1) it is useful, nevertheless, to allow t to vary.)

Equation (4.1) can now be rewritten using

$$\mathrm{d}t=\frac{\mathrm{d}t}{ds}\cdot\mathrm{d}s,$$

as:

$$\delta\int_{s_1}^{s_2}\mathrm{d}s\cdot\{x'\cdot p_x+z'\cdot p_z+\psi'\cdot J+t'\cdot(-\mathscr{H})$$
$$+p_s(x,z,t,\psi;p_x,p_z,-\mathscr{H},J;s)\}=0, \tag{4.2a}$$

with

$$\begin{cases}\delta x(s_1)=\delta z(s_1)=\delta t(s_1)=\delta\psi(s_1)=0,\\ \delta p_x(s_1)=\delta p_z(s_1)=\delta\mathscr{H}(s_1)=\delta J(s_1)=0,\\ \delta x(s_2)=\delta z(s_2)=\delta t(s_2)=\delta\psi(s_2)=0,\\ \delta p_x(s_2)=\delta p_z(s_2)=\delta\mathscr{H}(s_2)=\delta J(s_2)=0,\\ \delta s_1=\delta s_2=0,\end{cases} \tag{4.2b}$$

and

$$y'\equiv\frac{\mathrm{d}y}{\mathrm{d}s},\ (y\equiv x,z,t,\psi)$$

(where we make independent variations of the variables $x, z, t, \psi, p_x, p_z, -\mathscr{H}, J, s$ and where s is the independent variable).

The required equations with s as independent variable are then obtained from the Euler equations of the variational problem (4.2):

$$\frac{\mathrm{d}}{\mathrm{d}s}x=+\frac{\partial\mathscr{K}}{\partial p_x},\quad \frac{\mathrm{d}}{\mathrm{d}s}p_x=-\frac{\partial\mathscr{K}}{\partial x}, \tag{4.3a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}z=+\frac{\partial\mathscr{K}}{\partial p_z},\quad \frac{\mathrm{d}}{\mathrm{d}s}p_z=-\frac{\partial\mathscr{K}}{\partial z}, \tag{4.3b}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}t=+\frac{\partial\mathscr{K}}{\partial(-\mathscr{H})},\quad \frac{\mathrm{d}}{\mathrm{d}s}(-\mathscr{H})=-\frac{\partial\mathscr{K}}{\partial t}, \tag{4.3c}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\psi=+\frac{\partial\mathscr{K}}{\partial J},\quad \frac{\mathrm{d}}{\mathrm{d}s}J=-\frac{\partial\mathscr{K}}{\partial\psi}, \tag{4.3d}$$

with

$$\mathscr{K}\equiv-p_s. \tag{4.4}$$

So (3.7) must be solved for p_s. To come to that, we recall that in storage rings the total energy is very much greater than the energy due to SG forces and that our Hamiltonian (2.1) is based on semiclassical quantum mechanics where terms in $\hbar$ above first order are ignored. Thus here we also only keep zeroth and first order terms in $\hbar$ and make a perturbation calculation with respect to $\hbar$ [11]. Starting with zeroth order in $\hbar$, the term $\mathbf{\Omega}_0\cdot\boldsymbol{\xi}$ in

(3.7) does not appear and $\mathscr{H}=\mathscr{H}_{\text{orb}}$. Solving for p_s and using (3.11):

$$p_{s0}=[1+K_x\cdot x+K_z\cdot z]\cdot\left\{\frac{(\mathscr{H}-e\phi)^2}{c^2}-\left(p_x-\frac{e}{c}A_x\right)^2-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2}+[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{c}A_s\,. \tag{4.5}$$

Since we are interested only in terms up to order $\hbar$ [11], we make the ansatz:

$$p_s=p_{s0}+\hbar\cdot R_s, \tag{4.6}$$

where R_s is a function of $x, z, t, \psi, p_x, p_z, -\mathscr{H}, J$ to be determined.

Because $\boldsymbol{\xi}\cdot\boldsymbol{\Omega}_0$ is already $\mathcal{O}(\hbar)$, we can, in the argument of $\boldsymbol{\Omega}_0$, make the approximation:

$$p_s\Rightarrow p_{s0}\,. \tag{4.7}$$

This simplifies the problem because p_s now only appears in the orbital part of $\mathscr{H}$ (in $\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}$ the term p_s can be replaced by p_{s0}, i.e. by the known function (4.5) of $x, z, t, p_x, p_z, -\mathscr{H}, s$). Hence (3.7) becomes an equation quadratic in p_s and we obtain:

$$p_s(x,z,t,\psi,p_x,p_z,-\mathscr{H},J)=[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{c}A_s+[1+K_x\cdot x+K_z\cdot z]\cdot\left\{\frac{(\mathscr{H}-e\phi-\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi})^2}{c^2}-\left(p_x-\frac{e}{c}A_x\right)^2-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2}. \tag{4.8}$$

This can be simplified again by neglecting terms of $\mathcal{O}(\hbar^2)$:

$$p_s=p_{s0}-\frac{1}{c^2}\cdot[1+K_x\cdot x+K_z\cdot z]^2\cdot\frac{(\mathscr{H}-e\phi)}{p_{s0}-[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{c}A_s}\cdot[\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}]. \tag{4.9}$$

The second term in (4.9) is just $\hbar\cdot R_s$ and p_s is a well defined function of $x, z, t, \psi, p_x, p_z, -\mathscr{H}, J, s$.

For the new Hamiltonian $\mathscr{K}$ we obtain from (4.4), (4.5) and (4.9):

$$\mathscr{K}(x,z,t,\psi;p_x,p_z,-\mathscr{H},J;s)=\mathscr{K}_{\text{orb}}+\mathscr{K}_{\text{spin}}, \tag{4.10}$$

with

$$\mathscr{K}_{\text{orb}}\equiv -p_{s0}=-[1+K_x\cdot x+K_z\cdot z]\cdot\left\{\frac{(\mathscr{H}-e\phi)^2}{c^2}-\left(p_x-\frac{e}{c}A_x\right)^2-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2}-[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{c}A_s, \tag{4.11a}$$

$$\mathscr{K}_{\text{spin}}=[\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}]\cdot\frac{1}{c^2}[1+K_x\cdot x+K_z\cdot z]\cdot\frac{\mathscr{H}-e\phi}{\left(\frac{p_{s0}}{[1+K_x\cdot x+K_z\cdot z]}-\frac{e}{c}A_s\right)}=[\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}]\cdot\frac{[1+K_x\cdot x+K_z\cdot z]\cdot(\mathscr{H}-e\phi)}{c^2\cdot\left\{\frac{(\mathscr{H}-e\phi)^2}{c^2}-\left(p_x-\frac{e}{c}A_x\right)^2-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2}}. \tag{4.11b}$$

Note, that the factor after the quantity $[\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}]$ in (4.11b) is, apart from terms which only contribute to $\mathcal{O}(\hbar^2)$, just $(1/\dot{s})$, since we obtain from (3.11):

$$\dot{s}=\frac{\partial\mathscr{H}}{\partial p_s}=\frac{\partial\mathscr{H}_{\text{orb}}}{\partial p_s}+\mathcal{O}(\hbar)=\frac{c^2}{(\mathscr{H}-e\phi)}\cdot\frac{1}{[1+K_x\cdot x+K_z\cdot z]}\cdot\left(\frac{p_2}{[1+K_x\cdot x+K_z\cdot z]}-\frac{e}{c}A_s\right)+\mathcal{O}(\hbar). \tag{4.12}$$

This result could also have been obtained by much simpler means as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}=\boldsymbol{\Omega}_0\times\boldsymbol{\xi}\Rightarrow\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\xi}=\frac{1}{\dot{s}}\boldsymbol{\Omega}_0\times\boldsymbol{\xi}, \tag{4.13}$$

but we wanted to obtain it within a Hamiltonian formalism.

Thus, setting

$$p_t\equiv -\mathscr{H},$$

we have with (4.10–11) the Hamiltonian for the canonical variables

$$x, z, t, \psi;\ p_x, p_z, p_t, J,$$

and the arc length, s, of the design orbit acts as the independent variable.

We repeat that in a semiclassical treatment it is sufficient to evaluate $\boldsymbol{\Omega}_0$ using the substitution (4.7).

In the remaining part of this chapter we perform some further canonical transformations of the variables $x, z, t, \psi; p_x, p_z, p_t, J$ in order to prepare for the next chapters.

In the following we choose a gauge in which

$$\phi=0.$$

Then from (2.4) and (3.11) we obtain:

$$\mathscr{H}=\mathscr{H}_{\text{orb}}+\mathcal{O}(\hbar)=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}+\mathcal{O}(\hbar)=E+\mathcal{O}(\hbar),$$

($E\equiv\mathscr{H}_{\text{orb}}=$ the orbital energy of the particle)

and thus

$$p_t+E=\mathcal{O}(\hbar).$$

(Note that $v^2\equiv\mathbf{v}\cdot\mathbf{v}$ and $\mathbf{v}\equiv\mathrm{d}\mathbf{r}/\mathrm{d}t$.)

In order to describe the energy oscillations we use the design energy, E_0, to introduce the (small) quantity

$$\tilde{p}_t = p_t + E_0 \equiv -(E - E_0) + \mathcal{O}(\hbar) \equiv -\Delta E + \mathcal{O}(\hbar) \tag{4.14}$$

as a new (canonical) variable:

$$t, p_t \to \tilde{t}, \tilde{p}_t. \tag{4.15}$$

This transformation can be obtained using the generating function

$$F_2(t, \tilde{p}_t) = t \cdot (\tilde{p}_t - E_0). \tag{4.16}$$

The transformation equations read as:

$$p_t = \frac{\partial F_2}{\partial t} = \tilde{p}_t - E_0, \tag{4.17a}$$

$$\tilde{t} = \frac{\partial F_2}{\partial \tilde{p}_t} = t, \tag{4.17b}$$

$$\mathcal{H} \to \mathcal{H} + \frac{\partial F_2}{\partial s} = \mathcal{H}, \tag{4.17c}$$

whereby (4.17a) reproduces the defining equation (4.14) for $\tilde{p}_t$.

Finally, since the variable t increases without limit, it is more useful to introduce the variable

$$\sigma = s - v_0 \cdot t, \tag{4.18}$$

with

$$v_0 = \text{design speed} = c\beta_0; \quad \beta_0 = \sqrt{1 - \left(\frac{m_0 c^2}{E_0}\right)^2},$$

which describes the delay in arrival time at position s of a particle:

$$t, \tilde{p}_t \to \sigma, p_\sigma \tag{4.19}$$

(σ describes the longitudinal separation of the particle from the centre of the bunch.)

This point transformation can also be made by a canonical transformation (see Sect. 3.2). The generating function is

$$F_3(\tilde{p}_t, \sigma; s) = -\frac{1}{v_0} \tilde{p}_t \cdot (s - \sigma). \tag{4.20}$$

From this follows:

$$t = -\frac{\partial F_3}{\partial \tilde{p}_t} = \frac{1}{v_0} \cdot (s - \sigma), \tag{4.21a}$$

$$p_\sigma = -\frac{\partial F_3}{\partial \sigma} = -\frac{1}{v_0} \tilde{p}_t \equiv \frac{E - E_0}{v_0} + \mathcal{O}(\hbar) = \frac{\Delta E}{v_0} + \mathcal{O}(\hbar), \tag{4.21b}$$

and

$$\mathcal{H} \to \bar{\mathcal{H}}(x, z, \sigma, \psi; p_x, p_z, p_\sigma, J; s) = \mathcal{H} + \frac{\partial F_3}{\partial s} = \mathcal{H} + p_\sigma = \bar{K}_{\text{orb}} + \tilde{\mathbf{\Omega}}_0 \cdot \boldsymbol{\xi}, \tag{4.22}$$

with

$$\begin{aligned} \bar{\mathcal{H}}_{\text{orb}} = p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \\ \cdot \left\{ \beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c} A_x\right)^2 \right. \\ \left. - \left(p_z - \frac{e}{c} A_z\right)^2 - m_0^2 c^2 \right\}^{1/2} \\ - [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{e}{c} A_s, \end{aligned} \tag{4.23a}$$

$$\tilde{\mathbf{\Omega}}_0 = \mathbf{\Omega}_0 \cdot \frac{[1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0 \left(p_\sigma + \frac{E_0}{v_0}\right)}{c \cdot \left\{ \beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c} A_x\right)^2 - \left(p_z - \frac{e}{c} A_z\right)^2 - m_0^2 c^2 \right\}^{1/2}}. \tag{4.23b}$$

With (4.22–23) we have the Hamiltonian for the canonical variables

$$x, z, \sigma, \psi; p_x, p_z, p_\sigma, J.$$

In order to utilize the new Hamiltonian (4.22), the magnetic field $\mathcal{B}$ and the corresponding vector potential,

$$\mathbf{A} = \mathbf{A}(x, y, \sigma; s), \tag{4.24}$$

for commonly occurring types of accelerator magnet and for cavities must be given. Once $\mathbf{A}$ is known, the fields $\mathcal{E}$ and $\mathcal{B}$ can be found using (2.8a, b). In the variables x, z, s, σ these become (with $\phi = 0$):

$$\mathcal{E} = \beta_0 \cdot \frac{\partial}{\partial \sigma} \mathbf{A} \tag{4.25}$$

and

$$\mathcal{B}_x = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial z} [(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] - \frac{\partial}{\partial s} A_z \right\}, \tag{4.26a}$$

$$\mathcal{B}_z = \frac{1}{(1 + K_x \cdot x + K_z \cdot z)} \cdot \left\{ \frac{\partial}{\partial s} A_x - \frac{\partial}{\partial x} [(1 + K_x \cdot x + K_z \cdot z) \cdot A_s] \right\}, \tag{4.26b}$$

$$\mathcal{B}_s = \frac{\partial}{\partial x} A_z - \frac{\partial}{\partial z} A_x. \tag{4.26c}$$

In Appendix A the vector potential $\mathbf{A}$ is calculated for various types of lenses.

In the following we assume that the ring consists of bending magnets, quadrupoles, skew quadrupoles, solenoids, cavities and dipole correction coils. Then the vector potential $\mathbf{A}$ can be written as (see Appendix A):

$$\frac{e}{E_0} A_s = -\frac{1}{2}\beta_0 \cdot (1 + K_x \cdot x + K_z \cdot z) + \frac{1}{2} g \cdot \beta_0 \cdot (z^2 - x^2) + N \cdot \beta_0 \cdot xz - \frac{1}{\beta_0} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] + \frac{e}{E_0}[\Delta\mathcal{B}_x \cdot z - \Delta\mathcal{B}_z \cdot x], \tag{4.27}$$

(h = harmonic number) with

$$\Delta\mathcal{B}_x = \sum_\mu \Delta\hat{\mathcal{B}}_x^{(\mu)} \cdot \delta(s - s_\mu), \tag{4.28a}$$

$$\Delta\mathcal{B}_z = \sum_\mu \Delta\hat{\mathcal{B}}_z^{(\mu)} \cdot \delta(s - s_\mu), \tag{4.28b}$$

(dipole correction field in x- and z-direction), and

$$\frac{e}{E_0} A_x = -\beta_0 \cdot H \cdot z \,, \ \frac{e}{E_0} A_z = +\beta_0 \cdot H \cdot x \tag{4.29}$$

whereby the following abbreviations have been used:

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial \mathcal{B}_z}{\partial x}\right)_{x=z=0}, \tag{4.30a}$$

$$N = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial \mathcal{B}_x}{\partial x} - \frac{\partial \mathcal{B}_z}{\partial z}\right)_{x=z=0}, \tag{4.30b}$$

$$H = \frac{1}{2} \cdot \frac{e}{p_0 \cdot c} \cdot \mathcal{B}_s(0, 0, s), \tag{4.30c}$$

$$K_x = +\frac{e}{p_0 \cdot c} \cdot \mathcal{B}_z(0, 0, s), \ K_z = -\frac{e}{p_0 \cdot c} \cdot \mathcal{B}_x(0, 0, s), \tag{4.30d}$$

(p_0 = momentum corresponding to energy E_0).
In detail, one has:

a) $g \neq 0$, $N = K_x = K_z = H = V = 0$: quadrupole,
b) $N \neq 0$, $g = K_x = K_z = H = V = 0$: skew quadrupole,
c) $K_x^2 + K_z^2 \neq 0$, $g = N = H = V = 0$: bending magnet,
d) $H \neq 0$, $g = N = K_x = K_z = V = 0$: solenoid,
e) $V \neq 0$, $g = K_x = K_z = N = H = 0$: cavity.

Furthermore, for the magnetic field $\mathcal{B}$ we get (see Appendix A):

$$\frac{e}{E_0}\mathcal{B}_x = \beta_0\left[-K_z + \frac{e}{p_0 \cdot c}\Delta\mathcal{B}_x + (N - H') \cdot x + g \cdot z\right], \tag{4.31a}$$

$$\frac{e}{E_0}\mathcal{B}_z = \beta_0\left[+K_x + \frac{e}{p_0 \cdot c}\Delta\mathcal{B}_z - (N + H') \cdot z + g \cdot x\right], \tag{4.31b}$$

$$\frac{e}{E_0}\mathcal{B}_s = \beta_0 \cdot 2H, \tag{4.31c}$$

and for the electric field $\mathcal{E}$ we have:

$$\mathcal{E}_s = V(s)\sin\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] = V(s)\sin\varphi + \sigma(s) \cdot h \cdot \frac{2\pi}{L} \cdot V(s)\cos\varphi + \cdots, \tag{4.32a}$$

$$\mathcal{E}_x = \mathcal{E}_z = 0. \tag{4.32b}$$

Although $(x, z, \sigma, \psi; p_x, p_z, p_\sigma, J)$ are canonical variables, it is still useful to introduce the new quantities

$$\eta = \frac{v_0}{E_0} \cdot p_\sigma \equiv \frac{\Delta E}{E_0} + \mathcal{O}(\hbar), \tag{4.33}$$

and

$$\hat{\eta} = \frac{1}{\beta_0}\sqrt{(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2} - 1 = \frac{1}{\beta_0} \cdot \frac{p \cdot c}{E_0} - 1 + \mathcal{O}(\hbar) = \frac{p}{p_0} - 1 + \mathcal{O}(\hbar) = \frac{\Delta p}{p_0} + \mathcal{O}(\hbar), \tag{4.34}$$

where

$$p \equiv \frac{1}{c}\sqrt{E^2 - m_0^2 c^4}$$

= momentum corresponding to energy E,

$$p_0 \equiv \frac{1}{c}\sqrt{E_0^2 - m_0^2 c^4}$$

= momentum corresponding to energy E_0,

$$\Delta p \equiv p - p_0.$$

Then for the term

$$W \equiv \left\{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c}A_x\right)^2 - \left(p_z - \frac{e}{c}A_z\right)^2 - m_0^2 c^2\right\}^{1/2}, \tag{4.35a}$$

appearing in (4.23) we have:

$$W = \sqrt{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2 c^2} \cdot \left\{1 - \frac{\left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_z - \frac{e}{c}A_z\right)^2}{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2 c^2}\right\}^{1/2}$$

$$= \beta_0 \cdot \frac{E_0}{v_0} \cdot \sqrt{\left(\frac{v_0}{E_0}p_\sigma + 1\right)^2 - m_0^2 c^2 \cdot \frac{v_0^2}{E_0^2} \cdot \frac{1}{\beta_0^2}} \cdot \left\{1 - \left(\frac{E_0}{v_0}\right)^2 \cdot \frac{\left(\frac{v_0}{E_0}p_x - \frac{e}{E_0}\beta_0 \cdot A_x\right)^2 + \left(\frac{v_0}{E_0}p_z - \frac{e}{E_0}\beta_0 \cdot A_z\right)^2}{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - m_0^2 c^2}\right\}^{1/2}$$

$$=\frac{E_0}{c}\cdot\sqrt{(1+\eta)^2-\left(\frac{m_0c^2}{E_0}\right)^2}\left\{1-\left(\frac{E_0}{v_0}\right)^2\cdot\frac{\left(\frac{v_0}{E_0}p_x-\frac{e}{E_0}\beta_0\cdot A_x\right)^2+\left(\frac{v_0}{E_0}p_z-\frac{e}{E_0}\beta_0\cdot A_z\right)^2}{\beta_0^2\cdot\left(p_\sigma+\frac{E_0}{v_0}\right)^2-m_0^2c^2}\right\}^{1/2}$$

$$=\frac{E_0}{c}\cdot\beta_0\cdot(1+\hat{\eta})\left\{1-\left(\frac{E_0}{v_0}\right)^2\cdot\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\left[\frac{E_0}{c}\cdot\beta_0\cdot(1+\hat{\eta})\right]^2}\right\}^{1/2}$$

$$=\frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat{\eta})\cdot\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}. \tag{4.35b}$$

Thus, putting (4.27), (4.29), and (4.35b) into (4.23a), we obtain for the orbital part $\bar{K}_{orb}$ of the Hamiltonian:

$$\begin{aligned}\bar{\mathscr{K}}_{orb}=&\frac{E_0}{v_0}\cdot\frac{v_0}{E_0}p_\sigma-\frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\\&\cdot\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}\\&-[1+K_x\cdot x+K_z\cdot z]\cdot\frac{E_0}{c}\\&\cdot\left\{-\frac{1}{2}\beta_0\cdot(1+K_x\cdot x+K_z\cdot z)\right.\\&\quad+\frac{1}{2}g\cdot\beta_0\cdot(z^2-x^2)\;+N\cdot\beta_0\cdot xz.\\&\quad-\frac{1}{\beta_0}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\\&\quad\left.+\frac{e}{p_0\cdot c}\cdot\frac{p_0\cdot c}{E_0}\cdot[\varDelta\mathscr{B}_x\cdot z-\varDelta\mathscr{B}_z\cdot x]\right\},\end{aligned}$$

or

$$\begin{aligned}\frac{v_0}{E_0}\cdot\bar{\mathscr{K}}_{orb}=&\eta-\beta_0^2\cdot(1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\\&\cdot\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}\\&-[1+K_x\cdot x+K_z\cdot z]\cdot\beta_0^2\\&\cdot\left\{-\frac{1}{2}\cdot(1+K_x\cdot x+K_z\cdot z)\right.\\&\quad+\frac{1}{2}g\cdot(z^2-x^2)+N\cdot xz\\&\quad-\frac{1}{\beta_0^2}\cdot\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]\\&\quad\left.+\frac{e}{p_0\cdot c}\cdot[\varDelta\mathscr{B}_x\cdot z-\varDelta\mathscr{B}_z\cdot x]\right\}.\end{aligned} \tag{4.36}$$

The vector $\mathbf{\Omega}_0$ in (4.23b):

$$\mathbf{\Omega}_0=\Omega_{0s}\cdot\mathbf{e}_s+\Omega_{0x}\cdot\mathbf{e}_x+\Omega_{0z}\cdot\mathbf{e}_z$$

(see (3.12) and (3.13)) as a function of the variables $(x, z, \sigma, p_x, p_z, p_\sigma, s)$ now takes the form:

$$\begin{aligned}\frac{1}{c}\cdot\mathbf{\Omega}_{0s}=&-\frac{E_0}{m_0c^2}\left[\left(\frac{1}{\gamma}+a\right)\cdot\frac{e}{E_0}\mathscr{B}_s-\frac{aE_0^2}{\gamma(\gamma+1)\cdot m_0^2c^4\cdot\beta_0^2}\right.\\&\cdot\left(\frac{v_0}{E_0}\pi_s\cdot\frac{e}{E_0}\mathscr{B}_s+\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\mathscr{B}_x\right.\\&\left.\left.+\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\mathscr{B}_z\right)\frac{v_0}{E_0}\pi_s\right]\\=&-\gamma_0\left[\left(\frac{1}{\gamma}+a\right)\cdot\frac{e}{E_0}\mathscr{B}_s-\frac{a\gamma_0^2}{\gamma(\gamma+1)}\cdot\frac{1}{\beta_0^2}\right.\\&\cdot\left(\frac{v_0}{E_0}\pi_s\cdot\frac{e}{E_0}\mathscr{B}_s+\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\mathscr{B}_x\right.\\&\left.\left.+\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\mathscr{B}_z\right)\frac{v_0}{E_0}\pi_s\right],\end{aligned} \tag{4.37a}$$

$$\begin{aligned}\frac{1}{c}\cdot\mathbf{\Omega}_{0x}=&-\gamma_0\left[\left(\frac{1}{\gamma}+a\right)\cdot\frac{e}{E_0}\mathscr{B}_x-\frac{a\gamma_0^2}{\gamma(\gamma+1)}\cdot\frac{1}{\beta_0^2}\right.\\&\cdot\left(\frac{v_0}{E_0}\pi_s\cdot\frac{e}{E_0}\mathscr{B}_s+\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\mathscr{B}_x\right.\\&\left.+\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\mathscr{B}_z\right)\frac{v_0}{E_0}\pi_x\\&\left.-\frac{\gamma_0}{\gamma}\cdot\frac{1}{\beta_0}\left(a+\frac{1}{1+\gamma}\right)\cdot\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\mathscr{E}_s\right],\end{aligned} \tag{4.37b}$$

$$\frac{1}{c}\cdot\mathbf{\Omega}_{0z}=-\gamma_0\Bigg[\left(\frac{1}{\gamma}+a\right)\cdot\frac{e}{E_0}\mathscr{B}_z-\frac{a\gamma_0^2}{\gamma(\gamma+1)}\cdot\frac{1}{\beta_0^2}$$
$$\cdot\left(\frac{v_0}{E_0}\pi_s\cdot\frac{e}{E_0}\mathscr{B}_s+\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\mathscr{B}_x\right.$$
$$\left.+\frac{v_0}{E_0}\pi_z\cdot\frac{e}{E_0}\mathscr{B}_z\right)\frac{v_0}{E_0}\pi_z$$
$$+\frac{\gamma_0}{\gamma}\cdot\frac{1}{\beta_0}\left(a+\frac{1}{1+\gamma}\right)\cdot\frac{v_0}{E_0}\pi_x\cdot\frac{e}{E_0}\mathscr{E}_s\Bigg], \tag{4.37c}$$

whereby the fields $\mathscr{B}_s$, $\mathscr{B}_x$, $\mathscr{B}_z$ and $\mathscr{E}_s$ are taken from (4.31) and (4.32) and the term γ_0 is defined by

$$\gamma_0=\frac{E_0}{m_0c^2}. \tag{4.38}$$

For the Lorentz factor γ appearing in (4.37) one has:

$$\gamma=\frac{E}{m_0c^2}=\frac{E_0}{m_0c^2}\cdot\frac{v_0}{E_0}\cdot\left[p_\sigma+\frac{E_0}{v_0}\right]+\mathcal{O}(\hbar)$$
$$=\gamma_0\cdot\left(1+\eta\right)+\mathcal{O}(\hbar), \tag{4.39}$$

and for the quantity π_s we have ((3.9c), (4.5) and (4.6)):

$$\pi_s=\left\{\beta_0^2\cdot\left(p_\sigma+\frac{E_0}{v_0}\right)^2-\left(p_x-\frac{e}{c}A_x\right)^2\right.$$
$$\left.-\left(p_z-\frac{e}{c}A_z\right)^2-m_0^2c^2\right\}^{1/2}\equiv W$$
$$=\frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat{\eta})$$
$$\cdot\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}$$
$$=\frac{E_0}{v_0}\cdot\beta_0^2\cdot(1+\hat{\eta})$$
$$\cdot\left\{1-\frac{\left(\frac{v_0}{E_0}\pi_x\right)^2+\left(\frac{v_0}{E_0}\pi_z\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}, \tag{4.40}$$

(see (4.35b)) with

$$\frac{v_0}{E_0}\pi_x=\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz, \tag{4.41a}$$

$$\frac{v_0}{E_0}\pi_z=\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx. \tag{4.41b}$$

With (4.23b), (4.36–37), (4.39–41) we have rewritten the Hamiltonian for the canonical variables

$x, z, \sigma, \psi; p_x, p_z, p_\sigma, J$

in a more convenient form by replacing the terms in p_σ by the equivalent quantities η, $\hat{\eta}$.

Remark: Equation (4.36) is valid only for protons. For electrons one needs the extra-term in the Hamiltonian

$$\mathscr{H}_{\mathrm{rad}}=\frac{E_0}{v_0}\cdot C_1\cdot[K_x^2+K_z^2]\cdot\sigma, \tag{4.42}$$

$$\left(\text{where } C_1=\frac{2}{3}e^2\frac{\gamma_0^4}{E_0}\right)$$

(for $v_0\approx c$) in order to describe the energy loss by radiation in the bending magnets [10, 30]. In this case, the cavity phase φ in (4.32) is determined by the need to replace the energy radiated in the bending magnets. Thus:

$$\underbrace{\int_{s_0}^{s_0+L}\mathrm{d}s\cdot eV(s)\cdot\sin\varphi}$$

average energy uptake in the cavities;

$$=\underbrace{\int_{s_0}^{s_0+L}\mathrm{d}s\cdot E_0\cdot C_1\cdot[K_x^2+K_z^2].} \tag{4.43}$$

average energy loss due to radiation.

Note, that the $\mathscr{H}_{\mathrm{rad}}$ term only accounts for the average energy loss. Deviations from this average due to stochastic radiation effects and damping introduce non-symplectic terms into the equation of motion.

For proton storage rings, where radiation effects can be neglected, one has:

$$\sin\varphi=0\Rightarrow\varphi=0,\pi \tag{4.44}$$

(no average energy gain in the cavities) and the choice for φ is determined by the stability condition for synchrotron motion [2]

$$\begin{cases}\varphi=0 & \text{above "transition";}\\ \varphi=\pi & \text{below "transition".}\end{cases}$$

5 Spin motion in terms of the orthonormal dreibein ($\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_z$); canonical spin transformation

In this chapter we show how to describe the motion of the spin with respect to the ($\mathbf{e}_x, \mathbf{e}_z, \mathbf{e}_s$)-basis. The variables $x, z, \sigma; p_x, p_z, p_\sigma$ need no further transformation.

5.1 A new spin Hamiltonian

The transformation of the spin from the ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$)-basis to the ($\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_z$)- basis

$$\xi_1,\xi_2,\xi_3\Rightarrow\xi_s,\xi_x,\xi_z, \tag{5.1}$$

is merely a rotation and is defined by:

$$\boldsymbol{\xi}=\xi_1\cdot\mathbf{e}_1+\xi_2\cdot\mathbf{e}_2+\xi_3\cdot\mathbf{e}_3=\xi_s\cdot\mathbf{e}_s+\xi_x\cdot\mathbf{e}_x+\xi_z\cdot\mathbf{e}_z. \tag{5.2}$$

If, by analogy to (2.2), we introduce canonical variables ψ', J' for ξ_s, ξ_x, ξ_z:

$$\begin{cases}\xi_s=\sqrt{\xi^2-J'^2}\cdot\cos\psi',\\ \xi_x=\sqrt{\xi^2-J'^2}\cdot\sin\psi',\\ \xi_z=J',\end{cases} \tag{5.3}$$

then (5.2) becomes a canonical transformation:

$$x, z, \sigma, \psi, p_x, p_z, p_\sigma, J \Rightarrow x'=x, z'=z, \sigma'=\sigma, \psi', p'_x = p_x, p'_z=p_z, p'_\sigma=p_\sigma, J'. \quad (5.4)$$

Following Yokoya, who uses a Lie transform, the new Hamiltonian $\bar{\mathscr{H}}$ is [11]:

$$\bar{\mathscr{H}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s) = \bar{\mathscr{H}}_{\text{orb}}(x, z, \sigma; p_x, p_z, p_\sigma; s) + \sum_{\nu=1}^{3} [\tilde{\mathbf{\Omega}}_0(x, z, \sigma; p_x, p_z, p_\sigma; s) - \mathbf{U}(x, z, \sigma; p_x, p_z, p_\sigma; s)] \cdot \mathbf{u}_\nu(s)\, \xi'_\nu, \quad (5.5)$$

where

$$\mathbf{u}_1 \equiv \mathbf{e}_s, \quad \xi'_1 \equiv \xi_s, \quad (5.6a)$$

$$\mathbf{u}_2 \equiv \mathbf{e}_x, \quad \xi'_2 \equiv \xi_x, \quad (5.6b)$$

$$\mathbf{u}_3 \equiv \mathbf{e}_z, \quad \xi'_3 \equiv \xi_z, \quad (5.6c)$$

and

$$\mathbf{U} = \frac{1}{2} \sum_{\nu=1}^{3} \mathbf{u}_\nu \times \frac{\mathrm{d}\mathbf{u}_\nu}{\mathrm{d}s}. \quad (5.7)$$

From (3.3) we have:

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{u}_1(s) = -K_x(s) \cdot \mathbf{u}_2(s) - K_z(s) \cdot \mathbf{u}_3(s), \quad (5.8a)$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{u}_2(s) = +K_x(s) \cdot \mathbf{u}_1(s), \quad (5.8b)$$

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathbf{u}_3(s) = +K_z(s) \cdot \mathbf{u}_1(s). \quad (5.8c)$$

Putting (5.8) into (5.7) we obtain:

$$\begin{aligned}\mathbf{U} &= \frac{1}{2} \{\mathbf{e}_s \times [-K_x(s) \cdot \mathbf{e}_x(s) - K_z(s) \cdot \mathbf{e}_z(s)] \\ &\quad + \mathbf{e}_x \times (K_x \mathbf{e}_s) + \mathbf{e}_z \times (K_z \mathbf{e}_s)\} \\ &= -K_x \cdot \mathbf{e}_z + K_z \cdot \mathbf{e}_x,\end{aligned}$$

and it follows that:

$$\bar{\mathscr{H}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s) = \bar{\mathscr{H}}_{\text{orb}}(x, z, \sigma; p_x, p_z, p_\sigma; s) + \mathbf{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) \cdot (\xi_s \cdot \mathbf{e}_s + \xi_x \cdot \mathbf{e}_x + \xi_z \cdot \mathbf{e}_z), \quad (5.9)$$

with

$$\begin{aligned}\frac{v_0}{E_0} \cdot \bar{\mathscr{H}}_{\text{orb}} &= \frac{v_0}{E_0} \cdot \bar{\mathscr{H}}_{\text{orb}} = \eta - \beta_0^2 \cdot (1+\hat{\eta}) \cdot \left[1 + K_x \cdot x + K_z \cdot z\right] \\ &\quad \cdot \left\{1 - \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 \cdot (1+\hat{\eta})^2}\right\}^{1/2} \\ &\quad - [1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0^2 \left\{-\frac{1}{2} \cdot (1 + K_x \cdot x + K_z \cdot z) \right. \\ &\quad + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz - \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \\ &\quad \left. \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] + \frac{e}{p_0 \cdot c} \cdot [\Delta\mathscr{B}_x \cdot z - \Delta\mathscr{B}_z \cdot x]\right\} \\ &\quad + \frac{v_0}{E_0} \mathscr{H}_{\text{rad}}, \end{aligned} \quad (5.10)$$

and

$$\begin{aligned}\tilde{\mathbf{\Omega}}(x, z, \sigma; p_x, p_z, p_\sigma; s) &= \tilde{\mathbf{\Omega}} + K_x \cdot \mathbf{e}_z - K_z \cdot \mathbf{e}_x \\ &= \frac{[1 + K_x \cdot x + K_z \cdot z] \cdot \beta_0 \left(p_\sigma + \frac{E_0}{v_0}\right)}{c \cdot \left\{\beta_0^2 \cdot \left(p_\sigma + \frac{E_0}{v_0}\right)^2 - \left(p_x - \frac{e}{c} A_x\right)^2 - \left(p_z - \frac{e}{c} A_z\right)^2 - m_0^2 c^2\right\}^{1/2}} \\ &\quad \cdot \mathbf{\Omega}_0 + K_x \cdot \mathbf{e}_z - K_z \cdot \mathbf{e}_x \\ &= [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{(1+\eta)}{\beta_0 (1+\hat{\eta})} \\ &\quad \cdot \left\{1 - \frac{\left(\frac{v_0}{E_0} p_x + \beta_0^2 \cdot Hz\right)^2 + \left(\frac{v_0}{E_0} p_z - \beta_0^2 \cdot Hx\right)^2}{\beta_0^4 (1+\hat{\eta})^2}\right\}^{-1/2} \\ &\quad \cdot \frac{1}{c} \mathbf{\Omega}_0 + K_x \cdot \mathbf{e}_z - K_z \cdot \mathbf{e}_x. \end{aligned} \quad (5.11)$$

With (5.9–11) we have the Hamiltonian for the canonical variables

$$x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'.$$

Remark: The equation for spin motion corresponding to the Hamiltonian (4.22) reads as:

$$\frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{\xi} = \tilde{\mathbf{\Omega}}_0 \times \boldsymbol{\xi}. \quad (5.12)$$

Representing the spin vector $\boldsymbol{\xi}$ in the form

$$\boldsymbol{\xi} = \xi_s \cdot \mathbf{e}_s + \xi_x \cdot \mathbf{e}_x + \xi_z \cdot \mathbf{e}_z, \quad (5.13)$$

and using (3.3) we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} \boldsymbol{\xi} &= \xi'_s \cdot \mathbf{e}_s + \xi'_x \cdot \mathbf{e}_x + \xi'_z \cdot \mathbf{e}_z + \xi_x \cdot \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{e}_x + \xi_s \cdot \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{e}_s + \xi_z \cdot \frac{\mathrm{d}}{\mathrm{d}s} \mathbf{e}_z \\ &= \xi'_s \cdot \mathbf{e}_s + \xi'_x \cdot \mathbf{e}_x + \xi'_z \cdot \mathbf{e}_z - \xi_s \cdot (K_x \cdot \mathbf{e}_x + K_x \cdot \mathbf{e}_z) \\ &\quad + \xi_x \cdot K_x \mathbf{e}_s + \xi_z \cdot K_z \mathbf{e}_s \\ &= \xi'_s \cdot \mathbf{e}_s + \xi'_x \cdot \mathbf{e}_x + \xi'_z \cdot \mathbf{e}_z - \boldsymbol{\xi} \\ &\quad \times (K_z \cdot \mathbf{e}_x - K_x \cdot \mathbf{e}_z). \end{aligned} \quad (5.14)$$

Thus (5.12) can be rewritten as:

$$\mathbf{e}_s \cdot \frac{\mathrm{d}}{\mathrm{d}s} \xi_s + \mathbf{e}_x \cdot \frac{\mathrm{d}}{\mathrm{d}s} \xi_x + \mathbf{e}_z \cdot \frac{\mathrm{d}}{\mathrm{d}s} \xi_z = \mathbf{\Omega} \times \boldsymbol{\xi}, \quad (5.15)$$

with $\mathbf{\Omega}$ given by (5.11) which confirms the validity of the spin part $\mathbf{\Omega}\cdot\boldsymbol{\xi}$ in the Hamiltonian $\bar{\bar{\mathscr{H}}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s)$.

If the new spin basis had been an explicit function of the canonical orbital variables, then even at first order in $\hbar$ the orbital variables and the orbital Hamiltonian would have been modified by the canonical transformation (see (3.16), (3.17), (3.24) in [11]). However, at this stage in our treatment, the azimuthal variable, s, is the independent parameter, not a canonical variable. Therefore the variables $x, z, \sigma, p_x, p_z, p_\sigma$ remain unmodified by the transformation and $\bar{\bar{\mathscr{H}}}_{\mathrm{orb}}$ and $\bar{\mathscr{H}}_{\mathrm{orb}}$ are identical. Furthermore, since $\bar{\bar{\mathscr{H}}}$ and $\bar{\mathscr{H}}$ differ only by the term $(K_x\cdot\mathbf{e}_z - K_z\cdot\mathbf{e}_x)\cdot\boldsymbol{\xi}$ which is independent of the variables $x, z, \sigma, p_x, p_z, p_\sigma$, the Hamiltonians $\bar{\bar{\mathscr{H}}}$ and $\bar{\mathscr{H}}$ lead to the same equations of orbital motion.

5.2 *Series expansion of the Hamiltonian*

Since

$$\left|\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right| \ll 1,$$

the square root

$$\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2}$$

in (5.10) and (5.11) may be expanded in a series:

$$\left\{1-\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2+\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}\right\}^{1/2} = 1-\frac{1}{2}\cdot\frac{\left(\frac{v_0}{E_0}p_x+\beta_0^2\cdot Hz\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}-\frac{1}{2}\cdot\frac{\left(\frac{v_0}{E_0}p_z-\beta_0^2\cdot Hx\right)^2}{\beta_0^4\cdot(1+\hat{\eta})^2}+\cdots, \tag{5.16}$$

and the same can be done with the term

$$\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right],$$

resulting from the cavity field:

$$\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma+\varphi\right]=\frac{L}{2\pi\cdot h}\cdot\frac{eV(s)}{E_0}\cos\varphi - \sigma\cdot\frac{eV(s)}{E_0}\sin\varphi-\frac{1}{2}\sigma^2\cdot h\cdot\frac{2\pi}{L}\cdot\frac{eV(s)}{E_0}\cos\varphi+\cdots. \tag{5.17}$$

Furthermore, for the term

$$\frac{1}{1+\gamma},$$

appearing in (4.37) we may write:

$$\frac{1}{1+\gamma}=\frac{1}{(1+\gamma_0)+\gamma_0\cdot\eta}+\mathcal{O}(\hbar) = \frac{1}{1+\gamma_0}\cdot\left[1-\frac{\gamma_0}{1+\gamma_0}\cdot\eta\right]+\cdots+\mathcal{O}(\hbar), \tag{5.18a}$$

and for the quantity

$$\hat{\eta}=f(\eta)$$

one obtains from (4.34):

$$\hat{\eta}\equiv f(\eta) = f(0)+f'(0)\cdot\eta+f''(0)\cdot\frac{1}{2}\eta^2+\cdots = \frac{1}{\beta_0^2}\cdot\eta-\frac{1}{\beta_0^4\cdot\gamma_0^2}\cdot\frac{1}{2}\eta^2+\cdots, \tag{5.18b}$$

so that in practice the spin-orbit motion can be conveniently calculated to various orders of approximation in the orbit variables.

If we wish to obtain a symplectic linearised treatment of synchro-betatron motion (including SG effects) we expand the Hamiltonian up to second order in the orbit variables. Then from (5.10) and (5.11) we obtain:

a) For the orbital part $\bar{\bar{\mathscr{H}}}_{\mathrm{orb}}$ of the Hamiltonian:

$$\bar{\bar{\mathscr{H}}}_{\mathrm{orb}}=\mathscr{H}_0+\mathscr{H}_1, \tag{5.19}$$

where $\mathscr{H}_0$ and $\mathscr{H}_1$ are given by:

$$\begin{aligned}\frac{v_0}{E_0}\cdot\mathscr{H}_0 &= \frac{1}{2}\cdot\frac{1}{\gamma_0^2-1}\cdot\eta^2-[K_x\cdot x+K_z\cdot z]\cdot\eta \\ &+\frac{1}{2\beta_0^2}\cdot\left\{\left[\frac{v_0}{E_0}p_x+\beta_0^2H\cdot z\right]^2+\left[\frac{v_0}{E_0}p_z-\beta_0^2H\cdot x\right]^2\right\} \\ &+\frac{1}{2}\beta_0^2\cdot\{(K_x^2+g)\cdot x^2+(K_z^2-g)\cdot z^2-2N\cdot xz\} \\ &-\frac{1}{2}\sigma^2\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi,\end{aligned} \tag{5.20a}$$

$$\frac{v_0}{E_0}\cdot\mathscr{H}_1=-\sigma\cdot\left[\frac{eV}{E_0}\sin\varphi-C_1\cdot(K_x^2+K_z^2)\right] -\beta_0^2\cdot\frac{e}{p_0\cdot c}\left[\Delta\mathscr{B}_x\cdot z-\Delta\mathscr{B}_z\cdot x\right] \tag{5.20b}$$

(constant terms, $(L/2\pi h)\cdot(eV/E_0)\cdot\cos\varphi$ and $(-\beta_0^2/2)$, in the Hamiltonian, which have no influence on the motion have been dropped).

b) For the spin part $\bar{\bar{\mathscr{H}}}_{\mathrm{spin}}$ of the Hamiltonian:

$$\bar{\bar{\mathscr{H}}}_{\mathrm{spin}}=\mathbf{\Omega}\cdot\boldsymbol{\xi}, \tag{5.21}$$

with

$$\mathbf{\Omega}_s = -2H\cdot(1+a) + 2H\cdot(1+a)\cdot\frac{1}{\beta_0^2}\cdot\eta$$

$$-\frac{v_0}{E_0}p_x\cdot\frac{a\gamma_0^2}{1+\gamma_0}\left[K_z - \frac{e}{p_0\cdot c}\cdot\Delta\mathscr{B}_x\right]$$

$$+\frac{v_0}{E_0}p_z\cdot\frac{a\gamma_0^2}{1+\gamma_0}\left[K_x + \frac{e}{p_0\cdot c}\cdot\Delta\mathscr{B}_z\right], \tag{5.22a}$$

$$\mathbf{\Omega}_x = K_z\cdot a\gamma_0 - (1+a\gamma_0)\cdot\frac{e}{p_0\cdot c}\Delta\mathscr{B}_x$$

$$-(1+a\gamma_0)\cdot[(N-H')\cdot x - (K_z^2 - g)\cdot z]$$

$$+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}p_x + \beta_0^2\cdot Hz\right]$$

$$+\frac{1}{\beta_0^2}\cdot\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}p_z$$

$$-\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_z - \frac{e}{p_0 c}\Delta\mathscr{B}_x\right)\cdot\eta, \tag{5.22b}$$

$$\mathbf{\Omega}_z = -K_x\cdot a\gamma_0 - (1+a\gamma_0)\cdot\frac{e}{p_0\cdot c}\Delta\mathscr{B}_z$$

$$+(1+a\gamma_0)\cdot[(N+H')\cdot z - (K_x^2 + g)\cdot x]$$

$$+\frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}p_z - \beta_0^2\cdot Hx\right]$$

$$-\frac{1}{\beta_0^2}\cdot\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}p_x$$

$$+\left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_x + \frac{e}{p_0\cdot c}\Delta\mathscr{B}_z\right)\cdot\eta, \tag{5.22c}$$

(no solenoid field in the bending magnets and in the cavities$\Rightarrow K_x\cdot H = K_z\cdot H = 0;\ V\cdot H = 0$).

6 Introduction of an eight-dimensional closed orbit and a new pair of canonical variables for spin

As can be seen from (5.19–5.22), the series expansion for $\mathscr{H}_{\text{orb}}$ contains terms linear in the orbital coordinates and $\mathbf{\Omega}$ contains terms independent of the orbital coordinates. These and the linear terms can be eliminated by introducing a new 8-dimensional reference orbit. This orbit can then be used to construct a new reference frame for the spin motion and, as we show below, it is then possible to introduce new variables to describe the spin which are canonical and are related to the spin variables used by Chao [9].

6.1 Definition of the eight-dimensional closed orbit

We begin by defining the 8-dimensional closed orbit:

$$(\mathbf{y}_0(s), J_0(s), \psi_0(s)),$$

containing a periodic orbital part

$$\mathbf{y}_0^T = (x_0, p_{x0};\ z_0, p_{z0};\ \sigma_0, p_{\sigma 0}),$$

with

$$\mathbf{y}_0(s+L) = \mathbf{y}_0(s), \tag{6.1a}$$

and a spin part $J_0(s), \psi_0(s)$ which defines (see (5.3)) a periodic spin vector

$$\boldsymbol{\xi}_0(s) = \xi_{0s}\cdot\mathbf{e}_s + \xi_{0x}\cdot\mathbf{e}_x + \xi_{0z}\cdot\mathbf{e}_z,$$

with

$$\boldsymbol{\xi}_0(s+L) = \boldsymbol{\xi}_0(s), \tag{6.1b}$$

whereby the equations of motion read as:

$$\frac{d}{ds}\mathbf{y}_0 = -\underline{S}\cdot\frac{\partial}{\partial\mathbf{y}_0}\bar{\mathscr{H}}(\mathbf{y}_0;\psi_0, J_0; s), \tag{6.2a}$$

$$\mathbf{e}_s\cdot\frac{d}{ds}\xi_{0s} + \mathbf{e}_x\cdot\frac{d}{ds}\xi_{0x} + \mathbf{e}_z\cdot\frac{d}{ds}\xi_{0z} = \mathbf{\Omega}^{(0)}\times\boldsymbol{\xi}_0, \tag{6.2b}$$

with

$$\mathbf{\Omega}^{(0)} \equiv \mathbf{\Omega}(\mathbf{y}_0, s), \tag{6.3}$$

and

$$\underline{S} = \begin{pmatrix} \underline{S_2} & \underline{0} & \underline{0} \\ \underline{0} & \underline{S_2} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S_2} \end{pmatrix};\quad \underline{S_2} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \tag{6.4}$$

i.e. $(\mathbf{y}_0(s), J_0(s), \psi_0(s))$ is a periodic solution of the combined equations of motion.

Using $\boldsymbol{\xi}_0$ we can now construct a periodic spin frame $(\mathbf{n}_0, \mathbf{m}, \mathbf{l})$ along the closed orbit (see Appendix B):

$$[\mathbf{n}_0(s+L), \mathbf{m}(s+L), \mathbf{l}(s+L)] = [\mathbf{n}_0(s), \mathbf{m}(s), \mathbf{l}(s)],$$

with

$$\mathbf{n}_0 = \boldsymbol{\xi}_0/|\boldsymbol{\xi}_0|, \tag{6.5a}$$

$$\mathbf{n}_0(s)\perp\mathbf{m}(s)\perp\mathbf{l}(s), \tag{6.5b}$$

$$\mathbf{n}_0(s) = \mathbf{m}(s)\times\mathbf{l}(s), \tag{6.5c}$$

$$|\mathbf{n}_0(s)| = |\mathbf{m}(s)| = |\mathbf{l}(s)| = 1, \tag{6.5d}$$

and

$$\frac{d}{ds}\mathbf{n}_0(s) = \mathbf{\Omega}^{(0)}\times\mathbf{n}_0(s), \tag{6.6a}$$

$$\frac{d}{ds}\mathbf{m}(s) = \mathbf{\Omega}^{(0)}\times\mathbf{m}(s) + \mathbf{l}(s)\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{6.6b}$$

$$\frac{d}{ds}\mathbf{l}(s) = \mathbf{\Omega}^{(0)}\times\mathbf{l}(s) - \mathbf{m}(s)\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{6.6c}$$

$$\psi_{\text{spin}}(s+L) - \psi_{\text{spin}}(s) = 2\pi\cdot Q_{\text{spin}}. \tag{6.7}$$

6.2 Canonical transformations

The 8-dimensional closed orbit together with $\mathbf{l}(s)$, $\mathbf{m}(s)$ will now be used to construct new canonical spin-orbit variables. The canonical transformation for orbit and spin will be carried out separately.

6.2.1 Canonical transformation for the spin variables. To derive the new spin-Hamiltonian, we proceed in two steps:

1) *Canonical spin-transformation*:
Firstly we follow the method of Sect. (5.1) to transform from the $\mathbf{e}_x$, $\mathbf{e}_z$, $\mathbf{e}_s$ basis to the $\mathbf{n}_0$, $\mathbf{m}$, $\mathbf{l}$ basis:

$$\xi_s, \xi_x, \xi_z \Rightarrow \xi_n, \xi_m, \xi_l, \tag{6.8}$$

with

$$\boldsymbol{\xi} = \xi_s \cdot \mathbf{e}_s + \xi_x \cdot \mathbf{e}_x + \xi_z \cdot \mathbf{e}_z = \xi_n \cdot n_0 + \xi_m \cdot \mathbf{m} + \xi_l \cdot \mathbf{l}. \tag{6.9}$$

Introducing for ξ_n, ξ_m, ξ_l canonical variables ψ'', J'':

$$\begin{cases} \xi_m = \sqrt{\xi^2 - (J'')^2} \cdot \cos\psi'', \\ \xi_l = \sqrt{\xi^2 - (J'')^2} \cdot \sin\psi'', \\ \xi_n = J'', \end{cases} \tag{6.10}$$

(6.9) becomes a canonical transformation:

$$\psi', J' \Rightarrow \psi'', J'', \tag{6.11}$$

and the new Hamiltonian $\tilde{\mathscr{H}}$ reads as:

$$\begin{aligned} &\tilde{\mathscr{H}}(x, z, \sigma, \psi'; p_x, p_z, p_\sigma, J'; s) \\ &\quad = \bar{\bar{\mathscr{H}}}_{\text{orb}}(x, z, \sigma; p_x, p_z, p_\sigma; s) + \tilde{\mathscr{H}}_{\text{spin}}, \end{aligned} \tag{6.12}$$

with

$$\begin{aligned} &\tilde{\mathscr{H}}_{\text{spin}}(x, z, \sigma, \psi''; p_x, p_z, p_\sigma, J''; s) \\ &\quad = \{\boldsymbol{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) - \mathbf{U}'(x, z, \sigma; p_x, p_z, p_\sigma; s)\} \\ &\quad \cdot \left(\xi_n \cdot \mathbf{n}_0 + \xi_m \cdot \mathbf{m} + \xi_l \cdot \mathbf{l}\right), \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} \mathbf{U}' &= \frac{1}{2}\left[\mathbf{n}_0 \times \frac{d\mathbf{n}_0}{ds} + \mathbf{m} \times \frac{d\mathbf{m}}{ds} + \mathbf{l} \times \frac{d\mathbf{l}}{ds}\right] \\ &= \frac{1}{2}\left[\mathbf{n}_0 \times (\boldsymbol{\Omega}^{(0)} \times \mathbf{n}_0) + \mathbf{m} \times \left(\boldsymbol{\Omega}^{(0)} \times \mathbf{m} + \mathbf{l} \cdot \frac{d}{ds}\psi_{\text{spin}}(s)\right)\right. \\ &\quad \left. + \mathbf{l} \times \left(\boldsymbol{\Omega}^{(0)} \times \mathbf{l} - \mathbf{m} \cdot \frac{d}{ds}\psi_{\text{spin}}(s)\right)\right] \\ &= \frac{1}{2}\left[3\boldsymbol{\Omega}^{(0)} - \mathbf{n}_0 \cdot (\boldsymbol{\Omega}^{(0)} \cdot \mathbf{n}_0) - \mathbf{m} \cdot (\boldsymbol{\Omega}^{(0)} \cdot \mathbf{m}) - \mathbf{l} \cdot (\boldsymbol{\Omega}^{(0)} \cdot \mathbf{l})\right. \\ &\quad \left. + (\mathbf{n}_0 + \mathbf{n}_0) \cdot \frac{d}{ds}\psi_{\text{spin}}(s)\right] \\ &= \frac{1}{2}\left[3\boldsymbol{\Omega}^{(0)} - \boldsymbol{\Omega}^{(0)} + 2\mathbf{n}_0 \cdot \frac{d}{ds}\psi_{\text{spin}}(s)\right] \\ &= \boldsymbol{\Omega}^{(0)} + \mathbf{n}_0 \cdot \frac{d}{ds}\psi_{\text{spin}}(s). \end{aligned} \tag{6.14}$$

Thus we find:

$$\begin{aligned} &\tilde{\mathscr{H}}_{\text{spin}}(x, z, \sigma, \psi''; p_x, p_z, p_\sigma, J''; s) \\ &\quad = \left\{\boldsymbol{\Omega} - \boldsymbol{\Omega}^{(0)} - \mathbf{n}_0 \cdot \frac{d}{ds}\psi_{\text{spin}}(s)\right\} \cdot \left(\xi_n \cdot \mathbf{n}_0 + \xi_m \cdot \mathbf{m} + \xi_l \cdot \mathbf{l}\right) \\ &\quad = \boldsymbol{\omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) \\ &\qquad \times [\xi_n \cdot \mathbf{n}_0 + \xi_m \cdot \mathbf{m} + \xi_l \cdot \mathbf{l}] - \xi_n \cdot \frac{d}{ds}\psi_{\text{spin}}(s) \\ &\quad = [\xi_n \cdot (\mathbf{n}_0 \cdot \boldsymbol{\omega}) + \xi_m \cdot (\mathbf{m} \cdot \boldsymbol{\omega}) + \xi_l \cdot (\mathbf{l} \cdot \boldsymbol{\omega})] - \xi_n \cdot \frac{d}{ds}\psi_{\text{spin}}(s) \\ &\quad = (\xi_n, \xi_m, \xi_l) \cdot \begin{pmatrix} n_{0s}(s) & n_{0x}(s) & n_{0z}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \cdot \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \\ &\qquad - \xi_n \cdot \frac{d}{ds}\psi_{\text{spin}}(s), \end{aligned} \tag{6.15}$$

where we have introduced for abbreviation the vector

$$\boldsymbol{\omega} = \boldsymbol{\Omega} - \boldsymbol{\Omega}^{(0)}. \tag{6.16}$$

This is equivalent to the form for the spin Hamiltonian given by Derbenev [5].

Writing:

$$\boldsymbol{\omega} = \omega_s \cdot \mathbf{e}_s + \omega_x \cdot \mathbf{e}_x + \omega_z \cdot \mathbf{e}_z, \tag{6.17a}$$

and using (5.22) to expand $\boldsymbol{\omega}$ as a Taylor series in

$$\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0, \tag{6.17b}$$

the first order terms for the components ω_s, ω_x, ω_z of the vector $\boldsymbol{\omega}$ are:

$$\begin{aligned} \omega_s = &+2H \cdot (1+a) \cdot \frac{1}{\beta_0^2} \cdot \tilde{\eta} \\ &- \frac{v_0}{E_0}\tilde{p}_x \cdot \frac{a\gamma_0^2}{1+\gamma_0}\left[K_z - \frac{e}{p_0 \cdot c} \cdot \Delta\mathscr{B}_x\right] \\ &+ \frac{v_0}{E_0}\tilde{p}_z \cdot \frac{a\gamma_0^2}{1+\gamma_0}\left[K_x + \frac{e}{p_0 \cdot c} \cdot \Delta\mathscr{B}_z\right], \end{aligned} \tag{6.18a}$$

$$\begin{aligned} \omega_x = &-(1+a\gamma_0) \cdot [(N-H') \cdot \tilde{x} - (K_z^2 - g) \cdot \tilde{z}] \\ &+ \frac{a\gamma_0^2}{1+\gamma_0} \cdot 2H \cdot \left[\frac{v_0}{E_0}\tilde{p}_x + \beta_0^2 \cdot H\tilde{z}\right] \\ &+ \frac{1}{\beta_0^2} \cdot \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right] \cdot \frac{eV(s)}{E_0}\sin\varphi \cdot \frac{v_0}{E_0}\tilde{p}_z \\ &- \left[1 + \frac{1+a\gamma_0}{\gamma_0^2 - 1}\right] \cdot \left(K_z - \frac{e}{p_0 \cdot c}\Delta\mathscr{B}_x\right) \cdot \tilde{\eta}, \end{aligned} \tag{6.18b}$$

$$\omega_z = +(1+a\gamma_0)\cdot[(N+H')\cdot\tilde{z}-(K_x^2+g)\cdot\tilde{x}] + \frac{a\gamma_0^2}{1+\gamma_0}\cdot 2H\cdot\left[\frac{v_0}{E_0}\tilde{p}_z+\beta_0^2\cdot H\tilde{x}\right] - \frac{1}{\beta_0^2}\cdot\left[a\gamma_0+\frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{eV(s)}{E_0}\sin\varphi\cdot\frac{v_0}{E_0}\tilde{p}_x + \left[1+\frac{1+a\gamma_0}{\gamma_0^2-1}\right]\cdot\left(K_x+\frac{e}{p_0\cdot c}\Delta\mathscr{B}_z\right)\cdot\tilde{\eta}, \tag{6.18c}$$

with (see (4.33))

$$\tilde{\eta}=\frac{v_0}{E_0}\cdot\tilde{p}_\sigma. \tag{6.19}$$

In this paper only this linearised version of $\boldsymbol{\omega}$ will be used. This is sufficient for many purposes [9, 22]. The full $\boldsymbol{\omega}$ could be used if required.

With (5.10), (6.12), (6.15) we have the Hamiltonian (up to second order in the orbital varibles) for the canonical varibles

$x, z, \sigma, \psi''; p_x, p_z, p_\sigma, J''$.

2) *Introduction of a new pair of canonical spin variables*:

We now introduce the spin variables (α, β) defined by:

$$\alpha=\sqrt{2\cdot(\xi-J'')}\cdot\cos\psi'', \tag{6.20a}$$

$$\beta=\sqrt{2\cdot(\xi-J'')}\cdot\sin\psi''. \tag{6.20b}$$

From this definition we have:

$$\frac{\beta}{\alpha}=\tan\psi'', \tag{6.21a}$$

$$J''=\xi-\tfrac{1}{2}(\alpha^2+\beta^2), \tag{6.21b}$$

and

$$\xi_n=\xi-\tfrac{1}{2}(\alpha^2+\beta^2), \tag{6.22a}$$

$$\xi_m=\frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}, \tag{6.22b}$$

$$\xi_l=\frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}. \tag{6.22c}$$

The latter can be inverted to give:

$$\alpha=+\sqrt{\frac{2}{\xi+\xi_n}}\cdot\xi_m, \tag{6.23a}$$

$$\beta=+\sqrt{\frac{2}{\xi+\xi_n}}\cdot\xi_l. \tag{6.23b}$$

The transformation

$\psi'', J''\Rightarrow\alpha, \beta$

can be obtained from the generating function:

$$F_1(\alpha,\psi'')=\tfrac{1}{2}\alpha^2\cdot\tan\psi''-\xi\cdot\psi''. \tag{6.24}$$

$$+\frac{\partial F_1}{\partial\alpha}=\alpha\cdot\tan\psi''=\beta, \tag{6.25a}$$

$$-\frac{\partial F_1}{\partial\psi''}=-\frac{1}{2}\alpha^2\cdot(1+\tan^2\psi'')=-\frac{1}{2}\alpha^2\cdot\left(1+\frac{\beta^2}{\alpha^2}\right) = -\frac{1}{2}(\alpha^2+\beta^2)+\xi=J'', \tag{6.25b}$$

$$\tilde{\mathscr{H}}\to\tilde{\mathscr{H}}+\frac{\partial F_1}{\partial s}=\tilde{\mathscr{H}}=\bar{\tilde{\mathscr{H}}}_{\text{orb}}+\tilde{\mathscr{H}}_{\text{spin}}. \tag{6.25c}$$

Thus α, β are canonical variables.

From (6.15) and (6.22) we obtain:

$$\tilde{\mathscr{H}}_{\text{spin}}=\left(\xi-\frac{1}{2}(\alpha^2+\beta^2),\quad \frac{1}{\sqrt{2}}\cdot\alpha\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)},\quad \frac{1}{\sqrt{2}}\cdot\beta\cdot\sqrt{2\xi-\frac{1}{2}(\alpha^2+\beta^2)}\right) \cdot\begin{pmatrix} n_{0s}(s) & n_{0x}(s) & n_{0z}(s)\\ m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} -\left[\xi-\frac{1}{2}(\alpha^2+\beta^2)\right]\cdot\frac{d}{ds}\psi_{\text{spin}}(s). \tag{6.26}$$

With (5.10), (6.25c), (6.26) we have the Hamiltonian for the canonical variables

$x, z, \sigma, \alpha; p_x, p_z, p_\sigma, \beta$.

Remarks:

1) The values of α and β are restricted by the condition:

$\alpha^2+\beta^2\le 4\xi\Rightarrow\xi\ge\xi_n\ge-\xi$.

2) For

$\alpha^2+\beta^2<4\xi$,

the correspondence between α, β and ξ_n, ξ_m, ξ_l is one-one.

3) For

$\left|\frac{\alpha}{\sqrt{\xi}}\right|\ll 1;\quad\left|\frac{\beta}{\sqrt{\xi}}\right|\ll 1,$

we have:

$\xi_m\approx\alpha\cdot\sqrt{\xi},$
$\xi_l\approx\beta\cdot\sqrt{\xi},$

and in this case our canonical α and β behave like the spin-coordinates introduced by Chao in the SLIM-program [9].

4) For the Poisson-bracket*:

$$\{\alpha, \beta\}_{\psi'', J''} \equiv \frac{\partial\alpha}{\partial\psi''}\cdot\frac{\partial\beta}{\partial J''} - \frac{\partial\alpha}{\partial J''}\cdot\frac{\partial\beta}{\partial\psi''}$$

we obtain from (6.20):

$$\{\alpha, \beta\}_{\psi'', J''} = [-\sqrt{2\cdot(\xi - J'')}\cdot\sin\psi''] \cdot\frac{-2}{2\cdot\sqrt{2\cdot(\xi-J'')}}\cdot\sin\psi'' - \frac{-2}{2\cdot\sqrt{2\cdot(\xi-J'')}}\cdot\cos\psi'' \cdot[+\sqrt{2\cdot(\xi-J'')}\cdot\cos\psi''] = 1.$$

This relation demonstrates again that α and β are canonical variables.

5) The variables α and β could already have been introduced at the beginning in the starting Hamiltonian (2.1). They completely repalce ψ and J. See Paper II in this series [19].

6.2.2 Transformation of the oribital variables. The orbit vector $\mathbf{y}(s)$ can be separated into two components (see (6.17b)):

$$\mathbf{y}(s) = \mathbf{y}_0(s) + \tilde{\mathbf{y}}(s), \tag{6.27}$$

where the vector $\tilde{\mathbf{y}}(s)$ describes the synchro-betatron oscillations about the new closed equilibrium trajectory $\tilde{\mathbf{y}}_0(s)$.

The transformation

$$\mathbf{y};\ \alpha, \beta \Rightarrow \tilde{\mathbf{y}};\ \tilde{\alpha} = \alpha,\ \tilde{\beta} = \beta, \tag{6.28}$$

can be obtained from the generating function:

$$F_2(x, \tilde{p}_x; z, \tilde{p}_z; \sigma, \tilde{p}_\sigma; \alpha, \tilde{\beta}; s) = (x - x_0)\cdot(\tilde{p}_x + p_{x0}) + (z - z_0)\cdot(\tilde{p}_z + p_{z0}) + (\sigma - \sigma_0)\cdot(\tilde{p}_\sigma + p_{\sigma 0}) + \alpha\cdot\tilde{\beta} + f(s), \tag{6.29}$$

with an arbitrary function $f(s)$. The transformation equations read as:

$$p_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + p_{x0}, \quad \tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - x_0, \tag{6.30a}$$

$$p_z = \frac{\partial F_2}{\partial z_z} = \tilde{p}_x + p_{z0}, \quad \tilde{z} = \frac{\partial F_2}{\partial \tilde{p}_z} = z - z_0, \tag{6.30b}$$

$$p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma + p_{\sigma 0}, \quad \tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} = \sigma - \sigma_0, \tag{6.30c}$$

which reproduce the defining equation (6.27) for $\tilde{\mathbf{y}}$.

Furthermore we have $\left(\text{with } \frac{d}{ds} f(s) = x_0(s)\cdot\frac{d}{ds}p_{x0}(s) + z_0(s)\cdot\frac{d}{ds}p_{z0}(s) + \sigma_0(s)\cdot\frac{d}{ds}(s)\right)$:

$$\begin{aligned}\frac{\partial F_2}{\partial s} &= -\frac{dx_0}{ds}\cdot p_x + \frac{dp_{x0}}{ds}\cdot x - \frac{dz_0}{ds}\cdot p_z + \frac{dp_{z0}}{ds}\cdot z - \frac{d\sigma_0}{ds}\cdot p_\sigma + \frac{dp_{\sigma 0}}{ds}\cdot\sigma \\ &= -p_x\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial p_x}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} - x\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial x}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} \\ &= -p_z\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial p_z}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} - z\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial z}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} \\ &= -p_\sigma\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial p_\sigma}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} - \sigma\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial \sigma}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} \\ &= -\mathbf{y}\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial \mathbf{y}_x}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0},\end{aligned} \tag{6.31}$$

and therefore

$$\begin{aligned}\tilde{\mathscr{H}} \equiv \bar{\mathscr{H}}_{orb} + \tilde{\mathscr{H}}_{spin} \rightarrow \hat{\mathscr{H}} &= \tilde{\mathscr{H}} - \mathbf{y}\cdot\left(\frac{\partial\tilde{\mathscr{H}}}{\partial\mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} \\ &= \bar{\mathscr{H}}_{orb} - \mathbf{y}\cdot\left(\frac{\partial\bar{\mathscr{H}}_{orb}}{\partial\mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} + \tilde{\mathscr{H}}_{spin} - \mathbf{y}\cdot\left(\frac{\partial\tilde{\mathscr{H}}_{spin}}{\partial\mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0} \\ &= \hat{\mathscr{H}}_{orbit} + \hat{\mathscr{H}}_{spin},\end{aligned} \tag{6.32}$$

with

$$\hat{\mathscr{H}}_{orbit} = \bar{\mathscr{H}}_{orb} - \mathbf{y}\cdot\left(\frac{\partial\bar{\mathscr{H}}_{orb}}{\partial\mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0}, \tag{6.33a}$$

$$\hat{\mathscr{H}}_{spin} = \tilde{\mathscr{H}}_{spin} - \mathbf{y}\cdot\left(\frac{\partial\tilde{\mathscr{H}}_{spin}}{\partial\mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\ \alpha=\beta=0}. \tag{6.33b}$$

---

* It would be more correct to write:

$$\{\alpha, \beta\}_{y, p_y} \equiv \frac{\partial\alpha}{\partial x}\cdot\frac{\partial\beta}{\partial p_x} - \frac{\partial\alpha}{\partial p_x}\cdot\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial z}\cdot\frac{\partial\beta}{\partial p_z} - \frac{\partial\alpha}{\partial p_z}\cdot\frac{\partial\beta}{\partial z} + \frac{\partial\alpha}{\partial\sigma}\cdot\frac{\partial\beta}{\partial p_\sigma} - \frac{\partial\alpha}{\partial p_\sigma}\cdot\frac{\partial\beta}{\partial\sigma} + \frac{\partial\alpha}{\partial\psi''}\cdot\frac{\partial\beta}{\partial J''} - \frac{\partial\alpha}{\partial J''}\cdot\frac{\partial\beta}{\partial\psi''},$$

with y denoting (x, z, σ, ψ'') and $p_{\psi''} = J''$. Using the fact that α and β are independent of $(x, p_x, z, p_z, \sigma, p_\sigma)$, we obtain the result:

$$\{\alpha, \beta\}_{y, p_y} \equiv \frac{\partial\alpha}{\partial\psi''}\cdot\frac{\partial\beta}{\partial J''} - \frac{\partial\alpha}{\partial J''}\cdot\frac{\partial\beta}{\partial\psi''}.$$

This notation was already used in (2.10). Equation (2.17) could also be written in terms of Poisson brackets

For the linearised form of $\boldsymbol{\omega}$ (see (6.18)), (6.26) and (6.32) lead to:

$$\hat{\mathscr{H}}_{\text{spin}}(\tilde{x}, \tilde{z}, \tilde{\sigma}, \alpha; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \beta; s)$$

$$= \left(\xi - \frac{1}{2}(\alpha^2+\beta^2), \frac{1}{\sqrt{2}} \cdot \alpha \cdot \sqrt{2\xi - \frac{1}{2}(\alpha^2+\beta^2)}, \frac{1}{\sqrt{2}} \cdot \beta \cdot \sqrt{2\xi - \frac{1}{2}(\alpha^2+\beta^2)} \right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(x) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{1}{\sqrt{2}}(\alpha^2+\beta^2) \cdot \frac{d}{ds} \psi_{\text{spin}}(s)$$

$$-(\xi, 0, 0) \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$= \left(-\frac{1}{2}(\alpha^2+\beta^2), \frac{1}{\sqrt{2}} \cdot \alpha \cdot \sqrt{2\xi - \frac{1}{2}(\alpha^2+\beta^2)}, \frac{1}{\sqrt{2}} \cdot \beta \cdot \sqrt{2\xi - \frac{1}{2}(\alpha^2+\beta^2)} \right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{1}{\sqrt{2}}(\alpha^2+\beta^2) \cdot \frac{d}{ds} \psi_{\text{spin}}(s)$$

$$= \xi \cdot \left(-\frac{1}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right], \frac{\alpha}{\sqrt{\xi}} \cdot \sqrt{1 - \frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right]}, \frac{\beta}{\sqrt{\xi}} \cdot \sqrt{1 - \frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right]} \right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{\xi}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right] \cdot \frac{d}{ds} \psi_{\text{spin}}(s), \tag{6.34a}$$

and at second order the orbital Hamiltonian $\hat{\mathscr{H}}_{\text{orb}}$ takes the form, using (5.19) and (5.20):

$$\frac{v_0}{E_0} \cdot \hat{\mathscr{H}}_{\text{orb}}(\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, s)$$

$$= \frac{1}{2} \cdot \frac{1}{\gamma_0^2 - 1} \cdot \tilde{\eta}^2 - [K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \cdot \tilde{\eta}$$

$$+ \frac{1}{2\beta_0^2} \cdot \left\{ \left[\frac{v_0}{E_0} \tilde{p}_x + \beta_0^2 H \cdot \tilde{z} \right]^2 + \left[\frac{v_0}{E_0} \tilde{p}_z - \beta_0^2 H \cdot \tilde{x} \right]^2 \right\}$$

$$+ \frac{1}{2}\beta_0^2 \cdot \{ (K_x^2 + g) \cdot \tilde{x}^2 + (K_z^2 - g) \cdot \tilde{z}^2 - 2N \cdot \tilde{x}\tilde{z} \}$$

$$- \frac{1}{2}\tilde{\sigma}^2 \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos \varphi, \tag{6.34b}$$

(the constant terms

$$\left(-\xi \cdot \frac{d}{ds} \psi_{\text{spin}}(s) \right) \text{ and } \bar{\mathscr{H}}_{\text{orb}}(x_0, z_0, \sigma_0; p_{x0}, p_{z0}, p_{\sigma 0}; s)$$

in the Hamiltonian (6.34), which have no influence on the motion, have been neglected).

With (6.32), (6.34) we have the Hamiltonian for the canonical variables

$$\tilde{x}, \tilde{z}, \tilde{\sigma}, \alpha; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \beta,$$

and the canonical equations for spin-orbit motion are:

$$\frac{d}{ds}\tilde{x} = +\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{p}_x}, \quad \frac{d}{ds}\tilde{p}_x = -\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{x}}, \tag{6.35a}$$

$$\frac{d}{ds}\tilde{z} = +\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{p}_z}, \quad \frac{d}{ds}\tilde{p}_z = -\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{z}}, \tag{6.35b}$$

$$\frac{d}{ds}\tilde{\sigma} = +\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{p}_\sigma}, \quad \frac{d}{ds}\tilde{p}_\sigma = -\frac{\partial \hat{\mathscr{H}}}{\partial \tilde{\sigma}}, \tag{6.35c}$$

$$\frac{d}{ds}\alpha = +\frac{\partial \hat{\mathscr{H}}}{\partial \beta}, \quad \frac{d}{ds}\beta = -\frac{\partial \hat{\mathscr{H}}}{\partial \alpha}. \tag{6.35d}$$

As in (5.16) for the orbital motion, we can expand the square root

$$\sqrt{1 - \frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right]},$$

appearing in the spin-Hamiltonian (6.34a) in a series:

$$\sqrt{1 - \frac{1}{4}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right]} = 1 - \frac{1}{8}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2 \right] + \cdots, \tag{6.36}$$

so that the spin motion can be conveniently calculated to various orders of approximation.

If $\boldsymbol{\xi}$ is sufficiently parallel to $\mathbf{n}_0$ (i.e. $\alpha/\sqrt{\xi}$ and $\beta/\sqrt{\xi}$ are small)* an expression to linear order suffices and the Hamiltonian (6.34a) becomes:

$$\hat{\mathscr{K}}_{\text{spin}}(\tilde{x}, \tilde{z}, \tilde{\sigma}, \alpha; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \beta; s)$$

$$= \sqrt{\xi}\cdot(\alpha, \beta)\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix}\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} + \frac{\xi}{2}\left[\left(\frac{\alpha}{\sqrt{\xi}}\right)^2 + \left(\frac{\beta}{\sqrt{\xi}}\right)^2\right]\cdot\frac{\mathrm{d}}{\mathrm{d}s}\psi_{\text{spin}}(s), \tag{6.37}$$

and the corresponding canonical equations for α and β read:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\alpha}{\sqrt{\xi}}\right) = +(l_s(s), l_x(s), l_z(s))\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} + \left(\frac{\beta}{\sqrt{\xi}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\psi_{\text{spin}}(s), \tag{6.38a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\beta}{\sqrt{\xi}}\right) = -(m_s(s), m_x(s), m_z(s))\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} - \left(\frac{\alpha}{\sqrt{\xi}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\psi_{\text{spin}}(s). \tag{6.38b}$$

In this form (6.38) are the basic equations for spin motion used in the computer program SLIM [9, 10]. We have thus derived the SLIM-formalism from canonical equations based on a polynomial expansion of a spin Hamiltonian.

6.2.3 Scale transformation. In order to eliminate the factors (v_0/E_0) and β_0^2 appearing in the Hamiltonian (6.34), we define new relative variables:

$$\hat{x} \equiv \tilde{x}, \quad \hat{p}_x \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_x = \frac{\tilde{p}_x}{p_0}, \tag{6.39a}$$

$$\hat{z} \equiv \tilde{z}, \quad \hat{p}_z \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_z = \frac{\tilde{p}_z}{p_0}, \tag{6.39b}$$

$$\hat{\sigma} \equiv \tilde{\sigma}, \quad \hat{p}_\sigma \equiv \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\tilde{p}_\sigma \equiv \frac{1}{\beta_0^2}\tilde{\eta}, \tag{6.39c}$$

$$\hat{\alpha} \equiv \frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\alpha, \quad \hat{\beta} \equiv \frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\beta \tag{6.39d}$$

($\tilde{x}, \tilde{z}, \tilde{\sigma}$ unchanged).

Note that (6.39) is a combination of a scale transformation $\left(\text{using the scale factor } \frac{1}{\beta_0^2}\frac{v_0}{E_0}\right)$ and a canonical (point-) transformation (involving α, β only).

* This is the case that occurs for example in calculations of electron polarization far from spin orbit resonances [9, 22, 31]

Furthermore, the linearised vector (6.18)

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

in the spin-Hamiltonian (6.34a) can be written as:

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3\times 6)}\cdot\begin{pmatrix} \hat{x} \\ \hat{p}_x \\ \hat{z} \\ \hat{p}_z \\ \hat{\sigma} \\ \hat{p}_\sigma \end{pmatrix}, \tag{6.40}$$

with

$$F_{12} = -a(\gamma_0 - 1)\cdot\left[K_z - \frac{e}{p_0\cdot c}\cdot\Delta\mathscr{B}_x\right],$$

$$F_{14} = +a(\gamma_0 - 1)\cdot\left[K_x + \frac{e}{p_0\cdot c}\cdot\Delta\mathscr{B}_z\right],$$

$$F_{16} = +2H\cdot(1+a),$$

$$F_{21} = -(1+a\gamma_0)\cdot(N - H'),$$

$$F_{22} = +a(\gamma_0 - 1)\cdot 2H,$$

$$F_{23} = +(1+a\gamma_0)\cdot(K_z^2 - g) + \frac{a\gamma_0^2}{1+\gamma_0}\cdot 2\beta_0^2\cdot H^2,$$

$$F_{24} = \left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{e}{E_0}V(s)\sin\varphi,$$

$$F_{26} = -\left[1 + \frac{a}{\gamma_0}\right]\cdot\left(K_z - \frac{e}{p_0\cdot c}\Delta\mathscr{B}_x\right),$$

$$F_{31} = -(1+a\gamma_0)\cdot(K_x^2 + g) - \frac{a\gamma_0^2}{1+\gamma_0}\cdot 2\beta_0^2\cdot H^2,$$

$$F_{32} = -\left[a\gamma_0 + \frac{\gamma_0}{1+\gamma_0}\right]\cdot\frac{e}{E_0}V(s)\sin\varphi,$$

$$F_{33} = +(1+a\gamma_0)\cdot(N + H'),$$

$$F_{34} = +a(\gamma_0 - 1)\cdot 2H,$$

$$F_{36} = +\left[1 + \frac{a}{\gamma_0}\right]\cdot\left(K_x + \frac{e}{p_0\cdot c}\Delta\mathscr{B}_z\right),$$

$$F_{ik} = 0 \text{ otherwise.} \tag{6.41}$$

Introducing now

$$\hat{\mathscr{H}} = \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\hat{\mathscr{K}} = \hat{\mathscr{H}}_{\text{orb}} + \hat{\mathscr{H}}_{\text{spin}}, \tag{6.42}$$

with

$$\hat{\mathscr{H}}_{\text{orb}} = \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\hat{\mathscr{K}}_{\text{orb}}, \tag{6.43a}$$

$$\hat{\mathscr{H}}_{\text{spin}} = \frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\hat{\mathscr{K}}_{\text{spin}}, \tag{6.43b}$$

we can rewrite the canonical equations (6.35) in the form:

$$\frac{d}{ds}\hat{x}=+\frac{\partial\hat{\mathcal{H}}}{\partial\hat{p}_x},\quad \frac{d}{ds}\hat{p}_x=-\frac{\partial\hat{\mathcal{H}}}{\partial\hat{x}}, \tag{6.44a}$$

$$\frac{d}{ds}\hat{z}=+\frac{\partial\hat{\mathcal{H}}}{\partial\hat{p}_z},\quad \frac{d}{ds}\hat{p}_z=-\frac{\partial\hat{\mathcal{H}}}{\partial\hat{z}}, \tag{6.44b}$$

$$\frac{d}{ds}\hat{\sigma}=+\frac{\partial\hat{\mathcal{H}}}{\partial\hat{p}_\sigma},\quad \frac{d}{ds}\hat{p}_\sigma=-\frac{\partial\hat{\mathcal{H}}}{\partial\hat{\sigma}}, \tag{6.44c}$$

$$\frac{d}{ds}\hat{\alpha}=+\frac{\partial\hat{\mathcal{H}}}{\partial\hat{\beta}},\quad \frac{d}{ds}\hat{\beta}=-\frac{\partial\hat{\mathcal{H}}}{\partial\hat{\alpha}}, \tag{6.44d}$$

so that $\hat{\mathcal{H}}$ is the Hamiltonian for the canonical variables

$\hat{x}, \hat{z}, \hat{\sigma}, \hat{\alpha}; \hat{p}_x, \hat{p}_z, \hat{p}_\sigma, \hat{\beta}$.

By expanding the Hamiltonian $\hat{\mathcal{H}}$ in a power series in these variables, we can calculate spin-orbit motion in the required order of approximation and be sure that the equations of motion are symplectic.

To obtain linearised equations of motion we use (6.34b) and (6.37):

$$\begin{aligned}
&\hat{\mathcal{H}}_{\text{orb}}(\hat{x}, \hat{z}, \hat{\sigma}; \hat{p}_x, \hat{p}_z, \hat{p}_\sigma; s)\\
&\quad=\frac{1}{2}\cdot\frac{1}{\gamma_0^2}\cdot\hat{p}_\sigma^2-[K_x\cdot\hat{x}+K_z\cdot\hat{z}]\cdot\hat{p}_\sigma\\
&\quad+\frac{1}{2}\cdot\{[\hat{p}_x+H\cdot\hat{z}]^2+[\hat{p}_z-H\cdot\hat{x}]^2\}\\
&\quad+\frac{1}{2}\cdot\{G_1\cdot\hat{x}^2+G_2\cdot\hat{z}^2-2N\cdot\hat{x}\hat{z}\}\\
&\quad-\frac{1}{2}\hat{\sigma}^2\cdot\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi,
\end{aligned} \tag{6.45a}$$

$$\begin{aligned}
&\hat{\mathcal{H}}_{\text{spin}}(\hat{x}, \hat{z}, \hat{\sigma}; \hat{p}_x, \hat{p}_z, \hat{p}_\sigma; s)\\
&\quad=\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(\hat{\alpha},\hat{\beta})\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix}\\
&\qquad+\frac{1}{2}\cdot[\hat{\alpha}^2+\hat{\beta}^2]\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\\
&\quad=\sqrt{\xi}\cdot\frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot(\omega_s,\omega_x,\omega_z)\cdot\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}\\
&\qquad+\frac{1}{2}[\hat{\alpha}^2+\hat{\beta}^2]\cdot\frac{d}{ds}\psi_{\text{spin}}(s),
\end{aligned} \tag{6.45b}$$

where we have written for abbreviation:

$$G_1=K_x^2+g, \tag{6.46a}$$

$$G_2=K_z^2-g. \tag{6.46b}$$

The corresponding canonical equations take the form:

$$\frac{d}{ds}\hat{x}=\hat{p}_x+H\cdot\hat{z}+\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{12},F_{22},F_{32})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47a}$$

$$\frac{d}{ds}\hat{p}_x=K_x\cdot\hat{p}_\sigma+[\hat{p}_z-H\cdot\hat{x}]\cdot H-G_1\cdot\hat{x}+N\cdot\hat{z}-\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{11},F_{21},F_{31})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47b}$$

$$\frac{d}{ds}\hat{z}=\hat{p}_z-H\cdot\hat{x}+\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{14},F_{24},F_{34})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47c}$$

$$\frac{d}{ds}\hat{p}_z=K_z\cdot\hat{p}_\sigma-[\hat{p}_x+H\cdot\hat{z}]\cdot H-G_2\cdot\hat{z}+N\cdot\hat{x}-\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{13},F_{23},F_{33})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47d}$$

$$\frac{d}{ds}\hat{\sigma}=\frac{1}{\gamma_0^2}\cdot\hat{p}_\sigma-[K_x\cdot\hat{x}+K_z\cdot\hat{z}]+\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{16},F_{26},F_{36})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47e}$$

$$\frac{d}{ds}\hat{p}_\sigma=\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi\cdot\hat{\sigma}-\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(F_{15},F_{25},F_{35})\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.47f}$$

$$\frac{d}{ds}\hat{\alpha}=+\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(0,1)\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}\cdot\hat{\mathbf{y}}+\hat{\beta}\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{6.47g}$$

$$\frac{d}{ds}\hat{\beta} = -\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot(1,0)\begin{pmatrix} m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}\cdot\hat{\mathbf{y}}$$
$$-\hat{\alpha}\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{6.47h}$$

or in matrix-form:

$$\frac{d}{ds}\begin{pmatrix}\hat{\mathbf{y}}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix} = \underline{A}\cdot\begin{pmatrix}\hat{\mathbf{y}}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix}, \tag{6.48}$$

with

$$\underline{A}(s) = \begin{pmatrix}\underline{A}_{\text{orb}} & \underline{B}\\ \underline{C} & \underline{D}_0\end{pmatrix}, \tag{6.49}$$

and

$$\underline{A}_{\text{orb}}(s) = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2+H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 1/\gamma_0^2 \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot\frac{1}{\beta_0^2}\cdot\frac{2\pi h}{L}\cos\varphi & 0 \end{pmatrix}, \tag{6.50a}$$

$$\underline{B}(s) = -\sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot\underline{S}\cdot\underline{F}^T\cdot\begin{pmatrix} m_s(s) & l_s(s)\\ m_x(s) & l_x(s)\\ m_z(s) & l_z(s)\end{pmatrix}, \tag{6.50b}$$

$$\underline{C}(s) = \sqrt{\xi}\cdot\frac{1}{\beta_0}\cdot\sqrt{\frac{v_0}{E_0}}\cdot\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_z(s)\\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}, \tag{6.50c}$$

$$\underline{D}_0(s) = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\cdot\frac{d}{ds}\psi_{\text{spin}}(s). \tag{6.50d}$$

Here the matrix $\underline{B}(s)$ describes the influence of Stern-Gerlach forces on the orbital motion and the matrix $\underline{C}(s)$ the influence of orbital motion on the spin motion. The matrices $\underline{A}(s)$ and $\underline{D}_0(s)$ correspond to the "unperturbed" spin-orbit motion.

We emphasize again that the approximation in (6.45b) can only be used if the spin is almost parallel to $\mathbf{n}_0$.

Because the equations of motion (6.48) are linear and homogeneous, the solution can be written in the form:

$$\begin{pmatrix}\hat{\mathbf{y}}(s)\\ \hat{\alpha}(s)\\ \hat{\beta}(s)\end{pmatrix} = \underline{\hat{M}}(s,s_0)\cdot\begin{pmatrix}\hat{\mathbf{y}}(s_0)\\ \hat{\alpha}(s_0)\\ \hat{\beta}(s_0)\end{pmatrix}. \tag{6.51}$$

This defines the symplectic 8-dimensional transfer matrix $\underline{\hat{M}}(s, s_0)$ of linearised spin-orbit motion.

If the matrix $\underline{B}$ in (6.49) is retained but the matrix $\underline{C}$ is put to zero, i.e. if SG forces are included but the effect of orbital motion on spin is neglected, then $\underline{\hat{M}}$ will be non-symplectic and the phase space volume can, in principle, grow or shrink indefinitely – at least in this linearised description.

Another observation is that the matrices $\underline{B}$ and $\underline{C}$ serve to couple orbit and spin in a way analogous to the way that the off diagonal 2×2 blocks in solenoid and skew quadrupole matrices couple x and z motion. In the presence of orbital coupling and near resonance that x and z modes exchange energy and, depending on whether the system is at a sum or a difference resonance, the beam blows up or is stable as energy is exchanged between the modes indefinitely [32]. It will be interesting to see if analogous phenomena occur in the spin and orbit coordinates at spin orbit resonances. We will treat this case is another paper.

Remarks.

1) Neglecting the Stern-Gerlach terms coming from the component $\hat{\mathscr{H}}_{\text{spin}}$ the orbital part (6.44a, b, c) of the canonical equations (6.44) can be approximated as:

$$\frac{d}{ds}\hat{x} = +\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{p}_x},\quad \frac{d}{ds}\hat{p}_x = -\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{x}}, \tag{6.52a}$$

$$\frac{d}{ds}\hat{z} = +\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{p}_z},\quad \frac{d}{ds}\hat{p}_z = -\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{z}}, \tag{6.52b}$$

$$\frac{d}{ds}\hat{\sigma} = +\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{p}_\sigma},\quad \frac{d}{ds}\hat{p}_\sigma = -\frac{\partial\hat{\mathscr{H}}_{\text{orb}}}{\partial\hat{\sigma}}. \tag{6.52c}$$

This canonical system is then separate (and independent) from the spin motion and corresponds to the fully coupled 6-dimensional formalism [1, 2].

If the orbit vector

$$\hat{\mathbf{y}}(s) = \begin{pmatrix}\hat{x}\\ \hat{p}_x\\ \hat{z}\\ \hat{p}_z\\ \hat{\sigma}\\ \hat{p}_\sigma\end{pmatrix}$$

is known, we can calculate the spin motion from the equations:

$$\frac{d}{ds}\hat{\alpha} = +\frac{\partial\hat{\mathscr{H}}_{\text{spin}}}{\partial\hat{\beta}},\quad \frac{d}{ds}\hat{\beta} = -\frac{\partial\hat{\mathscr{H}}_{\text{spin}}}{\partial\hat{\alpha}}, \tag{6.53}$$

or

$$\frac{d}{ds}\alpha = +\frac{\partial \hat{\mathscr{K}}_{spin}}{\partial \beta}, \quad \frac{d}{ds}\beta = -\frac{\partial \hat{\mathscr{K}}_{spin}}{\partial \alpha}, \tag{6.54}$$

where $\hat{\mathscr{K}}_{spin}$ is given by (6.34a). These spin-equations are again in canonical form and the "forced solution" of (6.53) or (6.54) [20, 21] provides a method alternative to that in [22] for calculating the **n**-axis. By separating the equations of motion for the orbit and spin, we automatically ignore the second and higher orders in $\hbar$ when calculating the forced solution.

2) The perturbation of the orbit motion by SG forces is of $\mathcal{O}(\hbar)\cdot(a\gamma+1)$ but the effect of the orbit on spin of order $(a\gamma+1)$. The fact that $\underline{B}$ and $\underline{C}$ are of similar order of magnitude is an artefact of the choice of canonical variables [33].

Note also that the $(\hat{x}, \hat{p}_x)$, $(\hat{z}, \hat{p}_z)$, $(\hat{\sigma}, \hat{p}_\sigma)$ and $(\hat{\alpha}, \hat{\beta})$ phase space areas all have the dimension of length.

3) The formalism presented here describes the effect of SG forces in all three (x, z, s) planes. In particular it automatically describes the effect of longitudinal field gradients on the transverse motion [33].

7 Summary

Following earlier works of Yokoya and Derbenev, we have used a classical Hamiltonian in a fixed Cartesian coordinate system for a spin 1/2 charged particle to investigate a canonical formalism of spin-orbit motion expressed in machine coordinates, taking into account all kinds of coupling induced by skew quadrupoles and solenoids (coupling of betatron motion), by non-vanishing dispersion in the cavities (synchro-betatron coupling) and by Stern-Gerlach forces (spin-orbit coupling).

In addition to the well-known orbital variables $\hat{x}, \hat{p}_x, \hat{z}, \hat{p}_z, \hat{\sigma}, \hat{p}_\sigma$ of the fully coupled 6-dimensional formalism we introduce the canonical variables $\hat{\alpha}$ and $\hat{\beta}$ to describe the spin motion.

By expanding the Hamiltonian into a power series in these variables, one may obtain various orders of approximation for the canonical equations and the canonical structure of the formalism allows modern techniques such as Lie-algebra and differential algebra to be included in a natural way [34–40, 18]. For example, the $\hat{\alpha}$ and $\hat{\beta}$ variables might simplify discussion of the **n**-axis constructed by using normal forms.

The equations presented in this paper can serve to develop a nonlinear, 8-dimensional (symplectic) tracking program for the combined spin-orbit system.

Such a program may be used to study (in addition to orbital problems) chaotic behaviour of spin motion and to investigate the influence of Stern-Gerlach forces.

In this paper we have treated motion in a storage ring, i.e. the average energy E_0 of the particles is constant. But it is easy to encompass acceleration by cavity fields in this formalism. For more details see [41, 42].

Finally we remark that, starting from the variables $\hat{x}, \hat{p}_x, \hat{z}, \hat{p}_z, \hat{\sigma}, \hat{p}_\sigma, \hat{\alpha}, \hat{\beta}$ and using analytical techniques as described in [2, 43–45] one can also develop an 8-dimensional dispersion formalism.

Acknowledgements. We wish to thank Prof. Dr. G.-A. Voss and Prof. Dr. B. Wiik for continued encouragement. We thank Dr. S.R. Mane for useful comments.

Appendix A: Vector potentials for various lenses

Using the freedom to select a gauge, we can choose any vector potential which leads to the correct form of the fields. Suitable vector potentials are as follows and have been chosen for their simplicity [1].

A.1 Bending magnet

Since the design orbit

$$x(s) = z(s) \equiv 0 \tag{A.1}$$

is a solution of the equations of motion for

$$\mathscr{E} = 0, \quad E \equiv E_0 \tag{A.2}$$

by definition, the magentic bending field $\mathscr{B}_x^{(0)}(s)$ and $\mathscr{B}_z^{(0)}(s)$ is fixed by the curvatures K_x and K_z of the design orbit:

$$\frac{e}{p_0 \cdot c} \cdot \mathscr{B}_x^{(0)} = -K_z, \tag{A.3a}$$

$$\frac{e}{p_0 \cdot c} \cdot \mathscr{B}_z^{(0)} = +K_x. \tag{A.3b}$$

The corresponding vector potential can be written as:

$$\frac{e}{p_0 \cdot c} \cdot A_s = -\tfrac{1}{2}(1 + K_x \cdot x + K_z \cdot z), \tag{A.4a}$$

$$A_x = A_z = 0. \tag{A.4b}$$

A.2 Quadrupole

The quadrupole fields are

$$\mathscr{B}_x = z \cdot \left(\frac{\partial \mathscr{B}_z}{\partial x}\right)_{x=z=0}, \tag{A.5a}$$

$$\mathscr{B}_z = x \cdot \left(\frac{\partial \mathscr{B}_z}{\partial x}\right)_{x=z=0}, \tag{A.5b}$$

so that we may use the vector potential

$$A_s = \left(\frac{\partial \mathscr{B}_z}{\partial x}\right)_{x=z=0} \cdot \frac{1}{2}(z^2 - x^2), \tag{A.6a}$$

$$A_x = A_z = 0. \tag{A.6b}$$

We rewrite this as:

$$\frac{e}{p_0 \cdot c} A_s = \frac{1}{2} g \cdot (z^2 - x^2), \tag{A.7a}$$

with

$$g = \frac{e}{p_0 \cdot c} \cdot \left(\frac{\partial \mathscr{B}_z}{\partial x}\right)_{x=z=0}. \tag{A.7b}$$

A.3 Skew quadrupole

The fields are

$$\mathscr{B}_x = -\frac{1}{2}\cdot\left(\frac{\partial \mathscr{B}_z}{\partial z} - \frac{\partial \mathscr{B}_x}{\partial x}\right)_{x=z=0}\cdot x, \tag{A.8a}$$

$$\mathscr{B}_z = +\frac{1}{2}\left(\frac{\partial \mathscr{B}_z}{\partial z} - \frac{\partial \mathscr{B}_x}{\partial x}\right)_{x=z=0}\cdot z. \tag{A.8b}$$

Thus we may use

$$A_s = -\frac{1}{2}\cdot\left(\frac{\partial \mathscr{B}_z}{\partial z} - \frac{\partial \mathscr{B}_x}{\partial x}\right)_{x=z=0}\cdot xz, \tag{A.9a}$$

$$A_x = A_z = 0, \tag{A.9b}$$

and we write:

$$\frac{e}{p_0\cdot c} A_s = N\cdot xz, \tag{A.10a}$$

$$N = \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\left(\frac{\partial \mathscr{B}_x}{\partial x} - \frac{\partial \mathscr{B}_z}{\partial z}\right)_{x=z=0}. \tag{A.10b}$$

A.4 Solenoid fields

The field components in the current free region are given by [1, 46]:

$$\mathscr{B}_x(x, z, s) = x\cdot\sum_{\nu=0}^{\infty} b_{2\nu+1}\cdot(x^2+z^2)^\nu, \tag{A.11a}$$

$$\mathscr{B}_z(x, z, s) = z\cdot\sum_{\nu=0}^{\infty} b_{2\nu+1}\cdot(x^2+z^2)^\nu, \tag{A.11b}$$

$$\mathscr{B}_s(x, z, s) = x\cdot\sum_{\nu=0}^{\infty} b_{2\nu}\cdot(x^2+z^2)^\nu, \tag{A.11c}$$

where for consistency with Maxwell's equations the coefficients b_μ obey the recursion equations:

$$b_{2\nu+1}(s) = -\frac{1}{(2\nu+2)}\cdot b'_{2\nu}(s), \tag{A.12a}$$

$$b_{2\nu+2}(s) = +\frac{1}{(2\nu+2)}\cdot b'_{2\nu+1}(s), \tag{A.12b}$$

$(\nu = 0, 1, 2, \dots)$,

and where

$$b_0(s) \equiv \mathscr{B}_s(0, 0, s). \tag{A.13}$$

The vector potential leading to the solenoid field of (A.11) is then:

$$A_x(x, z, s) = -z\cdot\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+2)}\cdot b_{(2\nu)}(s)\cdot r^{2\nu}, \tag{A.14a}$$

$$A_z(x, z, s) = +x\cdot\sum_{\nu=0}^{\infty}\frac{1}{(2\nu+2)}\cdot b_{(2\nu)}(s)\cdot r^{2\nu}, \tag{A.14b}$$

$$A_s(x, z, s) = 0, \tag{A.14c}$$

with

$$r^2 = x^2 + z^2.$$

Thus we can write:

$$\frac{e}{E_0} A_x = -\beta_0\cdot H(s)\cdot z + \frac{1}{8}\beta_0\cdot H''(s)\cdot(x^2+z^2)\cdot z + \cdots, \tag{A.15a}$$

$$\frac{e}{E_0} A_z = +\beta_0\cdot H(s)\cdot x - \frac{1}{8}\beta_0\cdot H''(s)\cdot(x^2+z^2)\cdot x + \cdots \tag{A.15b}$$

with

$$H(s) = \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot b_0(s) \equiv \frac{1}{2}\cdot\frac{e}{p_0\cdot c}\cdot\mathscr{B}_s(0, 0, s). \tag{A.16}$$

Note that the cyclotron radius for the longitudinal field (A.13) is given by:

$$R = \frac{1}{2\cdot H}.$$

A.5 Dipole correction coils

$$A_s = \Delta\mathscr{B}_x\cdot z - \Delta\mathscr{B}_z\cdot x,$$

with

$$\Delta\mathscr{B}_x = \sum_\mu \Delta\hat{\mathscr{B}}_x^{(\mu)}\cdot\delta(s-s_\mu), \tag{A.17a}$$

$$\Delta\mathscr{B}_z = \sum_\mu \Delta\hat{\mathscr{B}}_z^{(\mu)}\cdot\delta(s-s_\mu), \tag{A.17b}$$

so that

$$\frac{e}{p_0\cdot c} A_s = \frac{e}{p_0\cdot c}\cdot\sum_\mu \delta(s-s_\mu)\cdot[\Delta\hat{\mathscr{B}}_x^{(\mu)}\cdot z - \Delta\hat{\mathscr{B}}_z^{(\mu)}\cdot x]. \tag{A.18}$$

A.6 Cavity field

For a longitudinal electric field

$$\mathscr{E}_x = 0, \quad \mathscr{E}_z = 0, \quad \mathscr{E}_s = \mathscr{E}(s, \sigma), \tag{A.19}$$

we write:

$$A_x = 0, \quad A_z = 0, \quad A_s = \frac{1}{\beta_0}\cdot\int_{\sigma_0}^{\sigma} d\tilde{\sigma}\cdot\mathscr{E}(s, \tilde{\sigma}), \tag{A.20}$$

which by (4.25) immediately gives $\mathscr{E}_s$.

Now the cavity field may be represented by

$$\mathscr{E}(s, \sigma) = V(s)\sin\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right], \tag{A.21}$$

and we obtain using (A.20):

$$A_s = -\frac{1}{\beta_0}\cdot\frac{L}{2\pi\cdot h}\cdot V(s)\cdot\cos\left[h\cdot\frac{2\pi}{L}\cdot\sigma + \varphi\right], \tag{A.22}$$

in which the phase φ is defined so that the average energy radiated away in the bending magnets is replaced by the cavities and h is the harmonic number.

Appendix B: The periodic spin frame ($\mathbf{n}_0, \mathbf{m}, \mathbf{l}$) along the closed orbit

In order to define the periodic spin frame, we first introduce a compact matrix notation. Rewriting an arbitrary vector

$$\mathbf{A}=A_s\cdot\mathbf{e}_s+A_x\cdot\mathbf{e}_x+A_z\cdot\mathbf{e}_z$$

as a column vector with components A_s, A_x, A_z:

$$A_s\cdot\mathbf{e}_s+A_x\cdot\mathbf{e}_x+A_z\cdot\mathbf{e}_z=\begin{pmatrix}A_s\\A_x\\A_z\end{pmatrix},$$

and defining the derivative of a column vector with respect to the arc length s as the derivative of the corresponding components A_i but not of the unit vectors:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix}A_s\\A_x\\A_z\end{pmatrix}\equiv\mathbf{e}_s\cdot\frac{\mathrm{d}}{\mathrm{d}s}A_s+\mathbf{e}_x\cdot\frac{\mathrm{d}}{\mathrm{d}s}A_x+\mathbf{e}_z\cdot\frac{\mathrm{d}}{\mathrm{d}s}A_z,$$

we get from (5.15) and (6.2b):

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{\xi}^{(0)}(s)=\underline{\Omega}^{(0)}\cdot\boldsymbol{\xi}^{(0)}(s), \tag{B.1}$$

where we have set

$$\boldsymbol{\xi}^{(0)}=\begin{pmatrix}\xi_{0s}\\\xi_{0x}\\\xi_{0z}\end{pmatrix}, \tag{B.2a}$$

and

$$\underline{\Omega}^{(0)}(s)=\begin{pmatrix}0 & -\Omega_z^{(0)} & \Omega_x^{(0)}\\ \Omega_z^{(0)} & 0 & -\Omega_s^{(0)}\\ -\Omega_x^{0} & \Omega_s^{(0)} & 0\end{pmatrix}. \tag{B.2b}$$

The transfer matrix $\underline{M}_{(\text{spin})}(s, s_0)$ for the spin motion defined by

$$\boldsymbol{\xi}^{(0)}(s)=\underline{M}_{(\text{spin})}(s, s_0)\cdot\boldsymbol{\xi}^{(0)}(s_0),$$

satisfies the relationships:

$$\underline{M}^T_{(\text{spin})}(s, s_0)\cdot\underline{M}_{(\text{spin})}(s, s_0)=\underline{1}, \tag{B.3a}$$

$$\det[\underline{M}_{(\text{spin})}(s, s_0)]=1, \tag{B.3b}$$

since (with (B.1))

$$\frac{\mathrm{d}}{\mathrm{d}s}\underline{M}_{(\text{spin})}(s, s_0)=\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(\text{spin})}(s, s_0),$$

$$\underline{M}_{(\text{spin})}(s_0, s_0)=\underline{1},$$

and therefore (with $[\underline{\Omega}^{(0)}]^T=-\underline{\Omega}^{(0)}$)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}[\underline{M}^T_{(\text{spin})}(s, s_0)\cdot\underline{M}_{(\text{spin})}(s, s_0)]\\
&=[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(\text{spin})}(s, s_0)]^T\cdot\underline{M}_{(\text{spin})}(s, s_0)\\
&\quad+\underline{M}^T_{(\text{spin})}(s, s_0)\cdot[\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(\text{spin})}(s, s_0)]\\
&=-\underline{M}_{(\text{spin})}(s, s_0)^T\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(\text{spin})}(s, s_0)\\
&\quad+\underline{M}^T_{(\text{spin})}(s, s_0)\cdot\underline{\Omega}^{(0)}(s)\cdot\underline{M}_{(\text{spin})}(s, s_0)\\
&=\underline{0},\end{aligned}$$

$$\det M(s, s_0)=\det M(s_0, s_0)=1,$$

i.e. $\underline{M}_{(\text{spin})}(s, s_0)$ is an orthogonal matrix with determinant 1.

Let us now consider the eigenvalue problem for the revolution matrix $\underline{M}(s_0+L, s_0)$ with the eigenvalues α_μ and eigenvectors $\mathbf{r}_\mu(s_0)$:

$$\underline{M}(s_0+L, s_0)\,\mathbf{r}_\mu(s_0)=\alpha_\mu\cdot\mathbf{r}_\mu(s_0), \quad (\mu=1, 2, 3). \tag{B.4}$$

Because of (B.3a, b) we can write [28, 47]:

$$\begin{aligned}&\alpha_1=1,\\
&\alpha_2=\mathrm{e}^{\mathrm{i}\cdot 2\pi\cdot Q_{\text{spin}}},\\
&\alpha_3=\mathrm{e}^{-\mathrm{i}\cdot 2\pi\cdot Q_{\text{spin}}}, \quad (Q_{\text{spin}}=\text{real number})\end{aligned} \tag{B.5}$$

and

$$\mathbf{r}_1(s_0)=\mathbf{n}_0(s_0), \tag{B.6a}$$

$$\mathbf{r}_2(s_0)=\mathbf{m}_0(s_0)+\mathrm{i}\cdot\mathbf{l}_0(s_0), \tag{B.6b}$$

$$\mathbf{r}_3(s_0)=\mathbf{m}_0(s_0)-\mathrm{i}\cdot\mathbf{l}_0(s_0), \; (\mathbf{n}_0, \mathbf{m}_0, \mathbf{l}_0=\text{real vectors}). \tag{B.6c}$$

If we require that

$$\mathbf{r}_1^+\cdot\mathbf{r}_1=1, \tag{B.7a}$$

$$\mathbf{r}_2^+\cdot\mathbf{r}_2\equiv\mathbf{r}_3^+\cdot\mathbf{r}_3=2, \quad (\text{normalizing conditions}) \tag{B.7b}$$

we find, using also (B.3a) [47]:

$$|\mathbf{n}_0(s_0)|=|\mathbf{m}_0(s_0)|=|\mathbf{l}_0(s_0)|=1, \tag{B.8a}$$

$$\mathbf{n}_0(s_0)\perp\mathbf{m}_0(s_0)\perp\mathbf{l}_0(s_0). \tag{B.8b}$$

Thus the vectors $\mathbf{n}_0(s_0)$, $\mathbf{m}_0(s_0)$ and $\mathbf{l}_0(s_0)$ form an orthogonal system of unit vectors. Choosing the direction of $\mathbf{n}_0(s_0)$ such that

$$\mathbf{n}_0(s_0)=\mathbf{m}_0(s_0)\times\mathbf{l}_0(s_0) \tag{B.8c}$$

these vectors form a right-handed coordinate system.

In this way we have found a coordinate frame for the position $s=s_0$.

An orthogonal system of unit vectors at an arbitrary position s can be defined by applying the transfer matrix $\underline{M}_{(\text{spin})}(s, s_0)$ to the vectors $\mathbf{n}_0(s_0)$, $\mathbf{m}_0(s_0)$ and $\mathbf{l}_0(s_0)$:

$$\mathbf{n}_0(s)=\underline{M}_{(\text{spin})}(s, s_0)\,\mathbf{n}_0(s_0), \tag{B.9a}$$

$$\mathbf{m}_0(s)=\underline{M}_{(\text{spin})}(s, s_0)\,\mathbf{m}_0(s_0), \tag{B.9b}$$

$$\mathbf{l}_0(s)=\underline{M}_{(\text{spin})}(s, s_0)\,\mathbf{l}_0(s_0). \tag{B.9c}$$

Because of (B.3a, b) the orthogonality relations remain unchanged:

$$\mathbf{n}_0(s)=\mathbf{m}_0(s)\times\mathbf{l}_0(s), \tag{B.10a}$$

$$\mathbf{m}_0(s)\perp\mathbf{l}_0(s), \tag{B.10b}$$

$$|\mathbf{n}_0(s)|=|\mathbf{m}_0(s)|=|\mathbf{l}_0(s)|=1. \tag{B.10c}$$

The coordinate frame defined by $\mathbf{n}_0(s)$, $\mathbf{m}_0(s)$ and $\mathbf{l}_0(s)$ is not yet appropriate for a description of the spin motion, because it does not transform into itself after one revolution of the particles:

$$\begin{aligned}\mathbf{m}_0(s_0+L)+\mathrm{i}\mathbf{l}_0(s_0+L)&=\underline{M}_{(\mathrm{spin})}(s_0+L, s_0)\,[\mathbf{m}_0(s_0)+\mathrm{i}\mathbf{l}_0(s_0)]\\&=\mathrm{e}^{\mathrm{i}\cdot 2\pi\cdot Q_{\mathrm{spin}}}\cdot[\mathbf{m}_0(s_0)+\mathrm{i}\mathbf{l}_0(s_0)]\\&\neq\mathbf{m}_0(s_0)+\mathrm{i}\mathbf{l}_0(s_0)\end{aligned}$$

(if $Q_{\mathrm{spin}}\neq$integer).

But by introducing a phase function $\psi_{\mathrm{spin}}(s)$ and using another orthogonal matrix $\underline{D}(s, s_0)$:

$$\underline{D}(s,s_0)=\begin{pmatrix}\cos[\psi_{\mathrm{spin}}(s)-\psi_{\mathrm{spin}}(s_0)] & \sin[\psi_{\mathrm{spin}}(s)-\psi_{\mathrm{spin}}(s_0)]\\ -\sin[\psi_{\mathrm{spin}}(s)-\psi_{\mathrm{spin}}(s_0)] & \cos[\psi_{\mathrm{spin}}(s)-\psi_{\mathrm{spin}}(s_0)]\end{pmatrix}, \tag{B.11}$$

with

$$\underline{D}^T(s, s_0)\cdot\underline{D}(s, s_0)=\underline{1}, \tag{B.12a}$$

$$\det[\underline{D}(s, s_0)]=1, \tag{B.12b}$$

we can construct a periodic orthogonal system of unit vectors from $\mathbf{n}_0(s)$, $\mathbf{m}_0(s)$ and $\mathbf{l}_0(s)$. Namely, if we put [47]:

$$\begin{aligned}&\begin{pmatrix}\mathbf{m}(s)\\ \mathbf{l}(s)\end{pmatrix}=\underline{D}(s, s_0)\begin{pmatrix}\mathbf{m}(s_0)\\ \mathbf{l}(s_0)\end{pmatrix}\\&\Rightarrow\mathbf{m}(s)+\mathrm{i}\mathbf{l}(s)=\mathrm{e}^{-\mathrm{i}\cdot[\psi_{\mathrm{spin}}(s)-\psi_{\mathrm{spin}}(s_0)]}\cdot[\mathbf{m}_0(s)+\mathrm{i}\mathbf{l}_0(s)]\\&\neq\mathbf{m}_0(s_0)+\mathrm{i}\mathbf{l}_0(s_0),\end{aligned} \tag{B.13}$$

we find, using (B.12a, b):

$$\mathbf{n}_0(s)=\mathbf{m}(s)\times\mathbf{l}(s), \tag{B.14a}$$

$$\mathbf{m}(s)\perp\mathbf{l}(s), \tag{B.14b}$$

$$|\mathbf{n}_0(s)|=|\mathbf{m}(s)|=|\mathbf{l}(s)|=1. \tag{B.14c}$$

Since,

$$\mathbf{m}(s_0+L)+\mathrm{i}\mathbf{l}(s_0+L)=\mathrm{e}^{-\mathrm{i}\cdot[\psi_{\mathrm{spin}}(s_0+L)-\psi_{\mathrm{spin}}(s_0)]}\cdot[\mathbf{m}(s_0)+i\mathbf{l}(s_0)],$$

it follows that the condition of periodicity for $\mathbf{n}_0$, $\mathbf{m}$ and $\mathbf{l}$:

$$(\mathbf{n}_0, \mathbf{m}, \mathbf{l})_{s=s_0+L}=(\mathbf{n}_0, \mathbf{m}, \mathbf{l})_{s=s_0}, \tag{B.15}$$

can indeed be fulfilled if the phase function $\psi_{\mathrm{spin}}(s)$ satisfies the relationship:

$$\psi_{\mathrm{spin}}(s_0+L)-\psi_{\mathrm{spin}}(s)=2\pi\cdot Q_{\mathrm{spin}}, \tag{B.16a}$$

($Q_{\mathrm{spin}}=$spin tune).

For instance we can choose:

$$\psi_{\mathrm{spin}}(s)=2\pi\cdot Q_{\mathrm{spin}}\cdot\frac{s}{L}. \tag{B.16b}$$

Taking the derivatives of $\mathbf{m}(s)$ and $\mathbf{l}(s)$ with respect to s, and taking into account (B.13), (B.9), and (B.1) we get:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{m}(s)=\underline{\Omega}^{(0)}(s)\,\mathbf{m}(s)+\psi'(s)\cdot\mathbf{l}(s), \tag{B.17a}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{l}(s)=\underline{\Omega}^{(0)}(s)\,\mathbf{l}(s)-\psi'(s)\cdot\mathbf{m}(s), \tag{B.17b}$$

and $\mathbf{n}_0(s)$ satisfies (see (B.9a)):

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{n}_0(s)=\underline{\Omega}^{(0)}(s)\,\mathbf{n}_0(s). \tag{B.17c}$$

Finally the vectors

$$\mathbf{r}_1(s)=\mathbf{n}_0(s)\equiv\underline{M}_{(\mathrm{spin})}(s, s_0)\mathbf{r}_1(s_0), \tag{B.18a}$$

$$\mathbf{r}_2(s)=\mathbf{m}_0(s)+\mathrm{i}\mathbf{l}_0(s)\equiv\underline{M}_{(\mathrm{spin})}(s, s_0)\mathbf{r}_2(s_0), \tag{B.18b}$$

$$\mathbf{r}_3(s)=\mathbf{m}_0(s)-\mathrm{i}\mathbf{l}_0(s)\equiv\underline{M}_{(\mathrm{spin})}(s, s_0)\mathbf{r}_3(s_0), \tag{B.18c}$$

are eigenvectors of the revolution matrix $\underline{M}_{\mathrm{spin}}$ with the same eigenvalues as in (B.5):

$$\underline{M}(s+L, s)\mathbf{r}_\mu(s)=\alpha_\mu\cdot\mathbf{r}_\mu(s). \tag{B.19}$$

Thus, the eigenvalues α_μ and the quantity Q_{spin} defined by (B.5) are independent of the chosen initial position s_0.

Remark: In order to solve (B.1), the 6-dimensional orbit vector $\mathbf{y}_0$ must be known (see (6.3)). This vector can be approximated by neglecting the Stern-Gerlach term

$$\frac{\partial}{\partial\mathbf{y}_0}\bar{\mathscr{H}}_{\mathrm{spin}}(\mathbf{y}_0;\psi_0, J_0; s)$$

in (6.2a), giving:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{y}_0=-\underline{S}\cdot\frac{\partial}{\partial\mathbf{y}_0}\bar{\mathscr{H}}_{\mathrm{orb}}(\mathbf{y}_0; s). \tag{B.20}$$

The error in calculating $\boldsymbol{\xi}_0(s)$ is of order $\hbar^2$ which we can neglect at our semiclassical level of approximation. A solution of (B.1) may then be obtained by using the method of thin-lens approximation as described in [9, 48].

References

1. G. Ripken: DESY 85-84, 1985
2. D.P. Barber, G. Ripken, F. Schmidt: DESY 87-36, 1987
3. M. Berz, E. Forest, J. Irwin: Part. Acc. 24 (1989) 91
4. A.J. Dragt et al.: Ann. Rev. Nucl. Part. Phys. 38 (1988) 455
5. Y.S. Derbenev: University of Michigan, Ann Arbor Preprint, UM HE 90-23, 1990; University of Michigan, Ann Arbor Preprint, UM HE 90-30, 1990; University of Michigan, Ann Arbor Preprint, UM HE 90-32, 1990
6. Ya.S. Derbenev, A.M. Kondratenko: Sov. Phys. JETP 37(6) (1973) 968
7. L. Thomas: Phil. Mag. 3 (1927) 1
8. V. Bargmann, L. Michel, V.L. Telegdi: Phys. Rev. Lett. 2 (1959) 435
9. A.W. Chao: Nucl. Instrum. Methods 180 (1981) 29; also in: Physics of High Energy Particle Accelerators, American Institute of Physics Proceedings 87, p. 395, 1982
10. H. Mais, G. Ripken: DESY 83-62, 1983
11. K. Yokoya: DESY Report 86-57, 1986
12. R. Schiller: Phys. Rev. 128 (1962) 1402
13. D.P. Barber: Symposium on High Energy Spin Physics, Minneapolis 1988, American Institute of Physics Proceedings 187, p. 855, 1988
14. D.B. Cline: in: Fundamental Symmetries, Conference at Erice, Italy 1986, p. 329, 1986; see also articles by: T.O. Niinikoski, Y. Onel et al., and H. Poth in the same procedings

15. R. Rossmanith: Proceedings of the 8th International Symposium on High Energy Spin Physics, Minneapolis 1988; American Institute of Physics Proceedings 187, p. 1162, 1988; T.O. Niinikoski, R. Rossmanith: Nucl. Instrum. Methods A255 (1987) 460
16. E. Steffens: Proceedings of the 8th International Symposium on High Energy Spin Physics, Minneapolis 1988; American Institute of Physics Proceedings 187, p. 1302, 1988
17. M. Conte et al.: INFN Report, INFN/TC-:93/04, 1993
18. K. Yokoya: Nucl. Instrum. Methods A258 (1987) 149
19. D.P. Barber, K. Heinemann, G. Ripken: Accepted for publication in Zeitschrift für Physik C; see also DESY 93-031 by the same authors
20. D.P. Barber, K. Heinemann, G. Ripken: DESY M-92-04, 1992
21. D.P. Barber, K. Heinemann,G. Ripken: Submitted to Z. Phys.C
22. S.R. Mane: Phys. Rev. A36 (1987) 105
23. A.O. Barut: Electrodynamics and classical theory of fields and particles. New York: Macmillan 1964
24. J.D. Jackson: Rev. Mod. Phys. 45 (1976) 417
25. K. Heinemann: Ph.D Thesis (in preparation)
26. D. Bohm: Quantum theory. New York: Prentice-Hall 1951
27. E. Courant, H. Snyder: Ann. Phys. 3 (1958) 1
28. H. Goldstein: Classical mechanics. New York: Addison-Wesley 1980
29. B.W. Montague: CERN Report 77-13, 1977
30. H. Mais, G. Ripken: DESY M-82-05, 1982
31. K. Yokoya: KEK Report 92-6, 1992
32. D.P. Barber, H. Mais, G. Ripken, F. Willeke: DESY 86-147, 1986
33. J. Schwinger: Science 262 (1993) 826
34. S.R. Mane: Phys. Lett. A177 (1993) 411
35. V. Balandin: Proceedings of the 1993 IEEE Particle Accelerator Conference, Washington DC, USA, p. 477, 1993
36. V. Balandin, N. Golubeva: Proceedings of the 1993 IEEE Particle Accelerator Conference, Washington DC, USA, p. 441, 1993
37. Yu. Eidelman, V. Yakimenko: Institute for Nuclear Physics, Novosibirsk, Preprint 90-127, 1990
38. Yu. Eidelman, V. Yakimenko: Proceedings of the 1993 IEEE Particle Accelerator Conference, Washington DC, USA, p. 450, 1993
39. M. Berz: COSY INFINITY, Version 6.1, Users Guide and Reference Manual, 1993
40. M. Berz: Nucl. Instrum. Methods, A298 (1990) 473
41. F. Schmidt: Ph.D. Thesis. DESY HERA 88-02, 1988
42. I. Borchardt, E. Karantzoulis, H. Mais, G. Ripken: DESY 88-013, 1988; Z. Phys. C41 (1988) 25
43. H. Mais, G. Ripken: DESY 86-29, 1986
44. G. Ripken, E. Karantzoulis: DESY 86-29, 1986
45. D.P. Barber, K. Heinemann, G. Ripken, F. Willeke: DESY HERA 94-02, 1994
46. E. Freytag, G. Ripken: DESY E3/R1-73/1, 1973
47. H. Mais, G. Ripken: DESY M-82-17, 1982
48. D.P. Barber, H. Mais, G. Ripken: DESY 84-17, 1984