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A canonical 8-dimensional formalism for classical spin-orbit motion in storage rings

II. Normal forms and the n-axis

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Abstract. The two real canonical spin variables α and β introduced in an earlier paper to describe spin motion in storage rings [1] are combined with the six canonical variables of coupled synchro-betatron motion to form a system of eight canonical spin-orbit variables in which spin and orbital motion are treated on the same level. In these variables one turn maps are origin preserving and the usual techniques of canonical perturbation theory can be applied. By writing the Hamiltonian in normal form the spin detuning terms as well as the so called **n**-axis, the semiclassical spin axis which is needed in the theory of radiative polarization, can be constructed. The equations derived are valid for arbitrary particle velocity (below and above transition energy).

1 Introduction

In paper I of this series [1] we introduced a pair of real canonical spin variables α and β which uniquely parametrize the classical spin over (almost) the whole 'spin sphere' and which behave in the small spin tilt limit like those used by Chao in the SLIM formalism [2]. By treating these variables on the same level as the canonical set, $(x, p_x, z, p_z, \sigma, p_\sigma)$ describing the coupled synchro-betatron motion in storage rings the Hamiltonian of combined spin-orbit motion can be expanded as a power series in small quantities. In this way the usual techniques of canonical perturbation theory may be applied simultaneously and consistently to the whole spin-orbit equation system. For example, at the linear level the motion is described using 8×8 symplectic transfer matrices. Furthermore, normal forms can be introduced and the mutual detuning of the orbital and spin motion may be investigated. The introduction of normal forms leads to the construction of the so called **n**-axis [3, 5] needed in the analytical calculation of spin polarization in electron storage rings.

In detail, the work is organized as follows:

The starting point is the semiclassical spin-orbit Hamiltonian of Derbenev and Kondratenko [4, 5] described in a fixed Cartesian coordinate system but rewritten in terms of α and β [1, 6]. This description is summarized in Sect. 2.

In Sect. 3 the spin-orbit Hamiltonian is expressed in machine coordinates within the framework of the 6-dimensional description of particle motion by using the arc length s of the design orbit as independent variable (instead of the time t), taking into account all kinds of coupling induced by skew quadrupoles and solenoids, by non-vanishing dispersion in the cavities and by Stern-Gerlach forces. The equations so derived are valid for arbitrary velocity of the particles (below and above transition energy).

In Sect. 4 we introduce the 8-dimensional closed orbit for the combined spin-orbit system as a new reference orbit for spin-orbit motion, defining the periodic 6-dimensional closed orbit for particle motion and the periodic $(\mathbf{n}_0, \mathbf{m}, \mathbf{l})$ -dreibein for spin motion. The oscillations around this closed orbit are investigated.

In Sect. 5 normal forms and the **n**-axis are defined.

In order to study the perturbative behaviour of spin-orbit motion we need the Hamiltonian of the linearised spin-orbit system in terms of action-angle variables. This can be obtained by variation of constants and is derived in Sect. 6.

After this preparation we then diagonalise the Hamiltonian using canonical perturbation theory which finally leads to the normal forms of spin-orbit motion (Sect. 7) and to a method of calculating the **n**-axis in storage rings.

A summary of the results is presented in Sect. 8.

This formalism contains as special cases the normal form analyses of coupled synchro-betatron oscillations or coupled betatron oscillations by restricting the complete variable set $(x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta)$ to $(x, p_x, z, p_z, \sigma, p_\sigma)$ or (x, p_x, z, p_z). Thus, in particular the impact of linear coupling on nonlinear dynamics can be investigated [7].

In contrast to paper I we write the spin-orbit Hamiltonian in terms of the new spin variables right from the beginning.This results in a small amount of repetition of some parts of paper I. But this procedure simplifies the derivation of the Hamiltonian and leads to a new insight into the behaviour of the canonical spin transformation. In particular a suitable scale transformation for the length of the spin vector can easily be found.

Finally we remark that other methods for calculating the **n**-axis and the corresponding computer algorithms have already been described in several papers [8–14]. Those based on normal forms could in principle be reformulated in terms of α and β.

2 Spin-orbit motion in a fixed coordinate system

2.1 The starting Hamiltonian

As in paper I we begin with the classical spin-orbit Hamiltonian:

$$\mathscr{H}(\mathbf{r}, \alpha; \mathbf{P}, \beta; t) = \mathscr{H}_{\mathrm{orb}}(\mathbf{r}, \mathbf{P}, t) + \mathscr{H}_{\mathrm{spin}}(\mathbf{r}, \alpha; \mathbf{P}, \beta; t), \tag{2.1}$$

with

$$\mathscr{H}_{\mathrm{orb}}(\mathbf{r}, \mathbf{P}, t) = c \cdot \{\boldsymbol{\pi}^2 + m_0^2 c^2\}^{1/2} + e\phi, \tag{2.2a}$$

$$\mathscr{H}_{\mathrm{spin}}(\mathbf{r}, \alpha; \mathbf{P}, \beta; t) = \boldsymbol{\Omega}_0(\mathbf{r}, \mathbf{P}, t) \cdot \boldsymbol{\xi}(\alpha, \beta), \tag{2.2b}$$

and

$$\boldsymbol{\Omega}_0 = -\frac{e}{m_0 c}\left[\left(\frac{1}{\gamma} + a\right) \cdot \mathscr{B} - \frac{a(\boldsymbol{\pi} \cdot \mathscr{B})}{\gamma(\gamma+1) m_0^2 c^2} \cdot \boldsymbol{\pi} - \frac{1}{m_0 c \gamma}\left(a + \frac{1}{1+\gamma}\right) \boldsymbol{\pi} \times \mathscr{E}\right], \tag{2.3}$$

where we use the same notation and where

$$\boldsymbol{\pi} = \mathbf{P} - \frac{e}{c}\mathbf{A} \ (\textit{kinetic momentum vector}), \tag{2.4}$$

and

$$\gamma = \frac{1}{m_0 c} \cdot \sqrt{m_0^2 c^2 + \boldsymbol{\pi}^2} \ \text{(Lorentz factor)}. \tag{2.5}$$

In terms of the three unit cartesian coordinate vectors in the fixed laboratory frame, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ we may write $\mathbf{r}$, $\mathbf{P}$ and $\boldsymbol{\xi}$ as:

$$\mathbf{r} = X_1 \cdot \mathbf{e}_1 + X_2 \cdot \mathbf{e}_2 + X_3 \cdot \mathbf{e}_3, \tag{2.6a}$$

$$\mathbf{P} = P_1 \cdot \mathbf{e}_1 + P_2 \cdot \mathbf{e}_2 + P_3 \cdot \mathbf{e}_3, \tag{2.6b}$$

$$\boldsymbol{\xi} = \xi_1 \cdot \mathbf{e}_1 + \xi_2 \cdot \mathbf{e}_2 + \xi_3 \cdot \mathbf{e}_3. \tag{2.6c}$$

The spin components ξ_1, ξ_2, ξ_3 as defined by (2.6c) are written in terms of the new canonical spin variables α and β:

$$\xi_1(\alpha, \beta) = \alpha \cdot \sqrt{\xi - \tfrac{1}{4}(\alpha^2 + \beta^2)}, \tag{2.7a}$$

$$\xi_2(\alpha, \beta) = \beta \cdot \sqrt{\xi - \tfrac{1}{4}(\alpha^2 + \beta^2)}, \tag{2.7b}$$

$$\xi_3(\alpha, \beta) = \xi - \tfrac{1}{2}(\alpha^2 + \beta^2), \tag{2.7c}$$

with

$$\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 = \frac{\hbar^2}{4}.$$

The canonical spin variables α and β are to be used on an equal basis with $\mathbf{r}$ and $\mathbf{P}$. The spin vector $\boldsymbol{\xi}$ is of constant length since it obeys a precession equation.

Remarks: 1) Equations (2.7a, b, c) can be inverted to give:

$$\alpha = +\sqrt{\frac{2}{\xi + \xi_3}} \cdot \xi_1, \tag{2.8a}$$

$$\beta = +\sqrt{\frac{2}{\xi + \xi_3}} \cdot \xi_2. \tag{2.8b}$$

2) The values of α and β are restricted by the condition:

$$\alpha^2 + \beta^2 \leq 4\xi \Rightarrow \xi \geq \xi_3 \geq -\xi.$$

3) For

$$\alpha^2 + \beta^2 < 4\xi,$$

the correspondence between α, β and ξ_1, ξ_2, ξ_3 is one-one.

4) For

$$\left|\frac{\alpha}{\sqrt{\xi}}\right| \ll 1, \quad \left|\frac{\beta}{\sqrt{\xi}}\right| \ll 1,$$

we have:

$$\xi_1 \approx \alpha \cdot \sqrt{\xi},$$

$$\xi_2 \approx \beta \cdot \sqrt{\xi},$$

and in this case our canonical α and β behave like the spin-coordinates introduced by Chao in the SLIM-program [2].

2.2 Orbital motion

In terms of the Hamiltonian (2.1) the orbital equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t} X_k = +\frac{\partial \mathscr{H}_{\mathrm{orb}}}{\partial P_k} + \frac{\partial \boldsymbol{\Omega}_0}{\partial P_k} \cdot \boldsymbol{\xi}, \tag{2.9a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_k = -\frac{\partial \mathscr{H}_{\mathrm{orb}}}{\partial X_k} - \frac{\partial \boldsymbol{\Omega}_0}{\partial X_k} \cdot \boldsymbol{\xi}, \tag{2.9b}$$

$(k = 1, 2, 3)$.

The first terms on the rhs of (2.9) are the Lorentz terms and the second terms describe the (very small) Stern-Gerlach force (SG) [15]. Thus our Hamiltonian includes the SG force automatically. Note that here we deal with the relativistic generalization of the SG effect.

2.3 Spin motion

Using (2.7a, b, c) and the relationship

$$\boldsymbol{\Omega}_0 = \Omega_{01} \cdot \mathbf{e}_1 + \Omega_{02} \cdot \mathbf{e}_2 + \Omega_{03} \cdot \mathbf{e}_3, \tag{2.10}$$

the spin-Hamiltonian $\mathscr{H}_{\mathrm{spin}}$

$$\mathscr{H}_{\mathrm{spin}} = \boldsymbol{\Omega}_0 \cdot \boldsymbol{\xi}$$

may also be written as:

$$\mathscr{H}_{\text{spin}} = \sqrt{\xi - \tfrac{1}{4}(\alpha^2+\beta^2)} \cdot [\Omega_{01}\cdot\alpha + \Omega_{02}\cdot\beta] + [\xi - \tfrac{1}{2}(\alpha^2+\beta^2)]\cdot\Omega_{03}. \tag{2.11}$$

Then we obtain the canonical equations of spin motion in the form:

$$\frac{d}{dt}\alpha = +\frac{\partial\mathscr{H}_{\text{spin}}}{\partial\beta}, \tag{2.12a}$$

$$\frac{d}{dt}\beta = -\frac{\partial\mathscr{H}_{\text{spin}}}{\partial\alpha}, \tag{2.12b}$$

leading to:

$$\frac{d}{dt}\alpha = +\frac{-\beta}{4\sqrt{\xi-\frac{1}{4}(\alpha^2+\beta^2)}}\cdot[\Omega_{01}\cdot\alpha+\Omega_{02}\cdot\beta] + \sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)}\cdot\Omega_{02} - \beta\cdot\Omega_{03}, \tag{2.13a}$$

$$\frac{d}{dt}\beta = -\frac{-\alpha}{4\sqrt{\xi-\frac{1}{4}(\alpha^2+\beta^2)}}\cdot[\Omega_{01}\cdot\alpha+\Omega_{02}\cdot\beta] - \sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)}\cdot\Omega_{01} + \alpha\cdot\Omega_{03}. \tag{2.13b}$$

In terms of the components ξ_i we have:

$$\begin{aligned}\frac{d}{dt}\xi_3 &= -\alpha\cdot\frac{d}{dt}\alpha - \beta\cdot\frac{d}{dt}\beta \\ &= -\alpha\sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)}\cdot\Omega_{02} + \beta\sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)}\cdot\Omega_{01} \\ &= \Omega_{01}\cdot\xi_2 - \Omega_{02}\cdot\xi_1.\end{aligned} \tag{2.14a}$$

Similarly we can show that:

$$\frac{d}{dt}\xi_1 = \Omega_{02}\cdot\xi_3 - \Omega_{03}\cdot\xi_2 \tag{2.14b}$$

and

$$\frac{d}{dt}\xi_2 = \Omega_{03}\cdot\xi_1 - \Omega_{01}\cdot\xi_3. \tag{2.14c}$$

Thus:

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix} = \begin{pmatrix}\Omega_{01}\\ \Omega_{02}\\ \Omega_{03}\end{pmatrix} \times \begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}, \tag{2.15a}$$

or

$$\frac{d}{dt}\boldsymbol{\xi} = \boldsymbol{\Omega}_0 \times \boldsymbol{\xi}. \tag{2.15b}$$

So as in [1] our Hamiltonian leads to the BMT-equation [16, 17].

Using (2.7), we can construct the Poisson brackets [3] for the spin components:

$$\{\xi_1, \xi_2\}_{\alpha,\beta} \equiv \frac{\partial\xi_1}{\partial\alpha}\cdot\frac{\partial\xi_2}{\partial\beta} - \frac{\partial\xi_1}{\partial\beta}\cdot\frac{\partial\xi_2}{\partial\alpha} = \xi_3, \tag{2.16a}$$

$$\{\xi_2, \xi_3\}_{\alpha,\beta} = \xi_1, \tag{2.16b}$$

$$\{\xi_3, \xi_1\}_{\alpha,\beta} = \xi_2. \tag{2.16c}$$

These are the classical analogues of the commutation relations among Pauli spin operators.

The result (2.15) can also be obtained by using the equation of motion in the form:

$$\frac{d}{dt}\xi = \{\xi, \mathscr{H}_{\text{spin}}\}_{\alpha,\beta} \tag{2.17}$$

together with the Poisson bracket relations (2.16) and the spin-Hamiltonian

$$\mathscr{H}_{\text{spin}} = \Omega_{01}\cdot\xi_1 + \Omega_{02}\cdot\xi_2 + \Omega_{03}\cdot\xi_3.$$

Remarks: 1) Instead of using the canonical spin variables (α, β) the spin Hamiltonian can be expressed in terms of the canonical variables (J, ψ) via the relations [1, 3]:

$$\alpha = \sqrt{2(\xi - J)}\cdot\cos\psi, \tag{2.18a}$$

$$\beta = \sqrt{2(\xi - J)}\cdot\sin\psi. \tag{2.18b}$$

Since

$$\frac{\beta}{\alpha} = \tan\psi, \tag{2.19a}$$

$$J = \xi - \tfrac{1}{2}(\alpha^2+\beta^2), \tag{2.19b}$$

we obtain the usual results:

$$\begin{aligned}\xi_1 &= \alpha\cdot\sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)} \\ &= \sqrt{2(\xi-J)}\cos\psi\cdot\sqrt{\xi-\tfrac{1}{2}(\xi-J)} \\ &= \sqrt{\xi^2-J^2}\cdot\cos\psi,\end{aligned} \tag{2.20a}$$

$$\xi_2 = \beta\cdot\sqrt{\xi-\tfrac{1}{4}(\alpha^2+\beta^2)} = \sqrt{\xi^2-J^2}\cdot\sin\psi, \tag{2.20b}$$

$$\xi_3 = \xi - \tfrac{1}{2}(\alpha^2+\beta^2) = J. \tag{2.20c}$$

The transformation

$$\alpha, \beta \Rightarrow \psi, J$$

can be obtained from the generating function

$$F_1(\alpha, \psi) = \tfrac{1}{2}\alpha^2\cdot\tan\psi - \xi\cdot\psi.$$

The transformation formulae are then:

$$\beta = +\frac{\partial F_1}{\partial\alpha} = \alpha\cdot\tan\psi, \tag{2.21a}$$

$$\begin{aligned}J = -\frac{\partial F_1}{\partial\psi} &= -\frac{1}{2}\alpha^2\cdot(1+\tan^2\psi) + \xi \\ &= -\frac{1}{2}\alpha^2\cdot\left(1+\frac{\beta^2}{\alpha^2}\right) + \xi \\ &= -\frac{1}{2}(\alpha^2+\beta^2) + \xi,\end{aligned} \tag{2.21b}$$

$$\begin{aligned}\mathscr{H}_{\text{spin}} \to \mathscr{K}_{\text{spin}}(\psi, J) &= \mathscr{H}_{\text{spin}} + \frac{\partial F_1}{\partial s} = \mathscr{H}_{\text{spin}} \\ &= \Omega_{01}\cdot\xi_1 + \Omega_{02}\cdot\xi_2 + \Omega_{03}\cdot\xi_3 \\ &= \sqrt{\xi^2-J^2}\cdot[\Omega_{01}\cdot\cos\psi + \Omega_{02}\cdot\sin\psi] + \Omega_{03}\cdot J,\end{aligned} \tag{2.21c}$$

and one sees that (2.21a, b) lead back to (2.19a, b) consistent with the fact that (ψ, J) are canonical [18].

2) It is easily checked that the pair

$\hat{J}=\xi-J,$

$\hat{\psi}=-\psi$

is also canonical. In terms of $(\hat{J},\hat{\psi})$ we have*

$$\alpha=+\sqrt{2\hat{J}}\cdot\cos\hat{\psi}, \tag{2.22a}$$

$$\beta=-\sqrt{2\hat{J}}\cdot\sin\hat{\psi}. \tag{2.22b}$$

2.4 The combined form of the spin-orbit equations

The combined equations of spin-orbit motion can be written in the form:

$$\frac{d}{dt}X_k=+\frac{\partial\mathscr{H}}{\partial P_k}, \tag{2.23a}$$

$$\frac{d}{dt}P_k=-\frac{\partial\mathscr{H}}{\partial X_k}, \tag{2.23b}$$

$(k=1, 2, 3, 4)$,

with

$$X_4\equiv\alpha, \tag{2.24a}$$

$$P_4\equiv\beta, \tag{2.24b}$$

and

$$\mathscr{H}=\mathscr{H}(X_1, X_2, X_3, X_4; P_1, P_2, P_3, P_4; t). \tag{2.25}$$

Remark. Neglecting the Stern–Gerlach (SG) terms coming from the component $\mathscr{H}_{\text{spin}}$ the orbital part (2.9a, b) of the canonical equations (2.9, 12) can be approximated as:

$$\frac{d}{dt}X_1=+\frac{\partial\mathscr{H}_{\text{orb}}}{\partial P_1},\quad \frac{d}{dt}P_1=-\frac{\partial\mathscr{H}_{\text{orb}}}{\partial X_1}, \tag{2.26a}$$

$$\frac{d}{dt}X_2=+\frac{\partial\mathscr{H}_{\text{orb}}}{\partial P_2},\quad \frac{d}{dt}P_2=-\frac{\partial\mathscr{H}_{\text{orb}}}{\partial X_2}, \tag{2.26b}$$

$$\frac{d}{dt}X_3=+\frac{\partial\mathscr{H}_{\text{orb}}}{\partial P_3},\quad \frac{d}{dt}P_3=-\frac{\partial\mathscr{H}_{\text{orb}}}{\partial X_3}. \tag{2.26c}$$

This canonical system is then separate (and independent) from the spin motion and corresponds to the fully coupled 6-dimensional formalism [19, 20].

If the orbit vector

$$\mathbf{y}=\begin{pmatrix}X_1\\P_1\\X_2\\P_2\\X_3\\P_3\end{pmatrix}$$

is known, we can calculate the spin motion from the equations:

$$\frac{d}{dt}\alpha=+\frac{\partial\mathscr{H}_{\text{spin}}}{\partial\beta},\quad \frac{d}{dt}\beta=-\frac{\partial\mathscr{H}_{\text{spin}}}{\partial\alpha}. \tag{2.27}$$

Methods for a numerical solution of the spin equation (2.27) are described in [6].

2.5 Transition to a new orthonormal dreibein $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ *for spin motion*

We now consider the transformation [3] :

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rightarrow\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3,$

with

$$\frac{d}{dt}u_k(t)=\mathbf{U}(t)\times\mathbf{u}_k(t)\Rightarrow\mathbf{U}=\frac{1}{2}\sum_{k=1}^{3}\mathbf{u}_k\times\frac{d}{dt}\mathbf{u}_k, \tag{2.28}$$

and

$$\boldsymbol{\xi}=\xi_1\cdot\mathbf{e}_1+\xi_2\cdot\mathbf{e}_2+\xi_3\cdot\mathbf{e}_3=\tilde{\xi}_1\cdot\mathbf{u}_1+\tilde{\xi}_2\cdot\mathbf{u}_2+\tilde{\xi}_3\cdot\mathbf{u}_3. \tag{2.29}$$

From (2.15b), (2.28) and (2.29) we obtain:

$$\begin{aligned}\frac{d}{dt}\boldsymbol{\xi}&=\sum_{k=1}^{3}\mathbf{e}_k\times\frac{d}{dt}\xi_k=\boldsymbol{\Omega}_0\times\boldsymbol{\xi}\\&=\sum_{k=1}^{3}\tilde{\xi}_k\cdot\frac{d}{dt}\mathbf{u}_k+\sum_{k=1}^{3}\mathbf{u}_k\cdot\frac{d}{dt}\tilde{\xi}_k\\&=\sum_{k=1}^{3}\tilde{\xi}_k\cdot[\mathbf{U}\times\mathbf{u}_k]+\sum_{k=1}^{3}\mathbf{u}_k\cdot\frac{d}{dt}\tilde{\xi}_k,\end{aligned}$$

and thus

$$\begin{aligned}\sum_{k=1}^{3}\mathbf{u}_k\cdot\frac{d}{dt}\tilde{\xi}_k&=\boldsymbol{\Omega}_0\times\boldsymbol{\xi}-\sum_{k=1}^{3}\tilde{\xi}_k\cdot[\mathbf{U}\times\mathbf{u}_k]\\&=\boldsymbol{\Omega}_0\times\boldsymbol{\xi}-\mathbf{U}\times\sum_{k=1}^{3}\tilde{\xi}_k\mathbf{u}_k\\&=\boldsymbol{\Omega}_0\times\boldsymbol{\xi}-\mathbf{U}\times\boldsymbol{\xi}=[\boldsymbol{\Omega}_0-\mathbf{U}]\times\boldsymbol{\xi}.\end{aligned} \tag{2.30a}$$

Therefore in the new dreibein the equation of spin motion is:

$$\Rightarrow\frac{d}{dt}\tilde{\xi}_k=\mathbf{u}_k\cdot\{[\boldsymbol{\Omega}_0-\mathbf{U}]\times\boldsymbol{\xi}\}. \tag{2.30b}$$

Writing:

$$\tilde{\xi}_1=\tilde{\alpha}\cdot\sqrt{\xi-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}, \tag{2.31a}$$

$$\tilde{\xi}_2=\tilde{\beta}\cdot\sqrt{\xi-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}, \tag{2.31b}$$

$$\tilde{\xi}_3=\xi-\tfrac{1}{2}(\tilde{\alpha}^2+\tilde{\beta}^2), \tag{2.31c}$$

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* The variables $\hat{J}$, $\hat{\psi}$ can be obtained directly from (α,β) using the generating function:

$F_1(\alpha,\hat{\psi})=-\frac{1}{2}\alpha^2\cdot\tan\hat{\psi}.$

In general one can write:

$$\begin{cases}\hat{J}=\xi-J,\\\hat{\psi}=c-\psi.\end{cases}\Rightarrow\begin{cases}\alpha=\sqrt{2\hat{J}}\cdot\cos(c-\hat{\psi}),\\\beta=\sqrt{2\hat{J}}\cdot\sin(c-\hat{\psi}),\end{cases}$$

with an arbitrary constant c (see (2.18a, b)). For $c=\pi/2$ one then has to take the generating function:

$F_1(\alpha,\hat{\psi})=\frac{1}{2}\alpha^2\cdot\cot\hat{\psi},$

leading to $\alpha=\sqrt{2\hat{J}}\cdot\sin\hat{\psi}$; $\beta=\sqrt{2\hat{J}}\cdot\cos\hat{\psi}$

and

$$\mathbf{\Omega}_0=\tilde{\Omega}_{01}\cdot\mathbf{u}_1+\tilde{\Omega}_{02}\cdot\mathbf{u}_2+\tilde{\Omega}_{03}\cdot\mathbf{u}_3, \tag{2.32a}$$

$$\mathbf{U}=\tilde{U}_1\cdot\mathbf{u}_1+\tilde{U}_2\cdot\mathbf{u}_2+\tilde{U}_3\cdot\mathbf{u}_3, \tag{2.32b}$$

we find the new spin-Hamiltonian by replacing the precession vector $\mathbf{\Omega}_0$ in (2.11) by $(\mathbf{\Omega}_0-\mathbf{U})$ (compare (2.30b) with (2.15b)) as in [6]:

$$\begin{aligned}\tilde{\mathscr{H}}_{\rm spin}&=[\tilde{\Omega}_{01}-\tilde{U}_1]\cdot\tilde{\xi}_1+[\tilde{\Omega}_{02}-\tilde{U}_2]\cdot\tilde{\xi}_2+[\tilde{\Omega}_{03}-\tilde{U}_3]\cdot\tilde{\xi}_3\\&=\sqrt{\xi-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}\{[\tilde{\Omega}_{01}-\tilde{U}_1]\cdot\tilde{\alpha}+[\tilde{\Omega}_{02}-\tilde{U}_2]\cdot\tilde{\beta}\}\\&\quad+[\tilde{\Omega}_{03}-\tilde{U}_3]\cdot[1-\tfrac{1}{2}(\tilde{\alpha}^2+\tilde{\beta}^2)\Big].\end{aligned} \tag{2.33}$$

It follows that:

$$\begin{aligned}\frac{d}{dt}\tilde{\alpha}&=+\frac{\partial\tilde{\mathscr{H}}_{\rm spin}}{\partial\tilde{\beta}}\\&=+\frac{-\tilde{\beta}}{4\sqrt{\xi-\frac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}}\cdot\{[\tilde{\Omega}_{01}-\tilde{U}_1]\cdot\tilde{\alpha}+[\tilde{\Omega}_{02}-\tilde{U}_2]\cdot\tilde{\beta}\}\\&\quad+\sqrt{\xi-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}\cdot[\tilde{\Omega}_{02}-\tilde{U}_2]-\tilde{\beta}\cdot[\tilde{\Omega}_{03}-\tilde{U}_3].\end{aligned} \tag{2.34a}$$

$$\begin{aligned}\frac{d}{dt}\tilde{\beta}&=-\frac{\partial\tilde{\mathscr{H}}_{\rm spin}}{\partial\tilde{\alpha}}\\&=-\frac{-\tilde{\alpha}}{4\sqrt{\xi-\frac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}}\cdot\{[\tilde{\Omega}_{01}-\tilde{U}_1]\cdot\tilde{\alpha}+[\tilde{\Omega}_{02}-\tilde{U}_2]\cdot\tilde{\beta}\}\\&\quad-\sqrt{\xi-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}\cdot[\tilde{\Omega}_{01}-\tilde{U}_1]+\tilde{\alpha}\cdot[\tilde{\Omega}_{03}-\tilde{U}_3].\end{aligned} \tag{2.34b}$$

Introducing (as in (2.19) or (2.21)) the spin variables $(\tilde{J},\tilde{\psi})$ via the relations:

$$\tilde{\alpha}=\sqrt{2(\xi-\tilde{J})}\cdot\cos\tilde{\psi}, \tag{2.35a}$$

$$\tilde{\beta}=\sqrt{2(\xi-\tilde{J})}\cdot\sin\tilde{\psi}, \tag{2.35b}$$

or

$$\tilde{\xi}_1=\sqrt{\xi^2-\tilde{J}^2}\cdot\cos\tilde{\psi}, \tag{2.36a}$$

$$\tilde{\xi}_2=\sqrt{\xi^2-\tilde{J}^2}\cdot\sin\tilde{\psi}, \tag{2.36b}$$

$$\tilde{\xi}_3=\tilde{J}, \tag{2.36c}$$

we get in analogy to (2.11c):

$$\begin{aligned}\tilde{\mathscr{H}}_{\rm spin}(\tilde{J},\tilde{\psi})&=\sqrt{\xi^2-\tilde{J}^2}\cdot\{[\tilde{\Omega}_{01}-\tilde{U}_1]\cos\tilde{\psi}\\&\quad+[\tilde{\Omega}_{02}-\tilde{U}_2]\sin\tilde{\psi}\}+[\tilde{\Omega}_{03}-\tilde{U}_3]\cdot\tilde{J},\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}\frac{d}{dt}\tilde{\psi}&=+\frac{\partial}{\partial\tilde{J}}\tilde{\mathscr{H}}_{\rm spin}(\tilde{J},\tilde{\psi})\\&=-\frac{\tilde{J}}{\sqrt{\xi^2-\tilde{J}^2}}\cdot\{[\tilde{\Omega}_{01}-\tilde{U}_1]\cos\tilde{\psi}\\&\quad+[\tilde{\Omega}_{02}-\tilde{U}_2]\sin\tilde{\psi}\}+[\tilde{\Omega}_{03}-\tilde{U}_3],\end{aligned} \tag{2.38a}$$

$$\begin{aligned}\frac{d}{dt}\tilde{J}&=-\frac{\partial}{\partial\tilde{\psi}}\tilde{\mathscr{H}}_{\rm spin}(\tilde{J},\tilde{\psi})\\&=\sqrt{\xi^2-\tilde{J}^2}\cdot\{-[\tilde{\Omega}_{01}-\tilde{U}_1]\sin\tilde{\psi}+[\tilde{\Omega}_{02}-\tilde{U}_2]\cos\tilde{\psi}\}.\end{aligned} \tag{2.38b}$$

Remarks. 1) If the rotation vector $\mathbf{U}$ defined by (2.28) is independent of the variables (X_k, P_k), then the orbital equations of motion can also be written as:

$$\frac{d}{dt}X_k=+\frac{\partial\mathscr{H}_{\rm orb}}{\partial P_k}+\boldsymbol{\xi}\cdot\frac{\partial}{\partial P_k}[\mathbf{\Omega}_0-\mathbf{U}], \tag{2.39a}$$

$$\frac{d}{dt}P_k=-\frac{\partial\mathscr{H}_{\rm orb}}{\partial X_k}-\boldsymbol{\xi}\cdot\frac{\partial}{\partial X_k}[\mathbf{\Omega}_0-\mathbf{U}], \tag{2.39b}$$

$(k=1, 2, 3)$.

With respect to the variables X_k, P_k, $\tilde{\alpha}$, $\tilde{\beta}$ we thus obtain (see (2.33) and (2.34)) the new Hamiltonian for the combined spin-orbit system in the form:

$$\tilde{\mathscr{H}}=\mathscr{H}_{\rm orb}+[\mathbf{\Omega}_0-\mathbf{U}]\cdot\boldsymbol{\xi}, \tag{2.40}$$

whereby in addition (2.29), (2.30) and (2.32) have to be used.

2) In the following chapters we introduce new sets of variables. But the modification of the Hamiltonian needed to affect a transformation of the dreibein has a form similar to that in (2.40) since the new BMT-equation will have a structure similar to that in (2.15b) [1].

3 Introduction of machine coordinates

3.1 Reference trajectory and coordinate frame

So far (2.1) we have been using a fixed coordinate system with the coordinates X_1, X_2 and X_3. We now wish to describe the motion in terms of the natural coordinates x, z, s in a suitable curvilinear coordinate system [21], i.e. in accelerator coordinates.

In this natural coordinate system an arbitrary orbit-vector $\mathbf{r}(s)$ can be written in the form:

$$\mathbf{r}(x,z,s)=\mathbf{r}_0(s)+x(s)\cdot\mathbf{e}_x(s)+z(s)\cdot\mathbf{e}_z(s), \tag{3.1a}$$

where:

$$\begin{cases}\dfrac{d}{ds}\mathbf{e}_x(s)=K_x(s)\cdot\mathbf{e}_s(s),\\[2mm]\dfrac{d}{ds}\mathbf{e}_z(s)=K_z(s)\cdot\mathbf{e}_s(s),\\[2mm]\dfrac{d}{ds}\mathbf{e}_z(s)=-K_x(s)\cdot\mathbf{e}_x(s)-K_z(s)\cdot\mathbf{e}_z(s),\ \mathbf{e}_s(s)\equiv\dfrac{d}{ds}\mathbf{r}_0(s)\end{cases} \tag{3.1b}$$

(for more details see [1, 19]).

The transformation of the spin components from the $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$-basis to the $(\mathbf{e}_s, \mathbf{e}_x, \mathbf{e}_z)$-basis

$$\xi_1, \xi_2, \xi_3 \Rightarrow \xi_s, \xi_x, \xi_z \tag{3.2}$$

is merely a rotation and is defined by:

$$\boldsymbol{\xi}=\xi_1\cdot\mathbf{e}_1+\xi_2\cdot\mathbf{e}_2+\xi_3\cdot\mathbf{e}_3=\xi_s\cdot\mathbf{e}_s+\xi_x\cdot\mathbf{e}_x+\xi_z\cdot\mathbf{e}_z. \tag{3.3}$$

If, by analogy to (2.7), we introduce canonical variables α', β' for ξ_s, ξ_x, ξ_z:

$$\begin{cases}\xi_s=\alpha'\cdot\sqrt{\xi-\frac{1}{4}(\alpha'^2+\beta'^2)},\\ \xi_x=\beta'\cdot\sqrt{\xi-\frac{1}{4}(\alpha'^2+\beta'^2)},\\ \xi_z=\xi-\frac{1}{2}(\alpha'^2+\beta'^2),\end{cases} \tag{3.4}$$

then (3.2) is equivalent to a canonical transformation:

$$\alpha, \beta \Rightarrow \alpha', \beta' \tag{3.5}$$

(see (2.40)).

3.2 The spin-orbit hamiltonian in terms of machine coordinates

The variables x and z in (3.1) describe the amplitudes of transverse motion.

To describe the longitudinal motion (synchrotron oscillations) we introduce two additional small oscillating variables σ and p_σ [1] with

$$\sigma=s-v_0\cdot t, \tag{3.6}$$

and

$$p_\sigma=\frac{1}{\beta_0^2}\cdot\eta, \tag{3.7}$$

where v_0 and η are given by

$$v_0=\text{design speed}=c\beta_0;\quad \beta_0=\sqrt{1-\left(\frac{m_0c^2}{E_0}\right)^2},$$

and

$$\eta=\frac{\Delta E}{E_0}. \tag{3.8}$$

The variable σ measures the delay in arrival time at position s of a particle and is the longitudinal separation of the particle from the centre of the bunch. The quantity η is the relative energy deviation of the particle.

Using this complete set of orbital variables defined in the machine coordinate system we are in a position to provide an analytical description for the orbital motion by a simultaneous treatment of longitudinal and transverse oscillations.

Starting then from the Hamiltonian (2.1) for the spin-orbit motion of a charged particle in an electromagnetic field, we can construct the Hamiltonian of the spin-orbit system with respect to the new variables

x, z, σ, α'

[1, 19, 20] by a succession of canonical transformations combined with an s-t exchange (introducing the length s along the design orbit as the independent variable instead of the time t) and a scale transformation leading to the modified spin variables

$$\bar{\alpha}\equiv\frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\alpha',\quad \bar{\beta}\equiv\frac{1}{\beta_0}\sqrt{\frac{v_0}{E_0}}\cdot\beta',$$

and the modified spin vector

$$\hat{\boldsymbol{\xi}}=\hat{\xi}_s\cdot\mathbf{e}_s+\hat{\xi}_x\cdot\mathbf{e}_x+\hat{\xi}_z\cdot\mathbf{e}_z=\frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\boldsymbol{\xi}, \tag{3.9}$$

$$\begin{cases}\hat{\xi}_s=\bar{\alpha}\cdot\sqrt{\hat{\xi}-\frac{1}{4}(\bar{\alpha}^2+\bar{\beta}^2)},\\ \hat{\xi}_x=\bar{\beta}\cdot\sqrt{\hat{\xi}-\frac{1}{4}(\bar{\alpha}^2+\bar{\beta}^2)},\\ \hat{\xi}_z=\hat{\xi}-\frac{1}{2}(\bar{\alpha}^2+\bar{\beta}^2),\end{cases} \tag{3.10}$$

of length*:

$$\hat{\xi}=\frac{1}{\beta_0^2}\frac{v_0}{E_0}\cdot\xi. \tag{3.11}$$

Choosing a gauge with $\phi=0$ (e.g. Coulomb gauge) we then obtain (to first order in $\hbar$)** [1]

$$\hat{\mathscr{H}}=\hat{\mathscr{H}}_{\text{orb}}+\mathscr{H}_{\text{spin}} \tag{3.12}$$

with

$$\begin{aligned}\hat{\mathscr{H}}_{\text{orb}}=&\,p_\sigma-(1+\hat{\eta})\cdot[1+K_x\cdot x+K_z\cdot z]\\ &\cdot\left\{1-\frac{\left(p_x-\frac{e}{p_0\cdot c}A_x\right)^2+\left(p_z+\frac{e}{p_0\cdot c}A_z\right)^2}{(1+\hat{\eta})^2}\right\}^{1/2}\\ &-[1+K_x\cdot x+K_z\cdot z]\cdot\frac{e}{p_0\cdot c}A_s,\end{aligned} \tag{3.13}$$

and***

$$\begin{aligned}\hat{\mathscr{H}}_{\text{spin}}&=\boldsymbol{\Omega}\cdot\hat{\boldsymbol{\xi}}\\ &=(\hat{\xi}_s\cdot\mathbf{e}_s+\hat{\xi}_x\cdot\mathbf{e}_x+\hat{\xi}_z\cdot\mathbf{e}_z)\cdot\boldsymbol{\Omega}(x,z,\sigma;p_x,p_z,p_\sigma;s)\\ &=\left(\alpha\cdot\sqrt{\hat{\xi}-\frac{1}{4}(\alpha^2+\beta^2)},\beta\right.\\ &\quad\left.\cdot\sqrt{\hat{\xi}-\frac{1}{4}(\alpha^2+\beta^2)},\hat{\xi}-\frac{1}{2}(\alpha^2+\beta^2)\right)\begin{pmatrix}\Omega_s\\ \Omega_x\\ \Omega_z\end{pmatrix},\end{aligned} \tag{3.14}$$

where the precession vector

$$\boldsymbol{\Omega}=\Omega_s\cdot\mathbf{e}_s+\Omega_x\cdot\mathbf{e}_x+\Omega_z\cdot\mathbf{e}_z,$$

* Using the relations $\xi=\frac{\hbar}{2}$, $E_0=\gamma_0 m_0c^2$, $\lambda_C=\frac{2\pi\hbar}{m_0c}$ (Compton-wavelength), the quantity $\hat{\xi}$ can be written in the form $\hat{\xi}=\frac{(\hbar/2)}{\gamma_0 m_0 v_0}=\frac{1}{4\pi}\cdot\lambda$ with $\lambda=\frac{1}{\beta_0\gamma_0}\cdot\lambda_C$ denoting the de Broglie wavelength of a particle with energy E_0. Note, that $\hat{\xi}$ as well as x, z, σ, $\bar{\alpha}^2$, $\bar{\beta}^2$ has the dimension of a length

** Since as in [1] the Hamiltonian (2.1) is based on a classical interpretation of a semiclassical Hamiltonian we work only to first order in $\hbar$

*** To simplify the notation we now write the spin coordinates as α, β instead of $\bar{\alpha}$, $\bar{\beta}$

is given by:

$$\mathbf{\Omega}(x, z, \sigma; p_x, p_z, p_\sigma; s)$$

$$= [1 + K_x \cdot x + K_z \cdot z] \cdot \frac{(1+\eta)}{\beta_0 (1+\hat{\eta})} \cdot \left\{ 1 - \frac{\left(p_x - \frac{e}{p_0 \cdot c} A_x\right)^2 + \left(p_z + \frac{e}{p_0 \cdot c} A_z\right)^2}{(1+\hat{\eta})^2} \right\}^{-1/2} \cdot \frac{1}{c} \mathbf{\Omega}_0 + K_x \cdot \mathbf{e}_z - K_z \cdot \mathbf{e}_x, \tag{3.15}$$

and where the quantity $\hat{\eta}$ appearing in (3.13) and (3.15) is defined by:

$$(1+\hat{\eta}) = \frac{1}{\beta_0} \sqrt{(1+\eta)^2 - \left(\frac{m_0 c^2}{E_0}\right)^2} = \frac{1}{\beta_0} \cdot \frac{p \cdot c}{E_0} = \frac{p}{p_0}, \tag{3.16a}$$

$$\hat{\eta} = \frac{p}{p_0} - 1 = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0}, \tag{3.16b}$$

$(p = m_0 \gamma v)$.

In the following we assume that the ring consists of bending magnets, quadrupoles, skew quadrupoles, solenoids, cavities and dipole correction coils. Then the vector potential $\mathbf{A}$ can be written as [22]:

$$\frac{e}{p_0 \cdot c} A_s = -\frac{1}{2}[1 + K_x \cdot x + K_z \cdot z] + \frac{1}{2} g \cdot (z^2 - x^2) + N \cdot xz - \frac{1}{\beta_0^2} \cdot \frac{L}{2\pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos\left[h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi\right] + \frac{e}{p_0 \cdot c} \cdot (\Delta \mathscr{B}_x \cdot z - \Delta \mathscr{B}_z \cdot x),$$

$$\frac{e}{p_0 \cdot c} A_x = -H \cdot z, \quad \frac{e}{p_0 \cdot c} A_z = +H \cdot x,$$

(h = harmonic number) with g, N, H and K defined in [1].

Expressing also the precession vector $\mathbf{\Omega}_0$ in (3.15) (see (2.3)) in machine coordinates, we obtain the Hamiltonian for the canonical variables

$$x, z, \sigma, \alpha; p_x, p_z, p_\sigma, \beta,$$

and the canonical equations of motion read as:

$$\frac{d}{ds} x = +\frac{\partial \hat{\mathscr{H}}}{\partial p_x}, \quad \frac{d}{ds} p_x = -\frac{\partial \hat{\mathscr{H}}}{\partial x}, \tag{3.17a}$$

$$\frac{d}{ds} z = +\frac{\partial \hat{\mathscr{H}}}{\partial p_z}, \quad \frac{d}{ds} p_z = -\frac{\partial \hat{\mathscr{H}}}{\partial z}, \tag{3.17b}$$

$$\frac{d}{ds} \sigma = +\frac{\partial \hat{\mathscr{H}}}{\partial p_\sigma}, \quad \frac{d}{ds} p_\sigma = -\frac{\partial \hat{\mathscr{H}}}{\partial \sigma}, \tag{3.17c}$$

$$\frac{d}{ds} \alpha = +\frac{\partial \hat{\mathscr{H}}}{\partial \beta}, \quad \frac{d}{ds} \beta = -\frac{\partial \hat{\mathscr{H}}}{\partial \alpha}. \tag{3.17d}$$

This can be written in matrix form:

$$\frac{d}{ds} \hat{\mathbf{y}} = -\underline{\hat{S}} \cdot \frac{\partial \hat{\mathscr{H}}}{\partial y}, \tag{3.18}$$

with

$$\hat{\mathbf{y}}^T = (x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta), \tag{3.19}$$

where the matrix $\underline{\hat{S}}$ is given by:

$$\underline{S} = \begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix}, \quad \underline{S}_2 = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}. \tag{3.20}$$

Remark. Equation (3.13) is valid only for 'protons', or more precisely, for all situations where radiation is negligible. For electrons we need the extra-term in the Hamiltonian [1]

$$\mathscr{H}_{\text{rad}} = C_1 \cdot [K_x^2 + K_z^2] \cdot \sigma, \tag{3.21}$$

$$\left(\text{where } C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0}\right)$$

(for $v_0 \approx c$) in order to describe the energy loss by radiation in the bending magnets [23, 24].

4 Introduction of an eight-dimensional closed orbit

As can be seen by a series expansion of the Hamiltonian, $\hat{\mathscr{H}}$ contains terms linear in the coordinates [1]. As in I these linear terms can be eliminated by introducing a new 8-dimensional reference orbit. This orbit can then also be used to construct a new reference frame for the spin motion. In the new variables spin-orbit maps are origin preserving*

4.1 Definition of the eight-dimensional closed orbit

The 8-dimensional closed orbit:

$$\hat{\mathbf{y}}_0 \equiv (\mathbf{y}_0(s), \alpha_0(s), \beta_0(s))$$

contains a periodic orbital part

$$\mathbf{y}_0^T = (x_0, p_{x0}; z_0, p_{z0}; \sigma_0, p_{\sigma 0}),$$

with

$$\mathbf{y}_0(s+L) = \mathbf{y}_0(s), \tag{4.1a}$$

and a periodic spin part α_0, β_0:

$$\alpha_0(s+L) = \alpha_0(s),$$

$$\beta_0(s+L) = \beta_0(s),$$

which defines (see (3.10)) a periodic spin vector

$$\hat{\boldsymbol{\xi}}_0(s) \equiv \hat{\xi}_{0s} \cdot \mathbf{e}_s + \hat{\xi}_{0x} \cdot \mathbf{e}_x + \hat{\xi}_{0z} \cdot \mathbf{e}_z = \hat{\boldsymbol{\xi}}_0(s+L), \tag{4.1b}$$

via

$$\begin{cases} \hat{\xi}_{0s} = \alpha_0 \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\alpha_0^2 + \beta_0^2)}, \\ \hat{\xi}_{0x} = \beta_0 \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\alpha_0^2 + \beta_0^2)}, \\ \hat{\xi}_{0z} = \hat{\xi} - \frac{1}{2}(\alpha_0^2 + \beta_0^2). \end{cases} \tag{4.2}$$

* To introduce normal forms we need origin preserving transformations

The equations of motion read as:

$$\frac{d}{ds}\hat{\mathbf{y}}_0 = -\hat{\underline{S}}\cdot\frac{\partial}{\partial\hat{\mathbf{y}}_0}\hat{\mathscr{H}}(\hat{\mathbf{y}}_0, s),$$

or

$$\frac{d}{ds}\mathbf{y}_0 = \underline{S}\cdot\frac{\partial}{\partial\mathbf{y}_0}\hat{\mathscr{H}}(\mathbf{y}_0;\alpha_0,\beta_0;s), \tag{4.3a}$$

$$\mathbf{e}_s\cdot\frac{d}{ds}\hat{\xi}_{0s}+\mathbf{e}_x\cdot\frac{d}{ds}\hat{\xi}_{0x}+\mathbf{e}_z\cdot\frac{d}{ds}\hat{\xi}_{0z}=\mathbf{\Omega}^{(0)}\times\hat{\boldsymbol{\xi}}_0, \tag{4.3b}$$

with

$$\mathbf{\Omega}^{(0)}\equiv\mathbf{\Omega}(\mathbf{y}_0, s) \tag{4.4}$$

$$\underline{S} = -\begin{pmatrix} \underline{S}_2 & \underline{0} & \underline{0} \\ \underline{0} & \underline{S}_2 & \underline{0} \\ \underline{0} & \underline{0} & \underline{S}_2 \end{pmatrix}, \tag{4.5}$$

and $\underline{S}_2$ given by (3.20). So $[\mathbf{y}_0(s), \alpha_0(s), \beta_0(s)]$ is a periodic solution of the combined equations of motion.

Using $\hat{\boldsymbol{\xi}}_0$ we can now construct a periodic spin frame $(\mathbf{n}_0, \mathbf{m}, \mathbf{l})$:

$$[\mathbf{n}_0(s+L), \mathbf{m}(s+L), \mathbf{l}(s+L)] = [\mathbf{n}_0(s), \mathbf{m}(s), \mathbf{l}(s)],$$

along the closed orbit [1] with

$$\mathbf{n}_0 = \hat{\boldsymbol{\xi}}_0/|\hat{\boldsymbol{\xi}}_0|, \tag{4.6a}$$

$$\mathbf{n}_0(s)\perp\mathbf{m}(s)\perp\mathbf{l}(s), \tag{4.6b}$$

$$\mathbf{n}_0(s)=\mathbf{m}(s)\times\mathbf{l}(s), \tag{4.6c}$$

$$|\mathbf{n}_0(s)|=|\mathbf{m}(s)|=|\mathbf{l}(s)|=1, \tag{4.6d}$$

and

$$\mathbf{e}_s\cdot\frac{d}{ds}n_{0s}+\mathbf{e}_x\cdot\frac{d}{ds}n_{0x}+\mathbf{e}_z\cdot\frac{d}{ds}n_{0z}=\mathbf{\Omega}^{(0)}\times\mathbf{n}_0(s), \tag{4.7a}$$

$$\mathbf{e}_s\cdot\frac{d}{ds}m_s+\mathbf{e}_x\cdot\frac{d}{ds}m_x+\mathbf{e}_z\cdot\frac{d}{ds}m_z=\mathbf{\Omega}^{(0)}\times\mathbf{m}(s)+\mathbf{l}(s)\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{4.7b}$$

$$\mathbf{e}_s\cdot\frac{d}{ds}l_s+\mathbf{e}_x\cdot\frac{d}{ds}l_x+\mathbf{e}_z\cdot\frac{d}{ds}l_z=\mathbf{\Omega}^{(0)}\times\mathbf{l}(s)-\mathbf{m}(s)\cdot\frac{d}{ds}\psi_{\text{spin}}(s), \tag{4.7c}$$

$$\psi_{\text{spin}}(s+L)-\psi_{\text{spin}}(s)=2\pi\cdot Q_{\text{spin}}, \tag{4.8}$$

whereby we have used:

$$\mathbf{n}_0=n_{0s}\cdot\mathbf{e}_s+n_{0x}\cdot\mathbf{e}_x+n_{0z}\cdot\mathbf{e}_z,$$

$$\mathbf{m}=m_s\cdot\mathbf{e}_s+m_x\cdot\mathbf{e}_x+m_z\cdot\mathbf{e}_z,$$

$$\mathbf{l}=l_s\cdot\mathbf{e}_s+l_x\cdot\mathbf{e}_x+l_z\cdot\mathbf{e}_z.$$

4.2 The oscillations around the closed orbit

We now use the 8-dimensional closed orbit together with $\mathbf{l}(s)$, $\mathbf{m}(s)$ to construct new canonical spin-orbit variables. The canonical transformation for orbit and spin will be carried out separately.

4.2.1 Canonical transformation for the spin variables. Following the method of Section 2.5 we transform from the $\mathbf{e}_x, \mathbf{e}_z, \mathbf{e}_s$ basis to the $\mathbf{n}_0, \mathbf{m}, \mathbf{l}$ basis:

$$\hat{\xi}_s, \hat{\xi}_x, \hat{\xi}_z \Rightarrow \hat{\xi}_n, \hat{\xi}_m, \hat{\xi}_l, \tag{4.9}$$

with

$$\hat{\boldsymbol{\xi}}=\hat{\xi}_s\cdot\mathbf{e}_s+\hat{\xi}_x\cdot\mathbf{e}_x+\hat{\xi}_z\cdot\mathbf{e}_z=\hat{\xi}_n\cdot\mathbf{n}_0+\hat{\xi}_m\cdot\mathbf{m}+\hat{\xi}_l\cdot\mathbf{l}, \tag{4.10}$$

and introduce for $\hat{\xi}_n, \hat{\xi}_m, \hat{\xi}_l$ the canonical variables $\tilde{\alpha}, \tilde{\beta}$:

$$\hat{\xi}_m=\tilde{\alpha}\cdot\sqrt{\hat{\xi}-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}, \tag{4.11a}$$

$$\hat{\xi}_l=\tilde{\beta}\cdot\sqrt{\hat{\xi}-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2)}, \tag{4.11b}$$

$$\hat{\xi}_n=\hat{\xi}-\tfrac{1}{4}(\tilde{\alpha}^2+\tilde{\beta}^2). \tag{4.11c}$$

Then (4.9) is a canonical transformation:

$$\alpha,\beta\Rightarrow\tilde{\alpha},\tilde{\beta}, \tag{4.12}$$

and the new Hamiltonian $\tilde{\mathscr{H}}$ reads as:

$$\tilde{\mathscr{H}}(x,z,\sigma,\tilde{\alpha};p_x,p_z,p_\sigma,\tilde{\beta};s)=\tilde{\mathscr{H}}_{\text{orb}}+\tilde{\mathscr{H}}_{\text{spin}}, \tag{4.13}$$

with

$$\tilde{\mathscr{H}}_{\text{orb}}(x,z,\sigma;p_x,p_z,p_\sigma;s)\equiv\hat{\mathscr{H}}_{\text{orb}}; \tag{4.14a}$$

$$\begin{aligned}\tilde{\mathscr{H}}_{\text{spin}}&(x,z,\sigma,\tilde{\alpha};p_x,p_z,p_\sigma,\tilde{\beta};s)\\ &=\{\mathbf{\Omega}(x,z,\sigma;p_x,p_z,p_\sigma;s)-\mathbf{U}(x,z,\sigma;p_x,p_z,p_\sigma;s)\}\\ &\quad\cdot(\hat{\xi}_n\cdot\mathbf{n}_0+\hat{\xi}_m\cdot\mathbf{m}+\hat{\xi}_l\cdot\mathbf{l}),\end{aligned} \tag{4.14b}$$

(see (2.28) and (2.40)) and

$$\begin{aligned}\mathbf{U}=\frac{1}{2}\bigg[&\mathbf{n}_0\times\left(\mathbf{e}_s\cdot\frac{d}{ds}n_{0s}+\mathbf{e}_x\cdot\frac{d}{ds}n_{0x}+\mathbf{e}_z\cdot\frac{d}{ds}n_{0z}\right)\\ &+\mathbf{m}\times\left(\mathbf{e}_s\cdot\frac{d}{ds}m_s+\mathbf{e}_x\cdot\frac{d}{ds}m_x+\mathbf{e}_z\cdot\frac{d}{ds}m_z\right)\\ &+\mathbf{l}\times\left(\mathbf{e}_s\cdot\frac{d}{ds}l_s+\mathbf{e}_x\cdot\frac{d}{ds}l_x+\mathbf{e}_z\cdot\frac{d}{ds}l_z\right)\bigg]\\ =\frac{1}{2}\bigg[&\mathbf{n}_0\times(\mathbf{\Omega}^{(0)}+\mathbf{n}_0)+\mathbf{m}\times\left(\mathbf{\Omega}^{(0)}\times\mathbf{m}+\mathbf{l}\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\right)\\ &+\mathbf{l}\times\left(\mathbf{\Omega}^{(0)}\times\mathbf{l}-\mathbf{m}\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\right)\bigg]\\ =\frac{1}{2}\bigg[&3\mathbf{\Omega}^{(0)}-\mathbf{n}_0\cdot(\mathbf{\Omega}^{(0)}\cdot\mathbf{n}_0)-\mathbf{m}\cdot(\mathbf{\Omega}^{(0)}\cdot\mathbf{m})\\ &-\mathbf{l}\cdot(\mathbf{\Omega}^{(0)}\cdot\mathbf{l})+(\mathbf{n}_0+\mathbf{n}_0)\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\bigg]\\ =\frac{1}{2}\bigg[&3\mathbf{\Omega}^{(0)}-\mathbf{\Omega}^{(0)}+2\mathbf{n}_0\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\bigg]\\ =\mathbf{\Omega}^{(0)}&+\mathbf{n}_0\cdot\frac{d}{ds}\psi_{\text{spin}}(s)\bigg].\end{aligned} \tag{4.15}$$

Thus we find:

$$
\begin{aligned}
&\tilde{\mathscr{H}}_{\text{spin}}(x, z, \sigma, \tilde{\alpha}; p_x, p_z, p_\sigma, \tilde{\beta}; s) \\
&= \left\{ \mathbf{\Omega} - \mathbf{\Omega}^{(0)} - \mathbf{n}_0 \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi_{\text{spin}}(s) \right\} \cdot (\hat{\xi} \cdot \mathbf{n}_0 + \hat{\xi}_m \cdot \mathbf{m} + \hat{\xi}_l \cdot \mathbf{l}) \\
&= \boldsymbol{\omega}(x, z, \sigma; p_x, p_z, p_\sigma; s) \cdot [\hat{\xi}_n \cdot \mathbf{n}_0 + \hat{\xi}_m \cdot \mathbf{m} + \hat{\xi}_l \cdot \mathbf{l}] \\
&\quad - \hat{\xi}_n \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi_{\text{spin}}(s) \\
&= [\hat{\xi}_n \cdot (\mathbf{n}_0 \cdot \boldsymbol{\omega}) + \hat{\xi}_m \cdot (\mathbf{m} \cdot \boldsymbol{\omega}) + \hat{\xi}_l \cdot (\mathbf{l} \cdot \boldsymbol{\omega})] - \hat{\xi}_n \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi_{\text{spin}}(s) \\
&= (\hat{\xi}_n, \hat{\xi}_m, \hat{\xi}_l) \cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} - \hat{\xi}_n \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi_{\text{spin}}(s) \\
&= \left(\hat{\xi} - \frac{1}{2}(\tilde{\alpha}^2 + \tilde{\beta}^2), \tilde{\alpha} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)}, \tilde{\beta} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)} \right) \\
&\quad \cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} \\
&\quad - \left[\hat{\xi} - \frac{1}{2}(\tilde{\alpha}^2 + \tilde{\beta}^2) \right] \cdot \frac{\mathrm{d}}{\mathrm{d}s} \psi_{\text{spin}}(s), \qquad (4.16)
\end{aligned}
$$

where we have introduced for abbreviation the vector

$$\boldsymbol{\omega} \equiv \omega_s \cdot \mathbf{e}_s + \omega_x \cdot \mathbf{e}_x + \omega_z \cdot \mathbf{e}_z = \mathbf{\Omega} - \mathbf{\Omega}^{(0)}. \qquad (4.17)$$

This is equivalent to the form for the spin Hamiltonian given by Derbenev [4].

Using (3.15) and (4.17) and writing:

$$\hat{\mathbf{y}} = \mathbf{y} - \mathbf{y}_0, \qquad (4.18)$$

the vector $\boldsymbol{\omega}$ can be linearised with respect to the orbital variables as in paper I so that in the spin-Hamiltonian (4.15) we can put:

$$\begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F}_{(3 \times 6)} \cdot \begin{pmatrix} \tilde{x} \\ \tilde{p}_x \\ \tilde{z} \\ \tilde{p}_z \\ \tilde{\sigma} \\ \tilde{p}_\sigma \end{pmatrix}, \qquad (4.19)$$

with the F_{ij} as in [1]*.

* The full $\boldsymbol{\omega}$ could be used if needed, but the linearized form is often sufficient [2, 41, 25]

With (3.13), (4.13), (4.14a), (4.16) and (4.19) we have the Hamiltonian for the canonical variables

$$x, z, \sigma, \tilde{\alpha}; p_x, p_z, p_\sigma, \tilde{\beta}.$$

4.2.2 Transformation of the orbital variables. The orbit vector $\mathbf{y}(s)$ can be written as a sum of two components (see (4.18)):

$$\mathbf{y}(s) = \mathbf{y}_0(s) + \tilde{\mathbf{y}}(s), \qquad (4.20)$$

where the vector $\tilde{\mathbf{y}}(s)$ describes the synchro-betatron oscillations about the new closed equilibrium trajectory $\mathbf{y}_0(s)$.

The transformation

$$\mathbf{y}; \tilde{\alpha}, \tilde{\beta} \Rightarrow \tilde{\mathbf{y}}; \tilde{\alpha}' = \tilde{\alpha}, \tilde{\beta}' = \tilde{\beta} \qquad (4.21)$$

can be obtained from the generating function [1]:

$$
\begin{aligned}
&F_2(x, \tilde{p}_x; z, \tilde{p}_z; \sigma, \tilde{p}_\sigma; \tilde{\alpha}, \tilde{\beta}'; s) \\
&\quad = (x - x_0) \cdot (\tilde{p}_x + p_{x0}) + (z - z_0) \cdot (\tilde{p}_z + p_{z0}) \\
&\quad\quad + (\sigma - \sigma_0) \cdot (\tilde{p}_\sigma + p_{\sigma 0}) + \tilde{\alpha} \cdot \tilde{\beta}' + f(s), \qquad (4.22)
\end{aligned}
$$

with an arbitrary function $f(s)$. The transformation equations read as:

$$p_x = \frac{\partial F_2}{\partial x} = \tilde{p}_x + p_{x0}, \quad \tilde{x} = \frac{\partial F_2}{\partial \tilde{p}_x} = x - x_0, \qquad (4.23a)$$

$$p_z = \frac{\partial F_2}{\partial z} = \tilde{p}_z + p_{z0}, \quad \tilde{z} = \frac{\partial F_2}{\partial \tilde{p}_z} = z - z_0, \qquad (4.23b)$$

$$p_\sigma = \frac{\partial F_2}{\partial \sigma} = \tilde{p}_\sigma + p_{\sigma 0}, \quad \tilde{\sigma} = \frac{\partial F_2}{\partial \tilde{p}_\sigma} = \sigma - \sigma_0, \qquad (4.23c)$$

which reproduce the defining equation (4.20) for $\tilde{\mathbf{y}}$.

Choosing the function $f(s)$ such that $\frac{\mathrm{d}}{\mathrm{d}s} f(s)$ becomes:

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s) = x_0(s) \cdot \frac{\mathrm{d}}{\mathrm{d}s} p_{x0}(s) + z_0(s) \cdot \frac{\mathrm{d}}{\mathrm{d}s} p_{z0}(s) + \sigma_0(s) \cdot \frac{\mathrm{d}}{\mathrm{d}s} p_{\sigma 0}(s),$$

we furthermore have:

$$
\begin{aligned}
\frac{\partial F_2}{\partial s} &= -\frac{\mathrm{d}x_0}{\mathrm{d}s} \cdot p_x + \frac{\mathrm{d}p_{x0}}{\mathrm{d}s} \cdot x - \frac{\mathrm{d}z_0}{\mathrm{d}s} \cdot p_z \\
&\quad + \frac{\mathrm{d}p_{z0}}{\mathrm{d}s} \cdot z - \frac{\mathrm{d}\sigma_0}{\mathrm{d}s} \cdot p_\sigma + \frac{\mathrm{d}p_{\sigma 0}}{\mathrm{d}s} \cdot \sigma \\
&= -p_x \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial p_x} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} - x \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial x} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} \\
&\quad - p_z \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial p_z} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} - z \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial z} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} \\
&\quad - p_\sigma \left(\frac{\partial \tilde{\mathscr{H}}}{\partial p_\sigma} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} - \sigma \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial \sigma} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0} \\
&= -\mathbf{y} \cdot \left(\frac{\partial \tilde{\mathscr{H}}}{\partial \mathbf{y}} \right)_{\mathbf{y}=\mathbf{y}_0; \tilde{\alpha}=\tilde{\beta}=0}, \qquad (4.24)
\end{aligned}
$$

and therefore:

$$\tilde{\mathcal{K}} \equiv \tilde{\mathcal{K}}_{\text{orb}} + \tilde{\mathcal{K}}_{\text{spin}} \to \tilde{\mathcal{H}} = \tilde{\mathcal{K}} - \mathbf{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}}{\partial \mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\tilde{\alpha}=\tilde{\beta}=0}$$

$$= \tilde{\mathcal{K}}_{\text{orb}} - \mathbf{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}_{\text{orb}}}{\partial \mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\tilde{\alpha}=\tilde{\beta}=0}$$

$$+ \tilde{\mathcal{K}}_{\text{spin}} - \mathbf{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}_{\text{spin}}}{\partial \mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\tilde{\alpha}=\tilde{\beta}=0}$$

$$= \tilde{\mathcal{H}}_{\text{orbit}} + \tilde{\mathcal{H}}_{\text{spin}}, \tag{4.25}$$

with

$$\tilde{\mathcal{H}}_{\text{orb}} = \tilde{\mathcal{K}}_{\text{orb}} - \mathbf{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}_{\text{orb}}}{\partial \mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\tilde{\alpha}=\tilde{\beta}=0}, \tag{4.26a}$$

$$\tilde{\mathcal{H}}_{\text{spin}} = \tilde{\mathcal{K}}_{\text{spin}} - \mathbf{y} \cdot \left(\frac{\partial \tilde{\mathcal{K}}_{\text{spin}}}{\partial \mathbf{y}}\right)_{\mathbf{y}=\mathbf{y}_0;\tilde{\alpha}=\tilde{\beta}=0}. \tag{4.26b}$$

For the linearised form of $\boldsymbol{\omega}$ (see (4.19)), (4.16) and (4.26) lead to:

$$\tilde{\mathcal{H}}_{\text{spin}}(\tilde{x}, \tilde{z}, \tilde{\sigma}, \tilde{\alpha}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \tilde{\beta}; s)$$

$$= \left(\hat{\xi} - \frac{1}{2}(\tilde{\alpha}^2 + \tilde{\beta}^2), \tilde{\alpha} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)}, \tilde{\beta} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)}\right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{1}{2}[\tilde{\alpha}^2 + \tilde{\beta}^2] \cdot \frac{d}{ds} \psi_{\text{spin}}(s)$$

$$- (\hat{\xi}, 0, 0) \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$= \left(-\frac{1}{2}(\tilde{\alpha}^2 + \tilde{\beta}^2), \tilde{\alpha} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)}, \tilde{\beta} \cdot \sqrt{\hat{\xi} - \frac{1}{4}(\tilde{\alpha}^2 + \tilde{\beta}^2)}\right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{1}{2}[\tilde{\alpha}^2 + \tilde{\beta}^2] \cdot \frac{d}{ds} \psi_{\text{spin}}(s)$$

$$= \hat{\xi} \cdot \left(-\frac{1}{2}\left[\left(\frac{\tilde{\alpha}}{\sqrt{\hat{\xi}}}\right)^2 + \left(\frac{\tilde{\beta}}{\sqrt{\hat{\xi}}}\right)^2\right], \frac{\tilde{\alpha}}{\sqrt{\hat{\xi}}} \cdot \sqrt{1 - \frac{1}{4}\left[\left(\frac{\tilde{\alpha}}{\sqrt{\hat{\xi}}}\right)^2 + \left(\frac{\tilde{\beta}}{\sqrt{\hat{\xi}}}\right)^2\right]}, \frac{\tilde{\beta}}{\sqrt{\hat{\xi}}} \cdot \sqrt{1 - \frac{1}{4}\left[\left(\frac{\tilde{\alpha}}{\sqrt{\hat{\xi}}}\right)^2 + \left(\frac{\tilde{\beta}}{\sqrt{\hat{\xi}}}\right)^2\right]}\right)$$

$$\cdot \begin{pmatrix} n_{os}(s) & n_{0x}(s) & n_{oz}(s) \\ m_s(s) & m_x(s) & m_z(s) \\ l_s(s) & l_x(s) & l_z(s) \end{pmatrix} \begin{pmatrix} \omega_s \\ \omega_x \\ \omega_z \end{pmatrix}$$

$$+ \frac{\hat{\xi}}{2}\left[\left(\frac{\tilde{\alpha}}{\sqrt{\hat{\xi}}}\right)^2 + \left(\frac{\tilde{\beta}}{\sqrt{\hat{\xi}}}\right)^2\right] \cdot \frac{d}{ds} \psi_{\text{spin}}(s), \tag{4.27a}$$

and at second order the orbital Hamiltonian $\tilde{\mathcal{H}}_{\text{orb}}$ takes the form [1]*:

$$\tilde{\mathcal{H}}_{\text{orb}}(\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma; s)$$

$$= \frac{1}{2} \cdot \frac{1}{\gamma_0^2} \cdot \tilde{p}_\sigma^2 - [K_x \cdot \tilde{x} + K_z \cdot \tilde{z}] \cdot \tilde{p}_\sigma$$

$$+ \frac{1}{2} \cdot \{[\tilde{p}_x + H \cdot \tilde{z}]^2 + [\tilde{p}_z - H \cdot \tilde{x}]^2\}$$

$$+ \frac{1}{2} \cdot \{(K_x^2 + g) \cdot \tilde{x}^2 + (K_z^2 - g) \cdot \tilde{z}^2 - 2N \cdot \tilde{x}\tilde{x}\}$$

$$- \frac{1}{2} \tilde{\sigma}^2 \cdot \frac{1}{\beta_0^2} \cdot \frac{eV(s)}{E_0} \cdot h \cdot \frac{2\pi}{L} \cdot \cos\varphi. \tag{4.27b}$$

With (4.27) we have the Hamiltonian for the canonical variables

$$\tilde{x}, \tilde{z}, \tilde{\sigma}, \tilde{\alpha}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \tilde{\beta}$$

and the canonical equations for spin-orbit motion are:

$$\frac{d}{ds}\tilde{x} = +\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_x}, \quad \frac{d}{ds}\tilde{p}_x = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{x}}, \tag{4.28a}$$

$$\frac{d}{ds}\tilde{z} = +\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_z}, \quad \frac{d}{ds}\tilde{p}_z = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{z}}, \tag{4.28b}$$

$$\frac{d}{ds}\tilde{\sigma} = +\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{p}_\sigma}, \quad \frac{d}{ds}\tilde{p}_\sigma = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{\sigma}}, \tag{4.28c}$$

$$\frac{d}{ds}\tilde{\alpha} = +\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{\beta}}, \quad \frac{d}{ds}\tilde{\beta} = -\frac{\partial \tilde{\mathcal{H}}}{\partial \tilde{\alpha}}. \tag{4.28d}$$

Remark. Using (4.11a, b, c) which determine the spin components $\hat{\xi}_n, \hat{\xi}_m, \hat{\xi}_l$ of the spin vector $\boldsymbol{\xi}$ (see (4.10)) in terms of the spin variables $\tilde{\alpha}$ and $\tilde{\beta}$, the spin Hamiltonian $\tilde{\mathcal{H}}_{\text{spin}}$ in (4.26a) may also be written as:

$$\tilde{\mathcal{H}}_{\text{spin}} = \hat{\Omega}_n \cdot \hat{\xi}_n + \hat{\Omega}_m \cdot \hat{\xi}_m + \hat{\Omega}_l \cdot \hat{\xi}_l - \hat{\xi} \cdot [\mathbf{n}_0 \cdot \boldsymbol{\omega} - \psi'_{\text{spin}}(s)], \tag{4.29}$$

with

$$\hat{\Omega}_n = \mathbf{n}_0 \cdot \boldsymbol{\omega} - \psi'_{\text{spin}}(s), \tag{4.30a}$$

$$\hat{\Omega}_m = \mathbf{m} \cdot \boldsymbol{\omega}, \tag{4.30b}$$

$$\hat{\Omega}_l = \mathbf{l} \cdot \boldsymbol{\omega}, \tag{4.30c}$$

and $\boldsymbol{\omega}$ given by (4.19).

* For simplicity we treat the orbital motion only in the linear form. But the construction of normal forms developed in Sects. 5 and 6 works also for a nonlinear orbital Hamiltonian up to an arbitrary order

Since the spin components $\hat{\xi}_n, \hat{\xi}_m, \hat{\xi}_l$ obey the Poisson bracket relations:

$$\{\hat{\xi}_n, \hat{\xi}_m\}_{\tilde{\alpha},\tilde{\beta}} \equiv \frac{\partial \hat{\xi}_n}{\partial \tilde{\alpha}} \cdot \frac{\partial \hat{\xi}_m}{\partial \tilde{\beta}} - \frac{\partial \hat{\xi}_1}{\partial \tilde{\beta}} \cdot \frac{\partial \hat{\xi}_2}{\partial \tilde{\alpha}} = \hat{\xi}_l, \tag{4.31a}$$

$$\{\hat{\xi}_m, \hat{\xi}_l\}_{\tilde{\alpha},\tilde{\beta}} = \hat{\xi}_n, \tag{4.31b}$$

$$\{\hat{\xi}_l, \hat{\xi}_n\}_{\tilde{\alpha},\tilde{\beta}} = \hat{\xi}_m, \tag{4.31c}$$

the equation of spin motion

$$\frac{d}{ds}\hat{\boldsymbol{\xi}} = \{\hat{\boldsymbol{\xi}}, \mathscr{H}_{\text{spin}}\}_{\tilde{\alpha},\tilde{\beta}}$$

takes the form:

$$\frac{d}{ds}\begin{pmatrix}\hat{\xi}_n\\ \hat{\xi}_m\\ \hat{\xi}_l\end{pmatrix} = \begin{pmatrix}\hat{\Omega}_n\\ \hat{\Omega}_m\\ \hat{\Omega}_l\end{pmatrix} \times \begin{pmatrix}\hat{\xi}_n\\ \hat{\xi}_m\\ \hat{\xi}_l\end{pmatrix}, \tag{4.32a}$$

or

$$\frac{d}{ds}\begin{pmatrix}\hat{\xi}_n\\ \hat{\xi}_m\\ \hat{\xi}_l\end{pmatrix} = \underline{\hat{\Omega}} \begin{pmatrix}\hat{\xi}_n\\ \hat{\xi}_m\\ \hat{\xi}_l\end{pmatrix}, \tag{4.32b}$$

with the notation:

$$\underline{\hat{\Omega}} = \begin{pmatrix} 0 & -\hat{\Omega}_l & \hat{\Omega}_m \\ \hat{\Omega}_l & 0 & -\hat{\Omega}_n \\ -\hat{\Omega}_m & \hat{\Omega}_n & 0 \end{pmatrix}, \tag{4.33}$$

representing the BMT equation in machine coordinates with respect to the dreibein

$(\mathbf{n}_0, \mathbf{m}, \mathbf{l})$.

Neglecting the SG-forces, the equations of orbital motion read as*:

$$\frac{d}{ds}\tilde{x} = +\frac{\partial}{\partial \tilde{p}_x}\tilde{\mathscr{H}}_{\text{orb}}, \quad \frac{d}{ds}\tilde{p}_x = -\frac{\partial}{\partial \tilde{x}}\tilde{\mathscr{H}}_{\text{orb}}, \tag{4.34a}$$

$$\frac{d}{ds}\tilde{z} = +\frac{\partial}{\partial \tilde{p}_z}\tilde{\mathscr{H}}_{\text{orb}}, \quad \frac{d}{ds}\tilde{p}_z = -\frac{\partial}{\partial \tilde{z}}\tilde{\mathscr{H}}_{\text{orb}}, \tag{4.34b}$$

$$\frac{d}{ds}\tilde{\sigma} = +\frac{\partial}{\partial \tilde{p}_\sigma}\tilde{\mathscr{H}}_{\text{orb}}, \quad \frac{d}{ds}\tilde{p}_\sigma = -\frac{\partial}{\partial \tilde{\sigma}}\tilde{\mathscr{H}}_{\text{orb}}. \tag{4.34c}$$

This canonical system is then separate (and independent) from the spin motion and corresponds to the fully coupled 6-dimensional formalism [19, 20] in the orbital variables

$(\tilde{x}, \tilde{p}_x, \tilde{z}, \tilde{p}_z, \tilde{\sigma}, \tilde{p}_\sigma)$.

These must be known in order to solve the equation of spin-motion (4.32)**.

Equations (4.32) and (4.34) were already derived in [6].

* The term $-\hat{\xi}\cdot[\mathbf{n}_0\cdot\boldsymbol{\omega}-\psi'_{\text{spin}}(s)]$ in (4.29), containing only orbital variables in linear form, has no influence on the spin motion. It may be subsumed under the orbital Hamiltonian $\tilde{\mathscr{H}}_{\text{orb}}$ in (4.27b) (instead of the spin Hamiltonian $\tilde{\mathscr{H}}_{\text{spin}}$) producing via the SG force a very small closed orbit shift which can be neglected

** The neglect of SG effects when calculating spin motion is consistent with the philosophy of working only to first order in $\hbar$ [1, 3]

4.2.3 Series expansion of the Hamiltonian and the linearised equations of motion. For many purposes in spin physics e.g. when calculating electron spin polarization far from spin-orbit resonances, it is sufficient to consider spin vectors which are almost parallel to $\mathbf{n}_0$ [2, 25]. In that case $(\tilde{\alpha}/\sqrt{\hat{\xi}}), (\tilde{\beta}/\sqrt{\hat{\xi}})$ are small and the combined Hamiltonian can be written as a series in the form:

$$\tilde{\mathscr{H}} = \tilde{\mathscr{H}}_2 + \tilde{\mathscr{H}}_3 + \cdots, \tag{4.35}$$

where $\tilde{\mathscr{H}}_n$ $(n=2, 3, \dots)$ contains terms of n^{th} order in the coordinates

$\tilde{x}, \tilde{z}, \tilde{\sigma}, \tilde{\alpha}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma, \tilde{\beta}$.

To obtain linearised equations of spin and orbit motion we only need the component $\tilde{\mathscr{H}}_2$ containing the quadratic terms of the variables. This is given by:

$$\tilde{\mathscr{H}}_2(\tilde{x}, \tilde{z}, \tilde{\sigma}; \tilde{p}_x, \tilde{p}_z, \tilde{p}_\sigma; s) = \tilde{\mathscr{H}}_2^{(\text{orb})} + \tilde{\mathscr{H}}_2^{(\text{spin})};$$

$$\begin{aligned}\tilde{\mathscr{H}}_2^{(\text{orb})} = {}& \frac{1}{2}\cdot\frac{1}{\gamma_0^2}\cdot\tilde{p}_\sigma^2 - [K_x\cdot\tilde{x} + K_z\cdot\tilde{z}]\cdot\tilde{p}_\sigma \\ &+ \frac{1}{2}\cdot\{[\tilde{p}_x + H\cdot\tilde{z}]^2 + [\tilde{p}_z - H\cdot\tilde{x}]^2\} \\ &+ \frac{1}{2}\cdot\{G_1\cdot\tilde{x}^2 + G_2\cdot\tilde{z}^2 - 2N\cdot\tilde{x}\tilde{z}\} \\ &- \frac{1}{2}\tilde{\sigma}^2\cdot\frac{1}{\beta_0^2}\cdot\frac{eV(s)}{E_0}\cdot h\cdot\frac{2\pi}{L}\cdot\cos\varphi,\end{aligned}$$

$$\begin{aligned}\tilde{\mathscr{H}}_2^{(\text{spin})} = {}& \sqrt{\hat{\xi}}\cdot(\tilde{\alpha}, \tilde{\beta})\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_x(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\omega_s\\ \omega_x\\ \omega_z\end{pmatrix} \\ &+ \frac{1}{2}\cdot[\tilde{\alpha}^2 + \tilde{\beta}^2]\cdot\frac{d}{ds}\psi_{\text{spin}}(s) \\ = {}& \sqrt{\hat{\xi}}\cdot(\omega_s, \omega_x, \omega_z)\cdot\begin{pmatrix} m_s(s) & l_s(s) \\ m_x(s) & l_x(s) \\ m_z(s) & l_z(s)\end{pmatrix}\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} \\ &+ \frac{1}{2}\cdot[\tilde{\alpha}^2 + \tilde{\beta}^2]\cdot\frac{d}{ds}\psi_{\text{spin}}(s),\end{aligned} \tag{4.36}$$

where we have written for abbreviation:

$$G_1 = K_x^2 + g, \tag{4.37a}$$

$$G_2 = K_z^2 - g \tag{4.37b}$$

(g, N, H, K are defined in [1]).

The corresponding canonical equations take the form [1]:

$$\frac{d}{ds}\begin{pmatrix}\tilde{\mathbf{y}}\\ \tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} = \underline{A}\cdot\begin{pmatrix}\tilde{\mathbf{y}}\\ \tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}, \tag{4.38}$$

with

$$\underline{A}(s) = \begin{pmatrix}\underline{A}_{\text{orb}} & \underline{B}\\ \underline{C} & \underline{D}_0\end{pmatrix}, \tag{4.39}$$

and

$$\underline{A}_{\text{orb}}(s)=\begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2+H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 1/\gamma_0^2 \\ 0 & 0 & 0 & 0 & \frac{eV(s)}{E_0}\cdot\frac{1}{\beta_0^2}\cdot\frac{2\pi h}{L}\cos\varphi & 0 \end{pmatrix}, \tag{4.40a}$$

$$\underline{B}(s)=-\sqrt{\hat{\xi}}\cdot\underline{S}\cdot\underline{F}^T\cdot\begin{pmatrix} m_s(s) & l_s(s) \\ m_x(s) & l_x(s) \\ m_z(s) & l_z(s)\end{pmatrix}, \tag{4.40b}$$

$$\underline{C}(s)=\sqrt{\hat{\xi}}\cdot\begin{pmatrix} 0 & +1 \\ -1 & 0\end{pmatrix}\cdot\begin{pmatrix} m_s(s) & m_x(s) & m_x(s) \\ l_s(s) & l_x(s) & l_z(s)\end{pmatrix}\underline{F}=\underline{S}_2\underline{B}^T\underline{S}, \tag{4.40c}$$

$$\underline{D}_0(s)=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\cdot\frac{d}{ds}\psi_{\text{spin}}(s). \tag{4.40d}$$

Here the matrix $\underline{B}(s)$ describes the influence of Stern–Gerlach forces on the orbital motion and the matrix $\underline{C}(s)$ the influence of orbital motion on the spin motion. The matrices $\underline{A}(s)$ and $\underline{D}_0(s)$ correspond to the "unperturbed" spin-orbit motion.

Because the equations of motion (4.38) are linear and homogeneous, the solution can be written as:

$$\begin{pmatrix}\tilde{\mathbf{y}}(s) \\ \tilde{\alpha}(s) \\ \tilde{\beta}(s)\end{pmatrix}=\underline{\hat{M}}(s,s_0)\cdot\begin{pmatrix}\tilde{\mathbf{y}}(s_0) \\ \tilde{\alpha}(s_0) \\ \tilde{\beta}(s_0)\end{pmatrix}. \tag{4.41}$$

This defines the symplectic 8-dimensional transfer matrix $\underline{\hat{M}}(s, s_0)$ of linearised spin-orbit motion.

Remark. Neglecting the SG forces by putting the matrix $\underline{B}$ to zero, the linearised equations of spin-orbit motion (4.38) take the form:

$$\frac{d}{ds}\tilde{\mathbf{y}}(s)=\underline{A}_{\text{orb}}(s)\,\tilde{\mathbf{y}}(s), \tag{4.42a}$$

$$\frac{d}{ds}\boldsymbol{\zeta}(s)=\sqrt{\hat{\xi}}\cdot\underline{G}_0(s)\,\tilde{\mathbf{y}}(s)+\underline{D}_0(s)\boldsymbol{\zeta}(s), \tag{4.42b}$$

whereby we have written:

$$\boldsymbol{\zeta}(s)=\begin{pmatrix}\tilde{\alpha} \\ \tilde{\beta}\end{pmatrix}, \tag{4.43}$$

and

$$\underline{C}(s)=\sqrt{\hat{\xi}}\cdot\underline{G}_0(s). \tag{4.44}$$

In this form (4.42b) and (4.43) are the basic equations for spin motion used in the computer program SLIM [2, 23]. We have thus derived the SLIM-formalism from canonical equations based on a polynomial expansion of a spin Hamiltonian.

The linearization of the spin motion is valid if the spin-vector $\hat{\boldsymbol{\xi}}$ defined by (4.9) is sufficiently parallel to $\mathbf{n}_0$. The solution of (4.42b) can be written as:

$$\boldsymbol{\zeta}(s)=\sqrt{\hat{\xi}}\cdot\underline{G}(s,s_0)\,\tilde{\mathbf{y}}(s_0)+\underline{D}(s,s_0)\,\boldsymbol{\zeta}(s_0), \tag{4.45}$$

with

$$\begin{aligned}\underline{G}(s,s_0)&=\underline{D}(s,s_0)\cdot\int_{s_0}^{s}d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\underline{M}(\tilde{s},s_0)\\ &=\int_{s_0}^{s}ds\cdot\underline{D}(s,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\underline{M}(\tilde{s},s_0),\end{aligned} \tag{4.46}$$

and

$$\underline{D}(s,s_0)=\begin{pmatrix}\cos[\psi_{\text{spin}}(s)-\psi_{\text{spin}}(s_0)] & \sin[\psi_{\text{spin}}(s)-\psi_{\text{spin}}(s_0)] \\ -\sin[\psi_{\text{spin}}(s)-\psi_{\text{spin}}(s_0)] & \cos[\psi(s)_{\text{spin}}-\psi_{\text{spin}}(s_0)]\end{pmatrix}, \tag{4.47}$$

where $\underline{M}(s, s_0)$ denotes the 6-dimensional orbital transfer matrix with respect to (4.42a) which is determined by the differential equation:

$$\frac{d}{ds}\underline{M}(s,s_0)=\underline{A}_{\text{orb}}(s)\underline{M}(s,s_0), \tag{4.48a}$$

$$\underline{M}(s,s_0)=\underline{1}. \tag{4.48b}$$

In this approximation, the 8-dimensional transfer matrix $\underline{\hat{M}}(s, s_0)$ defined by (4.41) takes the form:

$$\underline{\hat{M}}(s,s_0)=\begin{pmatrix}\underline{M}(s,s_0) & \underline{0} \\ \sqrt{\hat{\xi}}\cdot\underline{G}(s,s_0) & \underline{D}(s,s_0)\end{pmatrix}. \tag{4.49}$$

In particular, one finds the following expressions for the revolution matrix $\underline{\hat{M}}(s_0+L, s_0)$:

$$\underline{\hat{M}}(s_0+L,s_0)=\begin{pmatrix}\underline{M}(s_0+L,s_0) & \underline{0} \\ \sqrt{\hat{\xi}}\cdot\underline{G}(s_0+L,s_0) & \underline{D}(s_0+L,s_0)\end{pmatrix}, \tag{4.50}$$

with

$$\underline{D}(s_0+L,s_0)=\begin{pmatrix}\cos[2\pi Q_{\text{spin}}] & \sin[2\pi Q_{\text{spin}}] \\ -\sin[2\pi Q_{\text{spin}}] & \cos[2\pi Q_{\text{spin}}]\end{pmatrix}, \tag{4.51}$$

where the quantity Q_{spin} defines the (linear) spin tune on the closed orbit (see (4.7)).

5 The definition of normal forms and the n-axis

The nonlinear equations of spin-orbit motion represent a periodic canonical system described by a Hamiltonian

$$\mathscr{H}(x, p_x; z, p_z; \sigma, p_\sigma; \alpha, \beta; s) = \mathscr{H}^{(0)} + \mathscr{H}^{(1)}, \tag{5.1}$$

with an unperturbed part

$$\mathscr{H}^{(0)} \equiv \mathscr{K}_2 = \sum_{\mu_1+\mu_2+\dots+\mu_8=2} c_{\mu_1\mu_2\cdots\mu_8}(s) \cdot x^{\mu_1} p_x^{\mu_2} z^{\mu_3} p_z^{\mu_4} \sigma^{\mu_5} p_\sigma^{\mu_6} \alpha^{\mu_7} \beta^{\mu_8}, \tag{5.2a}$$

and a perturbative part

$$\mathscr{H}^{(1)} \equiv \sum_{n=3}^{\infty} \mathscr{K}_n = \sum_{\nu=3}^{\infty} \sum_{\mu_1+\mu_2+\dots+\mu_8=\nu} c_{\mu_1\mu_2\cdots\mu_8}(s) \cdot x^{\mu_1} p_x^{\mu_2} z^{\mu_3} p_z^{\mu_4} \sigma^{\mu_5} p_\sigma^{\mu_6} \alpha^{\mu_7} \beta^{\mu_8}, \tag{5.2b}$$

where the $c(s)$ coefficients are periodic functions*.

For a ring of length L the periodicity condition reads as:

$$\mathscr{H}(x, p_x; z, p_z; \sigma, p_\sigma; \alpha, \beta; s+L) = \mathscr{H}(x, p_x; z, p_z; \sigma, p_\sigma; \alpha, \beta; s). \tag{5.3}$$

As in Sect. 3 the coordinates x, z, σ, α and momenta $p_x, p_z, p_\sigma, \beta$ can be combined into the vector

$$\mathbf{y}(s) = \begin{pmatrix} x \\ p_x \\ z \\ p_z \\ \sigma \\ p_\sigma \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \\ y_8 \end{pmatrix}, \tag{5.4}$$

and the canonical equations of motion may be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{y} = -\underline{S} \cdot \frac{\partial \mathscr{H}}{\partial \mathbf{y}}. \tag{5.5}$$

Our aim now is to find a canonical transformation

$$x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta \rightarrow \Phi_I, J_I, \Phi_{II}, J_{II}, \Phi_{III}, J_{III}, \Phi_{IV}, J_{IV},$$

which brings the Hamiltonian into normal form:

$$\mathscr{H} \rightarrow \hat{\mathscr{H}} = \hat{\mathscr{H}}(J_I, J_{II}, J_{III}, J_{IV}),$$

where J_k and Φ_k are action-angle variables so that the spin-orbit vector $\mathbf{y}$ can be written as:

$$\mathbf{y} = \mathbf{f}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s), \tag{5.6a}$$

obeying the periodicity relations:

$$\begin{aligned} &\mathbf{f}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s) \\ &= \mathbf{f}(\Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s) \\ &= \mathbf{f}(\Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s) \\ &= \mathbf{f}(\Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s) \\ &= \mathbf{f}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV} + 2\pi; J_I, J_{II}, J_{III}, J_{IV}; s) \\ &= \mathbf{f}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV}; J_I, J_{II}, J_{III}, J_{IV}; s+L). \end{aligned} \tag{5.6b}$$

The content of the parametrization (5.6a) is as follows.

If the SG forces and $\boldsymbol{\omega}$ were to vanish, there would be no spin-orbit coupling. In this case the J_k, Φ_k, $(k=I, II, III)$ would parametrize just the orbital motion and J_{IV}, Φ_{IV} would parametrize the spin motion. But since in reality there is spin orbit coupling the $J_k, \Phi_k, (k=I, II, III)$ become slightly modified by the (very small) SG forces. For the same reason the orbital motion acquires a small dependence of J_{IV}, Φ_{IV}. Likewise the spin motion, described by the components y_7, y_8 becomes dependent additionally on the orbital motion through the $J_k, \Phi_k, (k=I, II, III)$.

Using this parametrization we can now obtain the **n**-axis, the special solution of the BMT equation on a particle trajectory needed in the analytical theory of radiative spin polarization. The **n**-axis is a unit vector obeying the following periodicity conditions [3, 5, 26]:

$$\begin{aligned} &\mathbf{n}(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III}; s) \\ &= \mathbf{n}(\Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, J_I, J_{II}, J_{III}; s) \\ &= \mathbf{n}(\Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, J_I, J_{II}, J_{III}; s) \\ &= \mathbf{n}(\Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, J_I, J_{II}, J_{III}; s) \\ &= \mathbf{n}(\Phi_I, \Phi_{II}, \Phi_{III}, J_I, J_{II}, J_{III}; s+L), \end{aligned} \tag{5.7}$$

and is thus a single valued function of the orbital phase space coordinates and azimuth. Within this formalism, **n** is simply the unit vector parallel to the spin vector $\boldsymbol{\zeta}_n$ obtained from the elements y_7, y_8 by setting $J_{IV}=0$**. At $J_{IV}=0$ the vector **y** becomes independent of Φ_{IV} so that we can write:

$$\boldsymbol{\zeta}_n \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_7 \\ y_8 \end{pmatrix}_{J_{IV}=0} = \boldsymbol{\zeta}_n(\Phi_I, \Phi_{II}, \Phi_{III}; J_I, J_{II}, J_{III}; s). \tag{5.8}$$

Using (5.8) the components of the **n**-axis with respect to the dreibein $(\mathbf{n}_0, \mathbf{m}, \mathbf{l})$ can be obtained from (4.10) and have to be multiplied by a normalisation factor such that it becomes a unit vector.

The action-angle variables $J_k, \Phi_k, (k=I, II, III, IV)$ in (5.6a) can be constructed iteratively**:

$$J_k = \lim_{\nu\to\infty} J_k^{(\nu)},$$

$$\Phi_k = \lim_{\nu\to\infty} \Phi_k^{(\nu)}.$$

* For convenience we have changed the notation from $(\tilde{x}, \tilde{p}_x; \tilde{z}, \tilde{p}_z; \tilde{\sigma}, \tilde{p}_\sigma; \tilde{\alpha}, \tilde{\beta})$ to $(x, p_x; z, p_z; \sigma, p_\sigma; \alpha, \beta)$, from $\tilde{\mathscr{H}}$ to $\mathscr{H}$ and from $\tilde{\mathscr{K}}_n$ in (4.34) to $\mathscr{K}_n$

* Since $\tilde{\mathscr{H}}$ is in normal form J_{IV} is an integral of motion

** In practise one would only calculate to a finite order $\nu = N$

As a first step we introduce the variables $(J_k^{(0)}, \Phi_k^{(0)})$ which are action-angle variables with respect to the linear motion. In this way we obtain the Hamiltonian in a form which can be used as the starting point for canonical perturbation theory (Sect. 7). In particular we show that the variables $J_k^{(0)}, \Phi_k^{(0)}$, $(k=I, II, III, IV)$ introduced to describe linear motion remain canonical in the presence of the perturbation $\mathscr{H}^{(1)}$.

6 Variation of constants in the coupled case

6.1 The unperturbed system

In this chapter for abbreviation we write:

$$J_k \equiv J_k^{(0)},$$

$$\Phi_k \equiv \Phi_k^{(0)}.$$

6.1.1 The equations of motion for the unperturbed system. Taking into account only the first component, $\mathscr{H}^{(0)}$, of the Hamiltonian (5.1) we obtain from (5.5) the equations of motion for the unperturbed system:

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{y}^{(0)} = -\underline{\hat{S}} \cdot \frac{\partial \mathscr{H}^{(0)}}{\partial \mathbf{y}^{(0)}},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{y}^{(0)} = \underline{A} \cdot \mathbf{y}^{(0)}, \tag{6.1a}$$

with

$$\underline{A} \cdot \mathbf{y}^{(0)} = -\underline{\hat{S}} \cdot \frac{\partial \mathscr{H}^{(0)}}{\partial \mathbf{y}^{(0)}}, \tag{6.1b}$$

and

$$\mathbf{y}^{(0)} = \begin{pmatrix} x \\ p_x \\ z \\ p_z \\ \sigma \\ p_\sigma \\ \alpha \\ \beta \end{pmatrix}.$$

Because the equations of motion (6.1) are linear, the solution can be written in the form:

$$\mathbf{y}^{(0)}(s) = \underline{\hat{M}}(s, s_0)\, \mathbf{y}^{(0)}(s_0), \tag{6.2}$$

which defines the transfer matrix $\underline{\hat{M}}(s, s_0)$.

From (6.1), $\underline{\hat{M}}(s, s_0)$ is determined by the differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}s}\underline{\hat{M}}(s, s_0) = \underline{A}(s) \cdot \underline{\hat{M}}(s, s_0), \tag{6.3a}$$

$$\underline{\hat{M}}(s, s_0) = \underline{1}. \tag{6.3b}$$

Since the variables $x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta$ are canonical, the transfer matrix is symplectic [27]:

$$\underline{\hat{M}}^T(s, s_0) \cdot \underline{\hat{S}} \cdot \underline{\hat{M}}(s, s_0) = \underline{\hat{S}}. \tag{6.4}$$

The symplecticity condition (6.4) ensures that the transfer matrix, $\underline{M}(s, s_0)$, contains complete information about the stability of the (linear) betatron motion.

Differentiating (6.4) with respect to s and using (6.3) one obtains an alternative relation for symplecticity in the form:

$$\underline{A}^T(s) \cdot \underline{\hat{S}} + \underline{\hat{S}} \cdot \underline{A}(s) = 0. \tag{6.5}$$

6.1.2 Eigenvectors for the particle motion; Floquet-theorem. To come further we need the eigenvalues and the eigenvectors of the matrix $\underline{\hat{M}}(s+L, s)$:

$$\underline{\hat{M}}(s+L, s)\, \mathbf{v}_\mu(s) = \lambda_\mu \cdot \mathbf{v}_\mu(s), \tag{6.6}$$

in order to study the normal modes. We proceed in the usual way [28]:

The vector $\mathbf{v}_\mu(s)$ in (6.6) is an eigenvector of the matrix $\underline{\hat{M}}(s+L, s)$ at point s with the eigenvalue λ_μ. The eigenvalues are independent of s.

If the eigenvector $\mathbf{v}_\mu(s_0)$ at a fixed point s_0 is known, the eigenvector at an arbitrary point s may be obtained by:

$$\mathbf{v}_\mu(s) = \underline{\hat{M}}(s, s_0)\, \mathbf{v}_\mu(s_0). \tag{6.7}$$

Since $\underline{\hat{M}}(s+L, s)$ is symplectic and we assume stability, the eigenvectors $\mathbf{v}_\mu(s)$ come in complex conjugate pairs

$$(\mathbf{v}_k, \mathbf{v}_{-k} = \mathbf{v}_k^*),\ (k = I, II, III, IV),$$

with complex conjugate eigenvalues.

In the following we put:

$$\begin{cases} \lambda_k = \mathrm{e}^{-\mathrm{i}\cdot 2\pi Q_k}, \\ \lambda_{-k} = \mathrm{e}^{-\mathrm{i}\cdot 2\pi Q_{-k}}, \end{cases} \tag{6.8}$$

$(k = I, II, III, IV)$,

with

$$Q_{-k} = -Q_k, \tag{6.9}$$

where Q_k is a real number.

Defining $\mathbf{u}_\mu(s)$ by

$$\mathbf{v}_\mu(s) = \mathbf{u}_\mu(s) \cdot \mathrm{e}^{-\mathrm{i}\cdot 2\pi Q_\mu \cdot (s/L)}, \tag{6.10a}$$

we find:

$$\mathbf{u}_\mu(s+L) = \mathbf{u}_\mu(s). \tag{6.10b}$$

Equation (6.10a, b) is a statement of the Floquet theorem: vectors $\mathbf{v}_\mu(s)$ are special solutions of the equations of motion (6.1) which can be expressed as the product of a periodic function $\mathbf{u}_\mu(s)$ and a harmonic function

$$\mathrm{e}^{-\mathrm{i}\cdot 2\pi Q_\mu \cdot (s/L)}.$$

The general solution of the equation of motion (6.1) is a linear combination of the special solutions (6.10a) and can be therefore written as:

$$\mathbf{y}(s) = \sum_{k=I, II, III, IV} \{A_k \cdot \mathbf{u}_k(s) \cdot \mathrm{e}^{-\mathrm{i}\cdot 2\pi Q_k \cdot (s/L)} + A_{-k} \cdot \mathbf{u}_{-k}(s) \cdot \mathrm{e}^{+\mathrm{i}\cdot 2\pi Q_k \cdot (s/L)}\}. \tag{6.11}$$

We have the orthogonality relations:

$$\begin{cases} \mathbf{v}_k^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_k(s) = -\mathbf{v}_{-k}^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_{-k}(s)\neq 0, \\ \mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_\nu(s)=0 \ \text{ for } \ \mu\neq\nu, \end{cases}$$

$(k=I, II, III, IV)$.

Furthermore the terms $\mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_\mu(s)$ in the last equation are pure imaginary:

$$[\mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_\mu(s)]^+ = \mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}^+\cdot\mathbf{v}_\mu(s) = -[\mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_\mu(s)],$$

(since $\hat{\underline{S}}^+ = -\hat{\underline{S}}$). We choose to normalise the vectors $\mathbf{v}_k(s)$ and $\mathbf{v}_{-k}(s)$ at a fixed point s_0 as:

$$\mathbf{v}_k^+(s_0)\cdot\hat{\underline{S}}\cdot\mathbf{v}_k(s_0) = -\mathbf{v}_{-k}^+(s_0)\cdot\hat{\underline{S}}\cdot\mathbf{v}_{-k}(s_0)=\mathrm{i},$$

$(k=I, II, III, IV)$.

This normalisation is valid for all s if we use the definition in (6.7) for $\mathbf{v}_\mu(s)$. Thus we obtain:

$$\begin{cases} \mathbf{v}_k^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_k(s) = -\mathbf{v}_{-k}^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_{-k}(s)=\mathrm{i}, \\ \mathbf{v}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{v}_\nu(s)=0 \ \text{ for } \ \mu\neq\nu. \end{cases} \tag{6.12}$$

Note that the Floquet-vectors

$$\mathbf{u}_\mu(s)=\mathbf{v}_\mu(s)\cdot e^{+\mathrm{i}\cdot 2\pi Q_\mu\cdot(s/L)}$$

then fulfill the same relationship:

$$\begin{cases} \mathbf{u}_k^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{u}_k(s) = -\mathbf{u}_{-k}^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{u}_{-k}(s)=\mathrm{i} \\ \mathbf{u}_\mu^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{u}_\nu(s)=0 \ \text{ for } \ \mu\neq\nu. \end{cases} \tag{6.13}$$

Remark. The eigenvectors can be approximated by neglecting the Stern-Gerlach forces (matrix $\underline{B}$ in (4.39)) and using the matrix $\hat{\underline{M}}(s_0+L, s_0)$ in (4.49).

For more details see Appendix A.

6.2 The perturbed system

Using these results we can now introduce a new set of canonical variables which will be needed later.

We first remark that the general solution of the unperturbed equation of motion (6.1) may be written in the form (see (6.10a) and (6.11)):

$$\mathbf{y}(s)=\sum_{k=I,II,III,IV}\{A_k\cdot\mathbf{v}_k(s)+A_{-k}\cdot\mathbf{v}_{-k}(s)\}, \tag{6.14}$$

where A_k, A_{-k} are constants of integration $(k=I, II, III, IV)$.

In order to solve the perturbed problem (5.5) we now make the following "ansatz" (variation of constants):

$$\mathbf{y}(s)=\sum_{k=I,II,III,IV}\{A_k(s)\cdot\mathbf{v}_k(s)+A_{-k}(s)\cdot\mathbf{v}_{-k}(s)\}. \tag{6.15}$$

Writing then for the coefficients A_k and A_{-k} $(k=I, II, III, IV)$:

$$A_k=\sqrt{J_k(s)}\cdot e^{-\mathrm{i}\psi_k(s)}, \tag{6.16a}$$

$$A_{-k}=\sqrt{J_k(s)}\cdot e^{+\mathrm{i}\psi_k(s)}, \tag{6.16b}$$

(6.15) takes the form:

$$\mathbf{y}=\sum_{k=I,II,III,IV}\sqrt{J_k(s)}\cdot\{\mathbf{v}_k(s)\cdot e^{-\mathrm{i}\psi_k(s)}+\mathbf{v}_{-k}(s)\cdot e^{+\mathrm{i}\psi_k(s)}\}$$
$$\equiv\mathbf{y}(\mathbf{s}, \psi_k, J_k). \tag{6.17}$$

Note that in (6.16) and (6.17) the s dependence in J_k, ψ_k is to be understood as implicit, not as explicit. We intend to treat the J_k, ψ_k as dynamical variables. The explicit s dependence of $\mathbf{y}(s, \psi_k, J_k)$ is incorporated in the eigenvectors $\mathbf{v}_{\pm k}(s)$ which obey the unperturbed equations of motion (6.1).

From (6.17) we now get:

$$\frac{d\mathbf{y}}{ds}=\frac{\partial\mathbf{y}}{\partial s}+\sum_k\frac{\partial\mathbf{y}}{\partial\psi_k}\cdot\psi_k'+\sum_k\frac{\partial\mathbf{y}}{\partial J_k}\cdot J_k'=-\hat{\underline{S}}\cdot\frac{\partial\mathscr{H}}{\partial\mathbf{y}}. \tag{6.18}$$

Then with

$$\frac{\partial\mathbf{y}}{\partial s}=-\hat{\underline{S}}\cdot\frac{\partial\mathscr{H}^{(0)}}{\partial\mathbf{y}}, \tag{6.19}$$

we obtain:

$$\sum_k\frac{\partial\mathbf{y}}{\partial\psi_k}\cdot\psi_k'+\sum_k\frac{\partial\mathbf{y}}{\partial J_k}\cdot J_k'=-\hat{\underline{S}}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}. \tag{6.20}$$

Furthermore from (6.17) we have:

$$\frac{\partial\mathbf{y}}{\partial\psi_k}=-\mathrm{i}\cdot\sqrt{J_k(s)}\cdot\{\mathbf{v}_k(s)\cdot e^{-\mathrm{i}\psi_k}-\mathbf{v}_{-k}(s)\cdot e^{+\mathrm{i}\psi_k}\}, \tag{6.21a}$$

$$\frac{\partial\mathbf{y}}{\partial J_k}=+\frac{1}{2\sqrt{J_k(s)}}\cdot\{\mathbf{v}_k(s)\cdot e^{-\mathrm{i}\psi_k}+\mathbf{v}_{-k}(s)\cdot e^{+\mathrm{i}\psi_k}\}. \tag{6.21b}$$

Taking into account (6.12) we obtain the equations:

$$\{\mathbf{v}_k^+\cdot e^{+\mathrm{i}\psi_k}+\mathbf{v}_{-k}^+\cdot e^{-\mathrm{i}\psi_k}\}\cdot\hat{\underline{S}}\cdot\frac{\partial\mathbf{y}}{\partial\psi_l}=2\cdot\sqrt{J_k}\cdot\delta_{kl}, \tag{6.22a}$$

$$\{\mathbf{v}_k^+\cdot e^{+\mathrm{i}\psi_k}-\mathbf{v}_{-k}^+\cdot e^{-\mathrm{i}\psi_k}\}\cdot\hat{\underline{S}}\cdot\frac{\partial\mathbf{y}}{\partial\psi_l}=0, \tag{6.22b}$$

$$\{\mathbf{v}_k^+\cdot e^{+\mathrm{i}\psi_k}+\mathbf{v}_{-k}^+\cdot e^{-\mathrm{i}\psi_k}\}\cdot\hat{\underline{S}}\cdot\frac{\partial\mathbf{y}}{\partial J_l}=0, \tag{6.22c}$$

$$\{\mathbf{v}_k^+\cdot e^{+\mathrm{i}\psi_k}-\mathbf{v}_{-k}^+\cdot e^{-\mathrm{i}\psi_k}\}\cdot\hat{\underline{S}}\cdot\frac{\partial\mathbf{y}}{\partial J_l}=2\mathrm{i}\cdot\frac{1}{2\cdot\sqrt{J_k}}\cdot\delta_{kl}. \tag{6.22d}$$

Then from (6.20) with the help of (6.22) and the relation $\mathbf{v}_{-k}=\mathbf{v}_k^*$:

$$2\cdot\sqrt{J_k}\cdot\psi_k'=\{\mathbf{v}_k^+\cdot e^{+\mathrm{i}\psi_k}+\mathbf{v}_{-k}^+\cdot e^{-\mathrm{i}\psi_k}\}\cdot\hat{\underline{S}}\cdot\left[-\hat{\underline{S}}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}\right]$$
$$=\{(\mathbf{v}_{-k})^T\cdot e^{+\mathrm{i}\psi_k}+(\mathbf{v}_k)^T\cdot e^{-\mathrm{i}\psi_k}\}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}},$$

or

$$\psi_k'=\frac{1}{2\cdot\sqrt{J_k}}\cdot\{\mathbf{v}_{-k}\cdot e^{+\mathrm{i}\psi_k}+\mathbf{v}_k\cdot e^{-\mathrm{i}\psi_k}\}^T\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}$$
$$=\frac{\partial\mathbf{y}^T}{\partial J_k}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}=\frac{\partial\mathscr{H}^{(1)}}{\partial J_k}, \tag{6.23a}$$

and

$$2i\cdot\frac{1}{2\cdot\sqrt{J_k}}\cdot J_k'=\{\mathbf{v}_k^+\cdot e^{+i\psi_k}-\mathbf{v}_{-k}^+\cdot e^{-i\psi_k}\}\cdot\underline{\hat{S}}\cdot\left[-\underline{\hat{S}}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}\right]$$
$$=\{(\mathbf{v}_{-k})^T\cdot e^{+i\psi_k}-(\mathbf{v}_k)^T\cdot e^{-i\psi_k}\}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}$$
$$=-\{\mathbf{v}_{-k}\cdot e^{-i\psi_k}-\mathbf{v}_k\cdot e^{+i\psi_k}\}^T\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}},$$

or

$$J_k'=i\cdot\sqrt{J_k}\cdot\{\mathbf{v}_k\cdot e^{-i\psi_k}-\mathbf{v}_{-k}\cdot e^{+i\psi_k}\}^T\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}$$
$$=-\frac{\partial\mathbf{y}^T}{\partial\psi_k}\cdot\frac{\partial\mathscr{H}^{(1)}}{\partial\mathbf{y}}=-\frac{\partial\mathscr{H}^{(1)}}{\partial\psi_k}. \tag{6.23b}$$

So, by this ansatz, the motion of J and ψ can be attributed entirely to the perturbative part $\mathscr{H}^{(1)}$ and the unperturbed motion is embodied in the motion of the eigenvectors $\mathbf{v}_k(s)$ [7].

The relations (6.23a, b) were already used in [7] and are similar to those of the uncoupled case [29, 30, 31] and can now be the starting point for detailed investigations of specific cases*.

Remarks. 1) From (6.23a, b) it follows that the quantities J_k and ψ_k $(k=I, II, III, IV)$ defined by (6.16a, b) are canonical variables and that (6.17) represents a canonical transformation

$$x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta\rightarrow\psi_I, J_I, \psi_{II}, J_{II}, \psi_{III}, J_{III}, \psi_{IV}, J_{IV}. \tag{6.24}$$

2) From (6.15) and (6.12) we obtain

$$A_k=-i\cdot\mathbf{v}_k^+(s)\cdot\underline{\hat{S}}\cdot\mathbf{z}(s), \tag{6.25}$$

and from (6.16a, b) we have:

$$J_k(s)=|\mathbf{v}_k^+(s)\cdot\underline{\hat{S}}\cdot\mathbf{z}(s)|^2. \tag{6.26}$$

In the special case of vanishing coupling (see (6.45) in [28]) we may thus write:

$$J_y(s)=\frac{1}{2\beta_y(s)}\cdot\{[\alpha_y\cdot y+\beta_y\cdot p_y]^2+y^2\}. \tag{6.27}$$

The terms on the r.h.s of (6.27) just represent the Courant-Snyder invariants for the linear uncoupled case. Therefore the term on the r.h.s of (6.26) may be interpreted as the generalized Courant-Snyder invariant for the linear coupled case.

3) Writing the Jacobian matrix

$$\underline{\mathscr{J}}=\left(\frac{\partial\mathbf{y}}{\partial\psi_I},\frac{\partial\mathbf{y}}{\partial J_I},\frac{\partial\mathbf{y}}{\partial\psi_{II}},\frac{\partial\mathbf{y}}{\partial J_{II}},\frac{\partial\mathbf{y}}{\partial\psi_{III}},\frac{\partial\mathbf{y}}{\partial J_{III}},\frac{\partial\mathbf{y}}{\partial\psi_{IV}},\frac{\partial\mathbf{y}}{\partial J_{IV}}\right) \tag{6.28}$$

as an 8×8-matrix written as a row of column vectors $(\partial\mathbf{y}/\partial\psi_I)$ etc. and taking into account (6.21), the relations (6.22) may be combined into the matrix form

$$\underline{\mathscr{J}}^T\cdot\underline{\hat{S}}\cdot\underline{\mathscr{J}}=\underline{\hat{S}}. \tag{6.29}$$

This matrix equation may also be written as:

$$\underline{\mathscr{J}}\cdot\underline{\hat{S}}\cdot\underline{\mathscr{J}}^T=\underline{\hat{S}}, \tag{6.30a}$$

or

$$\left(\frac{\partial\mathbf{y}}{\partial\psi_I},\frac{\partial\mathbf{y}}{\partial J_I},\frac{\partial\mathbf{y}}{\partial\psi_{II}},\frac{\partial\mathbf{y}}{\partial J_{II}},\frac{\partial\mathbf{y}}{\partial\psi_{III}},\frac{\partial\mathbf{y}}{\partial J_{III}},\frac{\partial\mathbf{y}}{\partial\psi_{IV}},\frac{\partial\mathbf{y}}{\partial J_{IV}}\right)\cdot\begin{pmatrix}-\left(\frac{\partial\mathbf{y}}{\partial J_I}\right)^T\\+\left(\frac{\partial\mathbf{y}}{\partial\psi_I}\right)^T\\-\left(\frac{\partial\mathbf{y}}{\partial J_{II}}\right)^T\\+\left(\frac{\partial\mathbf{y}}{\partial\psi_{II}}\right)^T\\-\left(\frac{\partial\mathbf{y}}{\partial J_{III}}\right)^T\\+\left(\frac{\partial\mathbf{y}}{\partial\psi_{III}}\right)^T\\-\left(\frac{\partial\mathbf{y}}{\partial J_{IV}}\right)^T\\+\left(\frac{\partial\mathbf{y}}{\partial\psi_{IV}}\right)^T\end{pmatrix}=\underline{\hat{S}}, \tag{6.30b}$$

since

$$\underline{\mathscr{J}}^T\cdot\underline{\hat{S}}\cdot\underline{\mathscr{J}}=\underline{\hat{S}}\Rightarrow[\underline{\hat{S}}^T\cdot\underline{\mathscr{J}}^T]\cdot[\underline{\hat{S}}\cdot\underline{\mathscr{J}}]=\underline{1}$$
$$\Rightarrow[\underline{\hat{S}}^T\cdot\underline{\mathscr{J}}^T]=[\underline{\hat{S}}\cdot\underline{\mathscr{J}}]^{-1}$$
$$\Rightarrow[\underline{\hat{S}}\cdot\underline{\mathscr{J}}]\cdot[\underline{\hat{S}}^T\cdot\underline{\mathscr{J}}^T]=\underline{1}$$
$$\Rightarrow\underline{\hat{S}}^2\cdot\underline{\mathscr{J}}\cdot[\underline{\hat{S}}^T\cdot\underline{\mathscr{J}}^T]=\underline{S}$$
$$\Rightarrow\underline{\mathscr{J}}\cdot\underline{\hat{S}}\cdot\underline{\mathscr{J}}^T=\underline{\hat{S}}.$$

In terms of components, one obtain from (6.30b):

$$\begin{cases}[u, p_u]_{(\psi,J)}=1,\\ [u, v]_{(\psi,J)}=[u, p_v]_{(\psi,J)}=[p_u, p_v]_{(\psi,J)}=0 \text{ otherwise,}\end{cases} \tag{6.31}$$

$(u, v=x, z, \sigma, \alpha)$,

where $[f, g]_{(\psi,J)}$ represents the Poisson bracket defined by:

$$[f,g]_{(\psi,J)}=\left[\frac{\partial f}{\partial\psi_I}\cdot\frac{\partial g}{\partial J_I}-\frac{\partial f}{\partial J_I}\cdot\frac{\partial g}{\partial\psi_I}\right]+\left[\frac{\partial f}{\partial\psi_{II}}\cdot\frac{\partial g}{\partial J_{II}}-\frac{\partial f}{\partial J_{II}}\cdot\frac{\partial g}{\partial\psi_{II}}\right]+\left[\frac{\partial f}{\partial\psi_{III}}\cdot\frac{\partial g}{\partial J_{III}}-\frac{\partial f}{\partial J_{III}}\cdot\frac{\partial g}{\partial\psi_{III}}\right]+\left[\frac{\partial f}{\partial\psi_{IV}}\cdot\frac{\partial g}{\partial J_{IV}}-\frac{\partial f}{\partial J_{IV}}\cdot\frac{\partial g}{\partial\psi_{IV}}\right].$$

These relations demonstrate again that (6.24) represents a canonical transformation [33]. The new Hamiltonian in terms of the variables J_k, ψ_k is just $\mathscr{H}^{(1)}$.

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* In the case that the linear motion is uncoupled the Hamiltonian in (6.23a, b) can also be obtained by a generating function [32]

4) Starting from the Floquet-form (6.11) of $\mathbf{y}(s)$ and using (6.16)*:

$$\mathbf{y}(s)=\sum_{k=I,II,III,IV}\sqrt{J_k}\cdot\{\mathbf{u}_k(s)\cdot e^{-i\Phi_k}+\mathbf{u}_{-k}(s)\cdot e^{+i\Phi_k}\}, \tag{6.32}$$

with

$$\Phi_k=\psi_k+2\pi Q_k\cdot\frac{s}{L}, \tag{6.33}$$

we may define another Jacobian matrix

$$\underline{\tilde{\mathscr{J}}}=\left(\frac{\partial\mathbf{y}}{\partial\Phi_I},\frac{\partial\mathbf{y}}{\partial J_I},\frac{\partial\mathbf{y}}{\partial\Phi_{II}},\frac{\partial\mathbf{y}}{\partial J_{II}},\frac{\partial\mathbf{y}}{\partial\Phi_{III}},\frac{\partial\mathbf{y}}{\partial J_{III}},\frac{\partial\mathbf{y}}{\partial\Phi_{IV}},\frac{\partial\mathbf{y}}{\partial J_{IV}}\right) \tag{6.34}$$

in terms of the variables J_k, Φ_k. This obeys the same relation as $\underline{\mathscr{J}}$:

$$\underline{\tilde{\mathscr{J}}}^T\cdot\underline{\hat{S}}\cdot\underline{\tilde{\mathscr{J}}}=\underline{\hat{S}}, \tag{6.35}$$

as may be seen by using (6.13). Therefore Φ_k, J_k are again canonical variables.

For the unperturbed case:

$$\mathscr{H}^{(1)}=0\Rightarrow(J_k=\text{const},\ \psi_k=\text{const}),$$

and

$$\frac{dJ_k}{ds}=0, \tag{6.36a}$$

$$\frac{d\Phi_k}{ds}=\frac{2\pi}{L}Q_k. \tag{6.36b}$$

So in that case, the quantities J, Φ appearing in the Floquet form (6.32) are standard action angle variables.

The transition

$$\psi_k,\ J_k\to\Phi_k,\ \tilde{J}_k=J_k$$

may be affected by a canonical transformation using a generating function of the form

$$F_3(J_k,\Phi_k;s)=\sum_{k=I,II,III,IV}\left\{-J_k\cdot\Phi_k+J_k\cdot\frac{2\pi}{L}Q_k\cdot s\right\}. \tag{6.37}$$

The corresponding transformation equations:

$$\psi_k=-\frac{\partial F_3}{\partial J_k}=\Phi_k-\frac{2\pi}{L}Q_k\cdot s, \tag{6.38a}$$

$$\tilde{J}_k=-\frac{\partial F_3}{\partial\Phi_k}=J_k \tag{6.38b}$$

are indeed identical with the defining equations for Φ_k and J_k (see (6.33)).

The new Hamiltonian $\tilde{\mathscr{H}}$ in terms of $\tilde{J}_k=J_k$ and Φ_k then reads as:

$$\tilde{\mathscr{H}}=\mathscr{H}^{(1)}+\frac{\partial F_3}{\partial s}$$

$$=\mathscr{H}^{(1)}+\sum_{k=I,II,III,IV}J_k\cdot\frac{2\pi}{L}Q_k. \tag{6.39}$$

This form of the Hamiltonian is useful for calculating the detuning terms [34, 32].

5) From (6.35) one obtains:

$$\underline{\tilde{\mathscr{J}}}\cdot\underline{\hat{S}}\cdot\underline{\tilde{\mathscr{J}}}^T=\underline{\hat{S}}, \tag{6.40}$$

or

$$\begin{cases}[u,p_u]_{(\Phi,J)}=1,\\ [u,v]_{(\Phi,J)}=[u,p_v]_{(\Phi,J)}=[p_u,p_v]_{(\Phi,J)}=0\quad\text{otherwise,}\end{cases} \tag{6.41}$$

$(u,v=x,z,\sigma,\alpha)$.

Moreover, since the transformation inverse to (6.24):

$$\psi_I,J_I,\psi_{II},J_{II},\psi_{III},J_{III},\psi_{IV},J_{IV}\to x,p_x,z,p_z,\sigma,p_\sigma,\alpha,\beta, \tag{6.42}$$

is also canonical one gets in the same way:

$$\begin{cases}[\Phi_k,J_l]_{(y,p_y)}=\delta_{kl},\\ [J_k,J_l]_{(y,p_y)}=[\Phi_k,\Phi_l]_{(y,p_y)}0,\end{cases} \tag{6.43}$$

$(y=x,z,\sigma,\alpha)$.

6) In linear order $\tilde{\mathscr{H}}$ in (6.39) takes the form:

$$\tilde{\mathscr{H}}=\sum_{k=I,II,III,IV}J_k\cdot\frac{2\pi}{L}Q_k. \tag{6.44}$$

Thus

$$J_k\equiv J_k^{(0)}$$

is an integral of motion. The $\mathbf{n}$-axis as defined through (5.8) and fulfilling the periodicity relations (5.7) is in this order proportional to (see (6.32)) [35]:

$$\zeta_n=\sum_{k=I,II,III}\sqrt{J_k}\cdot\left\{\begin{pmatrix}u_{k7}\\u_{k8}\end{pmatrix}\cdot e^{-i\Phi_k}+\begin{pmatrix}[u_{k7}]^*\\ [u_{k8}]^*\end{pmatrix}\cdot e^{+i\Phi_k}\right\}. \tag{6.45}$$

A general spin-vector can then be represented by:

$$\begin{pmatrix}y_7\\y_8\end{pmatrix}=\zeta_n+\sqrt{J_{IV}}\cdot\left\{\begin{pmatrix}u_{k7}\\u_{k8}\end{pmatrix}\cdot e^{-i\Phi_k}+\begin{pmatrix}[u_{k7}]^*\\ [u_{k8}]^*\end{pmatrix}\cdot e^{+i\Phi_k}\right\}_{k=IV}. \tag{6.46}$$

So in first order a spin precesses around the first order $\mathbf{n}$-axis with frequency Q_{IV} (see (6.33)).

7) In linear order (see (6.44)) we now introduce new canonical variables*:

$$q_k=+\sqrt{2J_k}\cdot\cos\Phi_k, \tag{6.47a}$$

$$p_k=-\sqrt{2J_k}\cdot\sin\Phi_k, \tag{6.47b}$$

by a canonical transformation

$$\Phi_k,\ J_k\to q_k,\ p_k,$$

using the generating function:

$$F_1(\Phi_k,q_k)=\tfrac{1}{2}q_k^2\cdot\tan\Phi_k \tag{6.48a}$$

$$\Rightarrow\begin{cases}J_k=+\dfrac{\partial F_1}{\partial\Phi_k}=+\dfrac{1}{2}q_k^2\cdot\dfrac{1}{\cos^2\Phi_k},\\ p_k=-\dfrac{\partial F_1}{\partial q_k}=-q_k\cdot\tan\Phi_k,\end{cases} \tag{6.48b}$$

---

* Recall, that in this chapter $J_k\equiv J_k^{(0)}$, $\Phi_k\equiv\Phi_k^{(0)}$. Thus:
$\mathbf{y}(s)=\sum_{k=I,II,III,IV}\sqrt{J_k^{(0)}}\cdot\{\mathbf{u}_k(s)\cdot e^{-i\Phi_k^{(0)}}+\mathbf{u}_{-k}(s)\cdot e^{+i\Phi_k^{(0)}}\}$

* We have already come across this form in (2.22a, b)

(note, that the defining equations (6.47) and q_k and p_k are reproduced by (6.48b)) to obtain the new Hamiltonian:

$$\mathscr{K}=\tilde{\mathscr{H}}+\frac{\partial F_1}{\partial s}=\tilde{\mathscr{H}}$$

$$=\sum_{k=I,II,III,IV}\frac{1}{2}[p_k^2+q_k^2]\cdot\frac{2\pi}{L}Q_k, \quad (6.49)$$

representing four s-independent uncoupled harmonic oscillators and have diagonalised the Hamiltonian (5.2a) which described four coupled Hill oscillators*.

Using the new variables q_k and p_k, the spin-orbit vector $\mathbf{y}(s)$ in (6.32) may be written as:

$$\mathbf{y}(s)=\frac{1}{\sqrt{2}}\cdot\sum_{k=I,II,III,IV}\{[q_k+\mathrm{i}p_k]\cdot\mathbf{u}_k(s)+[q_k-\mathrm{i}p_k]\cdot\mathbf{u}_{-k}(s)\}$$

$$=\frac{1}{\sqrt{2}}\cdot\sum_{k=I,II,III,IV}\{[\mathbf{u}_k(s)+\mathbf{u}_{-k}(s)]\cdot q_k+\mathrm{i}[\mathbf{u}_k(s)-\mathbf{u}_{-k}(s)]\cdot p_k\}, \quad (6.50)$$

or in matrix form:

$$\mathbf{y}(s)=\underline{R}(s)\cdot\mathbf{r}(s), \quad (6.51)$$

with

$$\underline{R}=([\mathbf{u}_I+\mathbf{u}_{-I}],\mathrm{i}[\mathbf{u}_I-\mathbf{u}_{-I}],\ldots,[\mathbf{u}_{IV}-\mathbf{u}_{-VI}],\cdot\mathrm{i}[\mathbf{u}_{VI}+\mathbf{u}_{-VI}]), \quad (6.52a)$$

and

$$\mathbf{r}=\begin{pmatrix} q_I\\ p_I\\ q_{II}\\ p_{II}\\ q_{III}\\ p_{III}\\ q_{IV}\\ p_{IV}\end{pmatrix}. \quad (6.52b)$$

It follows from (6.50), taking into account the orthogonality relations (6.13), that:

$$\mathbf{u}_k^+(s)\cdot\hat{\underline{S}}\cdot\mathbf{y}(s)=+\frac{\mathrm{i}}{\sqrt{2}}\cdot[q_k+\mathrm{i}p_k],$$

$$-\mathbf{u}_{-k}^{\pm}(s)\cdot\hat{\underline{S}}\cdot\mathbf{y}(s)=-\frac{\mathrm{i}}{\sqrt{2}}\cdot[q_k-\mathrm{i}p_k],$$

so that:

$$q_k=-\frac{\mathrm{i}}{\sqrt{2}}[\mathbf{u}_k^+(s)-\mathbf{u}_{-k}^{\pm}(s)]\,\hat{\underline{S}}\cdot\mathbf{y}(s), \quad (6.53a)$$

$$p_k=-\frac{\mathrm{i}}{\sqrt{2}}[\mathbf{u}_k^+(s)+\mathbf{u}_{-k}^{\pm}(s)]\,\hat{\underline{S}}\cdot\mathbf{y}(s), \quad (6.53b)$$

* The transfer matrix with respect to the Hamiltonian (6.49) is block diagonal

or

$$\mathbf{r}=-\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}[\mathbf{u}_I^+-\mathbf{u}_{-I}^{\pm}]\\ \mathrm{i}[\mathbf{u}_I^++\mathbf{u}_{-I}^{\pm}]\\ [\mathbf{u}_{II}^+-\mathbf{u}_{-II}^{\pm}]\\ \mathrm{i}[\mathbf{u}_{II}^++\mathbf{u}_{-II}^{\pm}]\\ [\mathbf{u}_{III}^+-\mathbf{u}_{-III}^{\pm}]\\ \mathrm{i}[\mathbf{u}_{III}^++\mathbf{u}_{-III}^{\pm}]\\ [\mathbf{u}_{IV}^+-\mathbf{u}_{-IV}^{\pm}]\\ \mathrm{i}[\mathbf{u}_{IV}^++\mathbf{u}_{-IV}^{\pm}]\end{pmatrix}\hat{\underline{S}}\cdot\mathbf{y}\equiv\hat{\underline{R}}^{-1}\mathbf{y}. \quad (6.54)$$

Equation (6.53) or (6.54) allows the quantities q_k, p_k to be calculated in terms of the starting variables y_ν.

From the relation

$$\mathbf{y}(s+L)=\underline{M}(s+L,s)\,\mathbf{y}(s),$$

we obtain, using (6.51):

$$\underline{R}(s+L)\cdot\mathbf{r}(s+L)=\underline{M}(s+L,s)\,\underline{R}(s)\cdot\mathbf{r}(s),$$

or

$$\mathbf{r}(s+L)=\underline{R}^{-1}(s+L)\,\underline{M}(s+L,s)\,\underline{R}(s)\cdot\mathbf{r}(s)$$

$$=\underline{R}^{-1}(s)\,\underline{M}(s+L,s)\,\underline{R}(s)\cdot\mathbf{r}(s), \quad (6.55)$$

(for the last step see (6.10b)).

The revolution matrix with respect to the variables p_k, q_k is thus given by

$$\underline{M}^{(r)}(s+L,s)=\underline{R}^{-1}(s)\,\underline{M}(s+L,s)\,\underline{R}(s). \quad (6.56)$$

This matrix is block diagonal as can be seen from (6.49).

Equations (6.50), (6.51) and (6.55) represent the first step in a normal form analysis [36].

In the next chapter we show how to put the Hamiltonian into normal form at the next and succeeding orders. Implicit use will be made of the fact that the Hamiltonian can be written as a power series in α and β (see Sect. 4.2.3).

7 Nonlinear perturbation theory and normal forms

The version of perturbation theory presented here is similar to that given by Courant et al. (CRW) [34]. The starting point is the Hamiltonian (6.39), written in terms of $J_k\equiv J_k^{(0)}$ and $\Phi_k\equiv\Phi_k^{(0)}$:

$$\tilde{\mathscr{H}}=\tilde{\mathscr{H}}_0(J)+\tilde{V}(\Phi,J;s), \quad (7.1)$$

with

$$\tilde{\mathscr{H}}_0=\sum_{k=I,II,III,IV}J_k\cdot\frac{2\pi}{L}Q_k, \quad (7.2a)$$

and

$$\tilde{V}\equiv\mathscr{H}^{(1)}, \quad (7.2b)$$

(see Appendix B) where the unperturbed part, $\tilde{\mathscr{H}}_0$, depends only on J_k ($k=I,II,III,IV$). The term $\tilde{V}(\Phi,J;s)$

resulting from (5.2b) and (6.14) which is in general nonlinear, describes the perturbation and is periodic in s and Φ_k:

$$\begin{aligned}
&\tilde{V}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV}, J_I, J_{II}, J_{III}, J_{IV}; s) \\
&= \tilde{V}(\Phi_I + 2\pi, \Phi_{II}, \Phi_{III}, \Phi_{IV}, J_I, J_{II}, J_{III}, J_{IV}; s) \\
&= \tilde{V}(\Phi_I, \Phi_{II} + 2\pi, \Phi_{III}, \Phi_{IV}, J_I, J_{II}, J_{III}, J_{IV}; s) \\
&= \tilde{V}(\Phi_I, \Phi_{II}, \Phi_{III} + 2\pi, \Phi_{IV}, J_I, J_{II}, J_{III}, J_{IV}; s) \\
&= \tilde{V}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV} + 2\pi, J_I, J_{II}, J_{III}, J_{IV}; s) \\
&= \tilde{V}(\Phi_I, \Phi_{II}, \Phi_{III}, \Phi_{IV}, J_I, J_{II}, J_{III}, J_{IV}; s + L).
\end{aligned} \tag{7.3}$$

From (5.2b) we may write:

$$\tilde{V}(J, \Phi; s) = \sum_{v=3}^{\infty} \tilde{V}_v(J, \Phi; s), \tag{7.4a}$$

with

$$\tilde{V}_n \equiv \mathscr{K}_n. \tag{7.4b}$$

The aim now is to transform the Hamiltonian (7.1) to normal form, i.e. a form in which the new Hamiltonian depends only on the new momenta. To achieve that, we look for a transformation which cancels the perturbative terms $\tilde{V}_n$ iteratively order by order.

At the $(n-2)^{\text{th}}$ step of iteration we have the Hamiltonian

$$\mathscr{H} = \check{\mathscr{H}}_0(J) + \check{V}(\Phi, J; s), \tag{7.5}$$

with

$$J_k \equiv J_k^{(n-3)},$$

$$\Phi_k \equiv \Phi_k^{(n-3)},$$

and

$$\check{V}(\Phi, J; s) = \sum_{v=n}^{\infty} \check{V}_v(\Phi, J; s) = \check{V}_n(\Phi, J; s) + W_n(\Phi, J; s), \tag{7.6}$$

$$W_n(\Phi, J; s) = \sum_{v=n+1}^{\infty} \check{V}_v(\Phi, J; s), \tag{7.7}$$

where $\check{V}_v$ is of order v and the higher order terms resulting from $\tilde{V}_3, \ldots, \tilde{V}_{n-1}$ at earlier stages of diagonalisation have been absorbed in $\check{V}_n$ and W_n.

At a first step we separate off the average of $\check{V}_n$:

$$\begin{aligned}
\langle \check{V}_n(J) \rangle = \frac{1}{(2\pi)^4 \cdot L} \cdot \int_{s_0}^{s_0+L} \mathrm{d}s \cdot \int_0^{2\pi} \mathrm{d}\Phi_I \cdot \int_0^{2\pi} \mathrm{d}\Phi_{II} \\
\cdot \int_0^{2\pi} \mathrm{d}\Phi_{III} \cdot \int_0^{2\pi} \mathrm{d}\Phi_{IV} \cdot \check{V}_n(\Phi_k, J_k; s),
\end{aligned} \tag{7.8}$$

and add it to $\check{\mathscr{H}}_0$ so that

$$\mathscr{H} = \mathscr{H}_0(J) + V(\Phi, J; s), \tag{7.9}$$

with

$$V(\Phi, J; s) = V_n(\Phi, J; s) + W_n(\Phi, J; s), \tag{7.10}$$

and

$$\mathscr{H}_0(J) = \check{\mathscr{H}}_0(J) + \langle \check{V}_n(J) \rangle, \tag{7.11a}$$

$$V_n(\Phi, J; s) = \check{V}_n(\Phi, J; s) - \langle \check{V}_n(J) \rangle. \tag{7.11b}$$

(We comment further on this separation at the end of this section.)

As is clear from (7.2a) and (7.11a) the term $\langle \check{V}_n(J) \rangle$ results in a tune shift of the form:

$$\delta Q_k = \frac{L}{2\pi} \cdot \frac{\partial}{\partial J_k} \langle \check{V}_n(J) \rangle. \tag{7.12}$$

If $\langle \check{V}_n(J) \rangle$ depends nonlinearly on J_k, the tune shift is amplitude dependent.

In a second step we make a canonical transformation*:

$$(\Phi_k, J_k) \to (\hat{\Phi}_k, \hat{J}_k), \tag{7.13}$$

with

$$\hat{J}_k \equiv J_k^{(n-2)},$$

$$\hat{\Phi}_k \equiv \Phi_k^{(n-2)},$$

designed to cancel the term $V_n(\Phi, J; s)$ by using the generating function:

$$F_2(\Phi, \hat{J}; s) = \sum_{k=I,II,III,IV} \Phi_k \cdot \hat{J}_k + G(\Phi, \hat{J}; s).$$

For the new variables $\hat{\Phi}_k, \hat{J}_k$:

$$\hat{\Phi}_k = \frac{\partial F_2}{\partial \hat{J}_k} = \Phi_k + G_{\hat{J}_k}, \tag{7.14a}$$

$$J_k = \frac{\partial F_2}{\partial \Phi_k} = \hat{J}_k + G_{\Phi_k}, \tag{7.14b}$$

the corresponding Hamiltonian:

$$\begin{aligned}
\hat{\mathscr{H}} &= \mathscr{H} + \frac{\partial F_2}{\partial s} = \mathscr{H}_0(\hat{J} + G_\Phi) + V(\Phi, \hat{J} + G_\Phi; s) + G_s \\
&= \mathscr{H}_0(\hat{J} + G_\Phi) + V_n(\Phi, \hat{J} + G_\Phi; s) + W_n(\Phi, \hat{J} + G_\Phi; s) + G_s
\end{aligned} \tag{7.15a}$$

is in n^{th} order only dependent on $\hat{J}_k$. In (7.14) and below we use the notation $G_\Phi = \partial G / \partial \Phi$ etc.

For this purpose, following CWR, we rewrite (7.15a) as:

$$\begin{aligned}
\hat{\mathscr{H}} = \mathscr{H}_0(\hat{J}) + \Bigg\{ &\mathscr{H}_0(\hat{J} + G_\Phi) - \mathscr{H}_0(\hat{J}) \\
&- \sum_{k=I,II,III,IV} \frac{2\pi}{L} \cdot Q_k(\hat{J}) \cdot G_{\Phi_k} \Bigg\} \\
&+ \{ V(\Phi, \hat{J} + G_\Phi; s) - V(\Phi, \hat{J}; s) \} \\
&+ \sum_{k=I,II,III,IV} \frac{2\pi}{L} \cdot Q_k(\hat{J}) \cdot G_{\Phi_k} + G_s \\
&+ V_n(\Phi, \hat{J}; s) + W_n(\Phi, \hat{J}; s),
\end{aligned} \tag{7.15b}$$

where for brevity we have written (see (7.2a), (7.11a) and (7.12)):

$$\frac{\partial \mathscr{H}_0(J)}{\partial J_k} = \frac{2\pi}{L} \cdot Q_k(J). \tag{7.16}$$

---

* In order to simplify the notation we drop the index 'n' in F_2 and G

We now require that the generating function G satisfies the partial differential equation:

$$\sum_{k=I,II,III,IV} \frac{2\pi}{L}\cdot Q_k(\hat{J})\cdot G_{\Phi_k}+G_s+V_n(\Phi,\hat{J};s)=0, \qquad (7.17)$$

so that

$$\hat{\mathscr{H}}=\mathscr{H}_0(\hat{J})+\hat{V}, \qquad (7.18)$$

where

$$\hat{V}=W_n(\Phi,\hat{J};s)$$
$$+\sum_{k=I,II,III,IV}\frac{1}{2}\cdot G_{\Phi_k}\cdot G_{\Phi_l}\cdot\frac{\partial^2}{\partial\hat{J}_k\partial\hat{J}_l}\mathscr{H}_0(\hat{J})$$
$$+\sum_{k=I,II,III,IV} G_{\Phi_k}\cdot\frac{\partial}{\partial\hat{J}_k}V(\Phi,\hat{J};s)+\cdots. \qquad (7.19a)$$

If the perturbation, V, in (7.9) is small compared to $\mathscr{H}$, then according to (7.17) we expect that G is small. It is then clear from (7.19a) that $\hat{V}$ in (7.18) is only a second order correction compared to $\mathscr{H}_0$, so that we may write:

$$\hat{V}(\Phi,\hat{J};s)=\sum_{\nu=n+1}^{\infty}\hat{V}_\nu(\Phi,\hat{J};s) \qquad (7.19b)$$

beginning the series expansion with the order $\nu=n+1^\star$.

For convenience we also require that the solution of (7.17) is periodic in s:

$$G(\Phi,\hat{J};s+L)=G(\Phi,\hat{J};s), \qquad (7.20)$$

so that $\hat{V}$ is also periodic with the same period L. In this case, the calculation embodied in (7.8–16) (and in the Fourier expansion below) can be repeated in a second iteration step, in which $\hat{V}$ replaces $\check{V}$ in (7.5).

In particular we can use the average of $\hat{V}$:

$$\langle\hat{V}_{n+1}(\hat{J})\rangle=\frac{1}{(2\pi)^4\cdot L}\cdot\int_{s_0}^{s_0+L}ds\cdot\int_0^{2\pi}d\Phi_I\cdot\int_0^{2\pi}d\Phi_{II}$$
$$\cdot\int_0^{2\pi}d\Phi_{III}\cdot\int_0^{2\pi}d\Phi_{IV}\cdot\hat{V}_{n+1}(\Phi,\hat{J};s)$$

to calculate the contribution to the Q-shift in the next order in analogy to (7.12).

A periodic solution to (7.17) can be obtained by writing V_n and G as:

$$V_n(\Phi,\hat{J};s)=\sum_{m_1,m_2,m_3,m_4} v^{(n)}_{m_1m_2m_3m_4}(\hat{J},s)$$
$$\cdot e^{i\cdot[m_1\Phi_I+m_2\Phi_{II}+m_3\Phi_{III}+m_4\Phi_{IV}]}, \qquad (7.21)$$

and

$$G(\Phi,\hat{J};s)=\sum_{m_1,m_2,m_3,m_4} g_{m_1m_2m_3m_4}(\hat{J},s)$$
$$\cdot e^{i\cdot[m_1\Phi_I+m_2\Phi_{II}+m_3\Phi_{III}+m_4\Phi_{IV}]}, \qquad (7.22a)$$

where $v^{(n)}_{m_1m_2m_3m_4}$ is periodic in s and where according to (7.20) we require:

$$g_{m_1m_2m_3m_4}(\hat{J},s+L)=g_{m_1m_2m_3m_4}(\hat{J},s). \qquad (7.22b)$$

On substituting (7.21) and (7.22a) into (7.17) we get the differential equation connecting the coefficients g and v:

$$\left\{i\cdot\frac{2\pi}{L}[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]+\frac{\partial}{\partial s}\right\}$$
$$g_{m_1m_2m_3m_4}(\hat{J},s)=-v^{(n)}_{m_1m_2m_3m_4}(\hat{J},s). \qquad (7.23)$$

This may also be written as:

$$\frac{\partial}{\partial s}\{e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]s}\cdot g_{m_1m_2m_3m_4}(\hat{J},s)\}$$
$$=-e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]s}\cdot v^{(n)}_{m_1m_2m_3m_4}(\hat{J},s). \qquad (7.24)$$

By integrating (7.24) from s to $s+L$ and using (7.22b) we then obtain:

$$g_{m_1m_2m_3m_4}(\hat{J},s)\cdot\{e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]\cdot(s+L)}$$
$$-e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]\cdot s}\}$$
$$=-\int_s^{s+L}d\tilde{s}\cdot v^{(n)}_{m_1m_2m_3m_4}(\hat{J},\tilde{s})$$
$$\cdot e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]\cdot\tilde{s}}, \qquad (7.25)$$

for which:

$$g_{m_1m_2m_3m_4}(\hat{J},s)$$
$$=\frac{i}{2\cdot\sin\pi[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]}$$
$$\cdot\int_s^{s+L}d\tilde{s}\cdot v^{(n)}_{m_1m_2m_3m_4}(\hat{J},\tilde{s})$$
$$\cdot e^{i\cdot(2\pi/L)\cdot[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]\cdot(\tilde{s}-s-L/2)}, \qquad (7.26)$$

so that (7.22) finally:

$$G(\Phi,\hat{J},s)$$
$$=\sum_{m_1,m_2,m_3,m_4}\frac{i}{2\cdot\sin\pi[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]}$$
$$\cdot e^{i\cdot[m_1\Phi_I+m_2\Phi_{II}+m_3\Phi_{III}+m_4\Phi_{IV}]}\cdot\int_s^{s+L}d\tilde{s}\cdot v^{(n)}_{m_1m_2m_3m_4}(\hat{J},\tilde{s})$$
$$\cdot e^{i\cdot(2\pi/L)[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}]\cdot(\tilde{s}-s-L/2)}. \qquad (7.27)$$

If the function $v^{(n)}_{m_1m_2m_3m_4}$ in (7.27) is furthermore expanded as a Fourier series in s:

$$v^{(n)}_{m_1m_2m_3m_4}(\hat{J},s)=\sum_q v^{(n)}_{m_1m_2m_3m_4q}(\hat{J})\cdot e^{-i\cdot q\frac{2\pi}{L}\cdot s}, \qquad (7.28)$$

---

* Since W_n is of order $(n+1)$ and J_k of order 2 and since G has the same order n as V (see (7.21) and (7.29)) $\hat{V}$ in (7.19a) becomes of order

$$\min(n+1,2n-2)=\begin{cases}2 & \text{for } n=2,\\ n+1 & \text{for } n\geq 3.\end{cases}$$

It follows that the canonical perturbation treatment does not work in the *linear* case where $n=2$ since the corrections of *linear* transformations remain *linear*. For that reason we have solved the linear problem separately before (Sect. 6) using another method (variation of constants) then G takes the form:

$$G(\Phi,\hat{J},s)=\mathrm{i}\cdot\frac{L}{2\pi}\sum_{m_1,m_2,m_3,m_4,q} v^{(n)}_{m_1m_2m_3m_4q}(\hat{J})\cdot\frac{\mathrm{e}^{\mathrm{i}\cdot[m_1\Phi_I+m_2\Phi_{II}+m_3\Phi_{III}+m_4\Phi_{IV}-q\cdot(2\pi/L)\cdot s]}}{[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}-q]}. \quad (7.29)$$

Since $\hat{\mathscr{H}}$ is approximately independent of $\hat{\Phi}_k$, the canonical equations:

$$\frac{\mathrm{d}}{\mathrm{d}s}\hat{J}_k=-\frac{\partial\hat{\mathscr{H}}}{\partial\hat{\Phi}_k} \quad (7.30)$$

predict that $\hat{J}_k$ are approximately constants of motion which together with (7.14b):

$$J_k=\hat{J}_k+\frac{\partial}{\partial\Phi_k}G(\Phi,\hat{J};s), \quad (7.31)$$

$(k=I, II, III, IV)$,

define invariant surfaces.

Remarks. 1) In separating off the average of $\check{V}_n$:

$\langle\check{V}_n\rangle=v^{(n)}_{00000}$

in (7.11b) we have ensured that the term in (7.29) for which m_1, m_2, m_3, m_4 and q in the denominator, $[m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}-q]$, are all zero, does not appear.

2) Taking into account the relation:

$$\int_0^{2\pi}\mathrm{d}\Phi\cdot\mathrm{e}^{\mathrm{i}\cdot m\Phi}=0 \text{ for } m\neq 0, \quad (7.32)$$

the quantities $\langle\check{V}_\nu\rangle$ vanish for odd values of ν. As a result, only integer powers of J_k $(k=I, II, III, IV)$ appear in the normal-form Hamiltonian. (So the half integer powers disappear; see (7.4a), (B.5) and (B.7) for the leading order and for higher orders [37]).

3) Since J_{IV} has the order of magnitude ξ, (see (6.46) and (A.4a, b) in Appendix A) we may neglect powers $(\hat{J}_{IV})^\nu$ for $\nu\geq 2$ in the final Hamiltonian $\hat{\mathscr{H}}$ which thus takes the form:

$$\hat{\mathscr{H}}=h_1(\hat{J}_I,\hat{J}_{II},\hat{J}_{III})+h_2(\hat{J}_I,\hat{J}_{II},\hat{J}_{III})\cdot\hat{J}_{IV}, \quad (7.33)$$

whereby h_1 and h_2 represent power series in $\hat{J}_k$ $(k=I, II, III)$ (we denote the action variables in the final form by $\hat{J}$)*. It follows that:

$$\frac{\mathrm{d}\hat{J}_k}{\mathrm{d}s}=-\frac{\partial\hat{\mathscr{H}}}{\partial\hat{\Phi}_k}=0, \quad (7.34a)$$

$$\frac{\mathrm{d}\hat{\Phi}_k}{\mathrm{d}s}=+\frac{\partial\hat{\mathscr{H}}}{\partial\hat{J}_k}=\text{const.}, \quad (7.34b)$$

* This corresponds to the fact that we work only to first order in $\hbar$. In fact terms of order $(J_{IV})^{N/2}$ $(N\geq 3)$ can be neglected at each stage of the perturbation procedure [8].

The form of the final Hamiltonian in (7.33) looks similar to that in [3]. However the meaning of $\hat{J}_{IV}$ differs from that of the action variable in [3]

and thus:

$$\hat{J}_k=\text{const.}, \quad (7.35a)$$

$$\hat{\Phi}_k=\frac{2\pi}{L}\cdot Q_k(\hat{J}_k)\cdot s, \quad (7.35b)$$

with

$$\frac{2\pi}{L}\cdot Q_k(\hat{J}_k)=+\frac{\partial}{\partial\hat{J}_k}[h_1+h_2\cdot\hat{J}_{IV}]. \quad (7.36)$$

In particular we obtain for the spin motion:

$$\hat{J}_{IV}=\text{const.}, \quad (7.37a)$$

$$\hat{\Phi}_{IV}=h_2(\hat{J}_I,\hat{J}_{II},\hat{J}_{III})\cdot s, \quad (7.37b)$$

and

$$Q_{IV}(\hat{J}_k)=\frac{L}{2\pi}\cdot h_2(\hat{J}_I,\hat{J}_{II},\hat{J}_{III}), \quad (7.38)$$

i.e. the spin tune depends only on the orbital action variables $\hat{J}_I, \hat{J}_{II}, \hat{J}_{III}$. See also [6] and paper III [40].

If the distribution function of the orbital action-angle variables is known, one can calculate the mean square spin tune spread $\langle(Q_{IV}-Q_{\rm spin})^2\rangle$ with $Q_{\rm spin}$ given by (4.7).

4) Since the transformation (7.13) is canonical, the Poisson bracket relations (6.43) remain valid for $\hat{\Phi}_k$ and $\hat{J}_k$:

$$\begin{cases}[\hat{\Phi}_k,\hat{J}_l]_{(y,p_y)}=\delta_{kl},\\ [\hat{J}_k,\hat{J}_l]_{(y,p_y)}=[\hat{\Phi}_k,\hat{\Phi}_l]_{(y,p_y)}=0,\end{cases} \quad (7.39)$$

$(y=x, z, \sigma, \alpha)$.

5) To calculate the **n**-axis (see (5.8)) we need the spin-orbit vector **y** in terms of the new action-angle variables. The **n**-axis is obtained from $\boldsymbol{\zeta}_n$ by taking $J_k^{(0)}$, $\Phi_k^{(0)}$ to be functions of $\hat{J}_k$, $\hat{\Phi}_k$, $(k=I, II, III, IV)$ in (6.32) (and footnote after eq. (5.7)) and putting $J_{IV}=0$. Clearly this can be a complicated procedure since in the CRW method the old and new variables in the generating function are mixed (see 7.14a, b)). It would then be more convenient to apply another kind of canonical perturbation theory, namely the method of Lie transforms (see [8, 10, 37]), whereby one obtains directly the old variables as functions of the new ones.

6) The above treatment of perturbation theory relies on the assumption that the perturbation G in (7.14) is small (see Sect. 4.2.3). From (7.29) it is clear that this condition is not valid if

$m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}$

is close to an integer. Thus the resonant case:

$$m_1Q_I+m_2Q_{II}+m_3Q_{III}+m_4Q_{IV}\approx\text{integer} \quad (7.40)$$

has to be investigated separately with other methods (see for instance [7])*. In this case the Hamiltonian becomes a function also of Φ_k $(k=I, II, III, IV)$. As a result, $J_{IV}=0$ is no longer a solution of the canonical equations of

* Since the SG forces are very small the spin motion can be considered to be a forced oscillation induced by the orbital motion (see [6]). Therefore only the resonances (7.40) for which $m_4=0,\pm 1$ are significant

motion as used in (5.8) and the method for calculating **n** described here breaks down (see also [3, 6]).

At linear order in this perturbation theory **n** contains first order resonances ($|m_1|+|m_2|+|m_3|=1$) and is of first order in orbit amplitudes (see (A.4a) and (6.45) as well as the Appendix in [23]). As can be seen in [6, 26, 40] at N^{th} order, **n** is of N^{th} order in orbital amplitudes and contains N^{th} order resonances with $|m_1+m_2+m_3|=N$.

8 Summary

Following an earlier paper [1], we have used a classical spin-orbit Hamiltonian for a spin 1/2 charged particle to construct a canonical formalism of spin-orbit motion expressed in machine coordinates, taking into account all kinds of coupling.

In addition to the orbital variables $x, p_x, z, p_z, \sigma, p_\sigma$ of the fully coupled 6-dimensional formalism we introduce the canonical variables α and β to describe the spin motion. All eight variables can be treated on the same level. In particular, the equations of spin-orbit motion can be linearised. Also the one turn maps are origin preserving.

By expanding the Hamiltonian into a power series in these variables, one may work to various orders of approximation for the canonical equations and the canonical structure of the formalism is well suited for the use of Lie algebra and normal forms.

In this paper we show how it is possible in principle to convert the spin-orbit Hamiltonian to normal form and how then to construct the **n**-axis applying a modified version of the canonical perturbation theory used by Courant, Ruth and Weng. The analysis is restricted to the non-resonant case but the resonant case could be incorporated in a natural way leading to a method of estimating the stopband width [30, 31] of spin-orbit resonances.

Finally we remark that, starting from the variables $x, p_x, z, p_z, \sigma, p_\sigma, \alpha, \beta$ and using analytical techniques as described in [20, 22, 38] one can also develop an 8-dimensional dispersion formalism.

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Appendix A: An approximative calculation of the eigenvectors for spin-orbit motion

The eigenvectors for linear spin-orbit motion in storage rings may approximated by neglecting the Stern-Gerlach forces. In this case the 8-dimensional transfer matrix $\underline{\hat{M}}(s, s_0)$ of linearised spin-orbit motion takes the form (see (4.49)):

$$\underline{\hat{M}}(s, s_0)=\begin{pmatrix}\underline{M}(s, s_0) & \underline{0} \\ \sqrt{\hat{\xi}}\cdot\underline{G}(s, s_0) & \underline{D}(s, s_0)\end{pmatrix}, \tag{A.1}$$

with $\underline{M}, \underline{G}, \underline{D}$ given by (4.46–48). Note that $\underline{M}$ is a symplectic matrix describing orbital motion.

The eigenvectors of the whole 8-dimensional revolution matrix $\underline{\hat{M}}(s_0+L, s_0)$ for spin and orbit which are defined by:

$$\underline{\hat{M}}(s_0+L, s_0)\cdot\mathbf{v}_\mu=\lambda_\mu\cdot\mathbf{v}_\mu \tag{A.2}$$

can now be written in the form:

$$\mathbf{v}_k(s_0)=\begin{pmatrix}\tilde{\mathbf{v}}_k(s) \\ \mathbf{w}_k(s)\end{pmatrix}, \quad \mathbf{v}_{-k}(s)=[\mathbf{v}_k(s)]^*, \tag{A.3a}$$

for $k=I, II, III$,

and

$$\mathbf{v}_{IV}(\mathbf{s}_0)=\begin{pmatrix}\mathbf{0}_6(s_0) \\ \mathbf{w}_{IV}(s_0)\end{pmatrix}, \quad \mathbf{q}_{-IV}(s_0)=[\mathbf{q}_{IV}(s_0)]^\star, \tag{A.3b}$$

for $k=IV$.

By combining (A.2), (A.3), (4.46), (4.50) and (4.51) we obtain the two-dimensional vectors $\mathbf{w}_k(s_0)$ ($k=I, II, III$) and $\mathbf{w}_{IV}(s_0)$ from the relation:

$$\sqrt{\hat{\xi}}\cdot\underline{G}(s_0+L, s_0)\,\tilde{v}_k(s_0)+\underline{D}(s_0+L, s_0)\,\mathbf{w}_k(s_0)=\lambda_k\cdot\mathbf{w}_k(s_0),$$

which leads to:

$$\begin{aligned}\mathbf{w}_k(s_0)&=-\sqrt{\hat{\xi}}\cdot[\underline{D}(s_0+L, s_0)-\lambda_k\cdot\underline{1}]^{-1}\\&\quad\cdot\underline{G}(s_0+L, s_0)\cdot\tilde{\mathbf{v}}_k(s_0)\\&=-\sqrt{\hat{\xi}}\cdot[\underline{D}(s_0+L, s_0)-\lambda_k\cdot\underline{1}]^{-1}\\&\quad\cdot\int_{s_0}^{s_0+L}d\tilde{s}\cdot\underline{D}(s_0+L, \tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\tilde{\mathbf{v}}_k(s),\end{aligned} \tag{A.4a}$$

for $k=I, II, III$,

and

$$\mathbf{w}_{IV}(s_0)=\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1 \\ -i\end{pmatrix}\cdot e^{-i\psi_{\text{spin}}(s_0)}, \tag{A.4b}$$

for $k=IV$,

with

$$\mathbf{w}_{-k}(s_0)=[\mathbf{w}_k(s_0)]^\star, \quad (k=I, II, III, IV), \tag{A.5}$$

where the vectors $\tilde{\mathbf{v}}_k(s_0)$ are determined by the relation:

$$\underline{M}(s_0+L, s_0)\cdot\tilde{\mathbf{v}}_k(s_0)=\lambda_k\cdot\tilde{\mathbf{v}}_k(s_0). \tag{A.6}$$

Thus $\tilde{\mathbf{v}}_k(s_0)$ ($k=I, II, III$) are eigenvectors of the (symplectic) orbital revolution matrix $\underline{M}(s_0+L, s_0)$ which may be normalised by:

$$\tilde{\mathbf{v}}_k^+(s_0)\cdot\underline{S}\cdot\tilde{\mathbf{v}}_k(s_0)=-\tilde{\mathbf{v}}_{-k}(s_0)\cdot\underline{S}\cdot\tilde{\mathbf{v}}_{-k}(s_0)=i. \tag{A.7}$$

As a result, the orthogonality relations (6.12) are then approximately fulfilled (due to the small value of $\hat{\xi}$).

The corresponding eigenvalues are

$$\lambda_k=e^{-i\cdot 2\pi Q_k}, \quad (k=I, II, III), \tag{A.8a}$$

and

$$\lambda_{IV}=e^{-i\cdot 2\pi Q_{IV}}, \quad \text{with } Q_{IV}=Q_{\text{spin}}. \tag{A.8b}$$

For the eigenvectors $\mathbf{v}_\mu(s)$ of the transfer matrix $\underline{\hat{M}}(s+L, s)$ (initial position s):

$$\underline{\hat{M}}(s+L, s)\cdot\mathbf{v}_\mu(s)=\lambda_\mu(s)\cdot\mathbf{v}_\mu(s), \tag{A.9}$$

we also have:

$$\mathbf{v}_\mu(s)=\underline{\hat{M}}(s,s_0)\,\mathbf{v}_\mu(s_0)\equiv\begin{pmatrix}\tilde{\mathbf{v}}_k(s)\\ \mathbf{w}_k(s)\end{pmatrix}. \tag{A.10}$$

In particular we get

$$\mathbf{v}_{IV}(s)=\begin{pmatrix}\mathbf{0}_6\\ \mathbf{w}_{IV}(s)\end{pmatrix},\quad \mathbf{q}_{-IV}(s)=[\mathbf{q}_{IV}(s)]^*, \tag{A.11a}$$

with

$$\mathbf{w}_{IV}(s)=\underline{D}(s,s_0)\,\mathbf{w}_{IV}(s_0)=\frac{1}{\sqrt{2}}\cdot\begin{pmatrix}1\\ -\mathrm{i}\end{pmatrix}\cdot e^{-\mathrm{i}\psi_{\mathrm{spin}}(s)},\ \mathbf{w}_{-IV}(s)=[\mathbf{w}_{IV}(s)]^*. \tag{A.11b}$$

The eigenvalues are independent of s:

$$\lambda_\mu(s)=\lambda_\mu(s_0). \tag{A.12}$$

Remarks. 1) Note, that the components $\mathbf{w}_k$ in (A.10)

$$\begin{aligned}\mathbf{w}_k(s)&=-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\cdot\underline{G}(s+L,s)\,\tilde{\mathbf{v}}_k(s)\\ &=-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\int_s^{s+L}d\tilde{s}\cdot\underline{D}(s+L,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\tilde{\mathbf{v}}_k(\tilde{s}),\end{aligned} \tag{A.13}$$

for ($k=I,II,III$) are solutions of (4.42b) (using the definition (4.43) with $\tilde{\mathbf{y}}(s)=\tilde{\mathbf{v}}_k(s)$:

$$\begin{aligned}\frac{d}{ds}\mathbf{w}_k(s)&=-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\int_s^{s+L}d\tilde{s}\cdot\frac{d}{ds}\underline{D}(s+L,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\tilde{\mathbf{v}}(\tilde{s})\\ &\quad-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\underline{D}(s+L,s+L)\cdot\underline{G}_0(s+L)\cdot\tilde{\mathbf{v}}_k(s+L)\\ &\quad+\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\underline{D}(s+L,s)\cdot\underline{G}_0(s)\cdot\tilde{\mathbf{v}}_k(s)\\ &=-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\int_s^{s+L}d\tilde{s}\cdot\underline{D}_0(s)\cdot\underline{D}(s+L,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\tilde{\mathbf{v}}_k(s)\\ &\quad-\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\underline{1}\cdot\underline{G}_0(s)\cdot\lambda_k\tilde{\mathbf{v}}_k(s)\\ &\quad+\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot\underline{D}(s+L,s)\cdot\underline{G}_0(s)\cdot\tilde{\mathbf{v}}_k(s)\\ &=\underline{D}_0(s)\cdot\mathbf{w}_k(s)+\sqrt{\tilde{\xi}}\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]^{-1}\\ &\quad\cdot[\underline{D}(s+L,s)-\lambda_k\cdot\underline{1}]\cdot\underline{G}_0(s)\cdot\tilde{\mathbf{v}}_k(s)\\ &=\underline{D}_0(s)\cdot\mathbf{w}_k(s)+\sqrt{\tilde{\xi}}\cdot\underline{G}_0(s)\cdot\tilde{\mathbf{v}}_k(s).\end{aligned}$$

This result agrees with the definition of $\mathbf{w}_k(s)$ in (A.10). That is, the spin-orbit eigenvector $\mathbf{v}_\mu(s)$ defined by (A.10) is a solution of (4.42), which represents the combined spin-orbit motion.

2) Introducing the spin vector

$$\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}=\begin{pmatrix}y_7\\ y_8\end{pmatrix}-\zeta_n(s)=\sqrt{J_{IV}}\cdot\begin{pmatrix}u_{k7}\\ u_{k8}\end{pmatrix}_{k=IV}\cdot e^{-\mathrm{i}\cdot\Phi_{IV}}+\text{compl. conj.}, \tag{A.14}$$

(see (6.45)) which describes the spin motion around the $\mathbf{n}$-axis we obtain from (6.10a) and (A.11a, b):

$$\begin{aligned}\tilde{\alpha}&=\frac{1}{\sqrt{2}}\sqrt{J_{IV}}\cdot e^{-\mathrm{i}\cdot[\Phi_{IV}+\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)]}+\text{compl.conj.}\\ &=\sqrt{2J_{IV}}\cdot\cos[\Phi_{IV}+\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)];\\ \tilde{\beta}&=\frac{1}{\sqrt{2}}\sqrt{J_{IV}}\cdot\frac{1}{\mathrm{i}}e^{-\mathrm{i}\cdot[\Phi_{IV}+\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)]}+\text{compl.conj.}\\ &=-\sqrt{2J_{IV}}\cdot\sin[\Phi_{IV}+\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)],\end{aligned}$$

and therefore:

$$J_{IV}=\tfrac{1}{2}[\tilde{\alpha}^2+\tilde{\beta}^2]. \tag{A.15}$$

Thus spins at the same point in the orbital phase space ($\tilde{x}$, $\tilde{p}_x$, $\tilde{z}$, $\tilde{p}_z$, $\tilde{\sigma}$, $\tilde{p}_\sigma$) and s, can be considered to precess around a common axis $\mathbf{n}$ with a tilt angle w.r.t. $\mathbf{n}$ proportional to J_{IV}. The quantity J_{IV} describes the spin component perpendicular to the $\mathbf{n}$-axis.*

Using the variables q_k and p_k defined by (6.47a, b), we can also write:

$$\begin{aligned}\tilde{\alpha}&=+q_{IV}\cdot\cos[\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)]\\ &\quad+p_{IV}\cdot\sin[\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)],\\ \tilde{\beta}&=-q_{IV}\cdot\sin[\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)]\\ &\quad+p_{IV}\cdot\cos[\psi_{\mathrm{spin}}-2\pi Q_{\mathrm{spin}}\cdot(s/L)].\end{aligned}$$

It follows that the quantities q_{IV} and p_{IV} oscillate around $\tilde{\alpha}$ and $\tilde{\beta}$ (see (4.7)).

Appendix B: The Hamiltonian in terms of $\mathbf{J}_k$ and $\mathbf{\Phi}_k$

In order to prepare a perturbation theory, the perturbative component $\mathscr{H}^{(1)}$ of the Hamiltonian defined in (5.2b), which may be written as:

$$\mathscr{H}^{(1)}\equiv\sum_{n=3}^{\infty}\mathscr{H}_n=\sum_{\nu=3}^{\infty}\sum_{\mu_1+\mu_2+\cdots+\mu_8=\nu}c_{\mu_1\mu_2\cdots\mu_8}(s)\cdot(y_1)^{\mu_1}(y_2)^{\mu_2}(y_3)^{\mu_3}(y_4)^{\mu_4}(y_5)^{\mu_5}(y_6)^{\mu_6}(y_7)^{\mu_7}(y_8)^{\mu_8}, \tag{B.1}$$

by using the notation of (5.4), should be expressed in terms of the new canonical variables J_k and Φ_k.*

This can be achieved by using (6.32):

$$\mathbf{y}(s)=\sum_{k=I,II,III,IV}\sqrt{J_k}\cdot[\mathbf{u}_k(s)\cdot e^{-\mathrm{i}\Phi_k}+\mathbf{u}_{-k}(s)\cdot e^{+\mathrm{i}\Phi_k}]\equiv\mathbf{y}(\Phi_k,J_k;s), \tag{B.2a}$$

---

* Outside the spin orbit resonances we may assume that the $\mathbf{n}$-axis is approximately parallel to $\mathbf{n}_0$

* In this Appendix as in Sect. 6 we write $J_k\equiv J_k^{(0)}$, $\Phi_k\equiv\Phi_k^{(0)}$

with

$$\mathbf{u}_k(s+L)=\mathbf{u}_k(s). \tag{B.2b}$$

From (B.2) one has the ν^{th} component of $\mathbf{y}$:

$$y_\nu=\sum_{k=I,II,III,IV}\sqrt{J_k}\cdot[u_{k\nu}\cdot e^{-i\Phi_k}+u^*_{k\nu}\cdot e^{+i\Phi_k}]. \tag{B.3}$$

Thus we get:

$$\begin{aligned}
(y_\nu)^n=&\sum_{l=0}^{n}\binom{n}{l}\cdot\left\{\sum_{k=I,II}\sqrt{J_k}\cdot[u_{k\nu}\cdot e^{-i\Phi_k}+u^*_{k\nu}\cdot e^{+i\Phi_k}]\right\}^l\\
&\cdot\left\{\sum_{k=III,IV}\sqrt{J_k}\cdot[u_{k\nu}\cdot e^{-i\Phi_k}+u^*_{k\nu}\cdot e^{+i\Phi_k}]\right\}^{n-l}\\
=&\sum_{l=0}^{n}\binom{n}{l}\sum_{p=0}^{l}\binom{p}{l}(J_I)^{p/2}\cdot[u_{I\nu}\cdot e^{-i\Phi_I}+u^*_{I\nu}\cdot e^{+i\Phi_I}]^p\\
&\cdot(J_{II})^{(l-p)/2}\cdot[u_{II\nu}\cdot e^{-i\Phi_{II}}+u^*_{II\nu}\cdot e^{+i\Phi_{II}}]^{l-p}\\
&\cdot\sum_{q=0}^{n-l}\binom{n-l}{q}(J_{III})^{q/2}\cdot[u_{III\nu}\cdot e^{-i\Phi_{III}}+u^*_{III\nu}\cdot e^{+i\Phi_{III}}]^q\\
&\cdot(J_{IV})^{(n-l-q)/2}\cdot[u_{IV\nu}\cdot e^{-i\Phi_{IV}}+u^*_{IV\nu}\cdot e^{+i\Phi_{IV}}]^{n-l-q}\\
=&\sum_{l=0}^{n}\binom{n}{l}\sum_{p=0}^{l}\binom{p}{l}\sum_{q=0}^{n-l}\binom{n-l}{q}(J_I)^{p/2}\\
&\cdot(J_{II})^{(l-p)/2}\cdot(J_{III})^{q/2}\cdot(J_{IV})^{(n-l-q)/2}\\
&\cdot\sum_{\lambda=0}^{p}\binom{\mathbf{p}}{\lambda}\cdot(u_{I\nu})^\lambda(u^*_{I\nu})^{p-\lambda}\cdot e^{-i\Phi_I(2\lambda-p)}\\
&\cdot\sum_{\lambda=0}^{l-p}\binom{l-p}{\lambda}\cdot(u_{II\nu})^\lambda(u^*_{II\nu})^{l-p-\lambda}\cdot e^{-i\Phi_{II}(2\lambda-l+p)}\\
&\cdot\sum_{\lambda=0}^{q}\binom{q}{\lambda}\cdot(u_{III\nu})^\lambda(u^*_{III\nu})^{q-\lambda}\cdot e^{-i\Phi_{III}(2\lambda-q)}\\
&\cdot\sum_{\lambda=0}^{n-l-q}\binom{n-l-q}{\lambda}\cdot(u_{IV\nu})^\lambda(u^*_{IV\nu})^{n-l-q-\lambda}\\
&\cdot e^{-i\Phi_{IV}(2\lambda-n+l+q)}.
\end{aligned} \tag{B.4}$$

Thus the terms of $\mathscr{H}^{(1)}$ can be factorized into a periodic and a harmonic function (see (B.2b)) and the Hamiltonian takes the form:

$$\begin{aligned}
\mathscr{H}^{(1)}=&\sum_{n_1}\sum_{n_2}\sum_{n_3}\sum_{n_4}\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3}\sum_{\lambda_4}H_{n_1n_2n_3n_4\lambda_1\lambda_2\lambda_3\lambda_4}(s)\\
&\cdot(J_I)^{n_1/2}\cdot(J_{II})^{n_2/2}\cdot(J_{III})^{n_3/2}\cdot(J_{IV})^{n_4/2}\\
&\cdot e^{-i\cdot\{\lambda_1\cdot\Phi_I+\lambda_2\cdot\Phi_{II}+\lambda_3\cdot\Phi_{III}+\lambda_4\cdot\Phi_{IV}\}},
\end{aligned} \tag{B.5}$$

with

$$H_{n_1n_2n_3n_4\lambda_1\lambda_2\lambda_3\lambda_4}(s+L)=H_{n_1n_2n_3n_4\lambda_1\lambda_2\lambda_3\lambda_4}(s), \tag{B.6}$$

and

$$\begin{aligned}
&\lambda_1\in\{-n_1,-n_1+2,\ldots,+n_1\},\\
&\lambda_2\in\{-n_2,-n_2+2,\ldots,+n_2\},\\
&\lambda_3\in\{-n_3,-n_3+2,\ldots,+n_3\},\\
&\lambda_4\in\{-n_4,-n_4+2,\ldots,+n_4\}.
\end{aligned} \tag{B.7}$$

With (B.5) and (B.6) we have established the connection with the canonical perturbation theory described in [31].

Note that the complex periodic functions $u_{k\nu}$ ($k=I, II, III, IV$) appearing in (B.4) are determined by (6.6), (6.12), (6.7) and (6.10a). They can be conveniently directly calculated using computer programs (for example SLIM)*. A description of a method to determine the eigenvectors of the transfer matrix may be found in [24, 39].

Remark: In terms of the variables Φ_k, J_k which are defined by (6.32) the Hamiltonian takes the form (see (6.39)):

$$\tilde{\mathscr{H}}=\tilde{\mathscr{H}}_0+\tilde{V}(\Phi_I,\Phi_{II},\Phi_{III},\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s), \tag{B.8}$$

with

$$\tilde{\mathscr{H}}_0=\sum_{k=I,II,III,IV}J_k\cdot\frac{2\pi}{L}Q_k, \tag{B.9a}$$

and

$$\tilde{V}\equiv\mathscr{H}^{(1)}. \tag{B.9b}$$

Here the term $\tilde{V}(\Phi,\hat{J};s)$ describes the perturbation and is periodic in s and Φ_k:

$$\begin{aligned}
&\tilde{V}(\Phi_I,\Phi_{II},\Phi_{III},\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s)\\
&\quad=\tilde{V}(\Phi_I+2\pi,\Phi_{II},\Phi_{III},\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s)\\
&\quad=\tilde{V}(\Phi_I,\Phi_{II}+2\pi,\Phi_{III},\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s)\\
&\quad=\tilde{V}(\Phi_I,\Phi_{II},\Phi_{III}+2\pi,\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s)\\
&\quad=\tilde{V}(\Phi_I,\Phi_{II},\Phi_{III},\Phi_{IV}+2\pi,J_I,J_{II},J_{III},J_{IV};s)\\
&\quad=\tilde{V}(\Phi_I,\Phi_{II},\Phi_{III},\Phi_{IV},J_I,J_{II},J_{III},J_{IV};s+L).
\end{aligned} \tag{B.10}$$

The corresponding canonical equations read as:

$$\frac{d\Phi_k}{ds}=-\frac{\partial\tilde{\mathscr{H}}}{\partial J_k}=\frac{2\pi}{L}Q_k+\frac{\partial\tilde{V}}{\partial J_k}, \tag{B.11a}$$

$$\frac{dJ_k}{ds}=-\frac{\partial\tilde{\mathscr{H}}}{\partial\Phi_k}=-\frac{\partial\tilde{V}}{\partial\Phi_k}. \tag{B.11b}$$

In this form the Hamiltonian can be used for a version of perturbation theory given by Courant et al. [34, 32].

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* As shown in Appendix A, the eigenvectors can be approximated by neglecting the SG effects

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