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A general harmonic spin matching formalism for the suppression of depolarisation caused by closed orbit distortion in electron storage rings

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Abstract

We present a general formalism for correcting perturbations to the equilibrium spin axis in electron storage rings due to the orbit errors so that depolarizing effects due to machine misalignments can be controlled. The method proposed is suitable for rings containing e.g. solenoids, skew quadrupoles and vertical bends and since it is based on a SLIM-like [1)] representation of the orbital and spin motion it can be conveniently realized as a straight forward extension to that program.

Contents

1. Introduction

In electron storage rings the electron spins become polarised antiparallel to the magnetic bending field as a result of synchrotron radiation emission (the Sokolov-Ternov effect) [2]. The maximum polarization obtainable from this effect is 92.4 % and occurs when the ring contains no vertical bends and the equilibrium spin vector lies everywhere along the guide field.

In addition to the polarizing effect there are a number of sources of depolarization resulting from the coupling of the spin motion with the transverse and longitudinal orbital motion. Thus, in practice, the equilibrium polarization is less than the Sokolov-Ternov prediction and can be strongly dependent on the precise optical state of the machine. Therefore, if high polarizations are to be consistently obtained, it is at least necessary that steps are taken to suppress the depolarizing effects and to do so in a way which is convenient and reproducible.

At the level of the linear theory of depolarizing effects used in the program SLIM by A. Chao [1,3] two kinds of measures are available:

- To begin with, the depolarization effects which occur in the ideal machine must be minimized. The required optimization techniques are now well-known [4,5,6,7] and can, for example, be inferred from the equations describing the rate of depolarization [8]. In the case of the ideal flat machine, the equilibrium spin axis, the so-called $\vec{n}$-axis, is vertical in the arcs and an important source of depolarization resulting from horizontal particle oscillations can be neglected [9].

- Unfortunately, in a real machine these "spin matching" conditions are not sufficient. As a result of unavoidable errors in the fields and the positioning of the machine elements the closed orbit becomes distorted and this causes the $\vec{n}$-axis to become tilted from its ideal direction. In this case, the spin motion can again become strongly coupled to the particle oscillations (which can be considered as receiving contributions from both betatron motion and dispersion motion) and further steps must be taken [7,9,10,11]. Furthermore, gradient errors in the quadrupoles can also spoil the spin matches.

This work will be devoted to a discussion only of methods for the correction of the depolarizing effects caused by the closed orbit distortion. It will be shown how the method already proposed by R. Schmidt et.al. [10] for a decoupled flat machine such as PETRA (whereby vertical correction coils are used to correct the closed orbit so that the tilt of the $\vec{n}$-axis is reduced), can be generalized to cover machines containing skew-quadrupoles, solenoids and vertical bends. Thus, the formalism presented here will be applicable to rings containing spin rotators [9,12,13] by means of which the spins can be made longitudinal at the interaction point.

2. Equations of motion

The starting point for the study of the general harmonic correction scheme is the specification of the equations of spin-orbit motion [3].

2.1 The equations of motion for the orbit

Using the notation of Ref. 3), the linearized equations of orbital motion are written as

$$\frac{d}{ds}\vec{y} = \underline{A}\vec{y} + \vec{c}_0 + \vec{c}_1 \tag{2.1}$$

with

$$\underline{A} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2+H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cos\Phi\cdot\sum\limits_{\nu}\delta(s-s_\nu) & 0 \end{pmatrix}; \tag{2.2}$$

$$\vec{c}_0^{\,T} = (0,\ 0,\ 0,\ 0,\ 0,\ \frac{e\hat{V}}{E_0}\sin\Phi\cdot\sum_{\nu}\delta(s-s_\nu) - C_1(K_x^2+K_z^2))\ ; \tag{2.3}$$

$$\vec{c}_1^{\,T} = (0,\ -\frac{e}{E_0}\,\Delta B_z,\ 0,\ \frac{e}{E_0}\,\Delta B_x,\ 0,\ 0)\ ; \tag{2.4}$$

$$H = \frac{1}{2}\frac{e}{E_o} \cdot B_\tau^{(o)} \quad ;$$

$$N = \frac{1}{2}\frac{e}{E_o}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=o} \quad ;$$

$$g = \frac{e}{E_o}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o} \quad ;$$

$$G_1 = K_x^2 + g \quad ;$$

$$G_2 = K_z^2 - g \quad ;$$

$$C_1 = \frac{2}{3} e^2 \frac{\gamma_o^2}{E_o} \quad ;$$

$$\vec{y}^T = (x, p_x, z, p_z, \sigma, \eta) \quad ;$$

$$p_x = x' - H \cdot z \quad ;$$

$$p_z = z' + H \cdot x \quad .$$

In this form, the matrix $\underline{A}$ describes the effect of lenses and cavities and the vector $\vec{c}_1$ the effect of field "errors" ΔB_x, ΔB_z caused by magnet misalignments etc. and by orbit correction magnets. Field errors ΔB_τ have been neglected here because they only appear in second order in the equation of orbit motion. The vector $\vec{c}_0$ describes the effect of energy variations caused by radiation in the bending magnets and energy uptake in the cavities. In detail, one has:

a) $g \neq 0$; $N = H = \hat{V} = 0$; $K_x = K_z = 0$: quadrupole ;

b) $N \neq 0$; $H = g = \hat{V} = 0$; $K_x = K_z = 0$: skew quadrupole ;

c) $G_1 = K_x^2 + g$; $G_2 = -g$ or $G_1 = g$; $G_2 = K_z^2 - g$; $H = \hat{V} = 0$:

combined function magnet ;

d) $H \neq 0$; $g = N = \hat{V} = 0$; $K_x = K_z = 0$: solenoid ;

e) $\hat{V} \neq 0$; $g = N = H = 0$; $K_x = K_z = 0$: cavity.

2.2 Spin motion

Spin motion in a storage ring is described by the BMT [14] precession equation

$$\frac{d}{ds}\vec{\xi} = \underline{\Omega} \cdot \vec{\xi}$$

where

$$\vec{\xi} = \begin{pmatrix} \xi_\tau \\ \xi_x \\ \xi_z \end{pmatrix} \quad ;$$

describes the spin vector and (Ref. 3)

$$\underline{\Omega} = \begin{pmatrix} 0 & -\Omega_z & \Omega_x \\ \Omega_z & 0 & -\Omega_\tau \\ -\Omega_x & \Omega_\tau & 0 \end{pmatrix} \quad ; \tag{2.5}$$

$$\Omega_\tau = -\,2H \cdot (1 + a) - \frac{e}{E_0}\,\Delta B_\tau \cdot (1 + a) + $$
$$+\,2H \cdot \eta \cdot (1 + a) - a\gamma_0 \cdot (x' \cdot K_z - z' \cdot K_x) \quad ; \tag{2.5a}$$

$$\Omega_x = K_z \cdot a\gamma_0 + (1 + a\gamma_0) \cdot K_z^2 \cdot z - K_z \cdot \eta - $$
$$-\,(1 + a\gamma_0) \cdot [(N - H') \cdot x + g \cdot z] + a\gamma_0 \cdot 2H \cdot x' + $$
$$+\,(1 + a\gamma_0) \cdot \frac{e}{E_0}\,\hat{V}\,\sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \cdot z' - $$
$$-\,(1 + a\gamma_0) \cdot \frac{e}{E_0} \cdot \Delta B_x \quad ; \tag{2.5b}$$

$$\Omega_z = -K_x \cdot a\gamma_0 - (1 + a\gamma_0) \cdot K_x^2 \cdot x + K_x \cdot \eta + $$
$$+\,(1 + a\gamma_0) \cdot [(N + H') \cdot z - g \cdot x] + a\gamma_0 \cdot 2H \cdot z' - $$
$$-\,(1 + a\gamma_0) \cdot \frac{e}{E_0}\,\hat{V}\,\sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \cdot x' - $$
$$-\,(1 + a\gamma_0) \cdot \frac{e}{E_0} \cdot \Delta B_z \quad . \tag{2.5c}$$

The "spin matrix" Ω in (2.5) can be decomposed into two parts:

$$\underline{\Omega} = \underline{\Omega}^{(o)} + \underline{\omega} \qquad (2.6)$$

with

$$\Omega_\tau^{(o)} = -2H \cdot (1 + a) \quad ; \qquad (2.7a)$$

$$\Omega_x^{(o)} = K_z \cdot a\gamma_0 \quad ; \qquad (2.7b)$$

$$\Omega_z^{(o)} = -K_x \cdot a\gamma_0 \quad ; \qquad (2.7c)$$

and

$$\omega_\tau = -\frac{e}{E_0} \Delta B_\tau \cdot (1 + a) \; + \; + 2H \cdot \eta \cdot (1 + a) - a\gamma_0 \cdot (x' \cdot K_z - z' \cdot K_x) \quad ; \qquad (2.8a)$$

$$\omega_x = -(1 + a\gamma_0) \cdot \frac{e}{E_0} \Delta B_x + (1 + a\gamma_0) \cdot K_z^2 \cdot z - K_z \cdot \eta \; - \; - (1 + a\gamma_0) \cdot [(N - H') \cdot x + g \cdot z] + a\gamma_0 \cdot 2H \cdot x' \; + \; + (1 + a\gamma_0) \cdot \frac{e}{E_0} \hat{V} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \cdot z' \quad ; \qquad (2.8b)$$

$$\omega_z = -(1 + a\gamma_0) \cdot \frac{e}{E_0} \Delta B_z - (1 + a\gamma_0) \cdot K_x^2 \cdot x + K_x \cdot \eta \; + \; + (1 + a\gamma_0) \cdot (N + H') \cdot z - g \cdot x] + a\gamma_0 \cdot 2H \cdot z' \; - \; - (1 + a\gamma_0) \cdot \frac{e}{E_0} \hat{V} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \cdot x' \quad . \qquad (2.8c)$$

where $\Omega^{(o)}$ is due to spin precession on the design orbit.

Furthermore, equation (2.8) can be written in the form

$$\begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} = \underline{F} \cdot \vec{y} + \vec{c} \qquad (2.9)$$

with

$$\vec{c} = -\frac{e}{E_0} \cdot \begin{pmatrix} \Delta B \cdot (1 + a) \\ \Delta B_x \cdot (1 + a\gamma_0) \\ \Delta B_z \cdot (1 + a\gamma_0) \end{pmatrix} \qquad (2.10)$$

and

$$F = ((F_{\mu\nu})) \quad ;$$

$$F_{12} = - a\gamma_0 \cdot K_z \quad ;$$

$$F_{14} = a\gamma_0 \cdot K_x \quad ;$$

$$F_{16} = 2H \cdot (1 + a) \quad ;$$

$$F_{21} = - (1 + a\gamma_0)(N - H') \quad ;$$

$$F_{22} = a\gamma_0 \cdot 2H \quad ;$$

$$F_{23} = (1 + a\gamma_0) \cdot (K_z^2 - g) + a\gamma_0 \cdot 2H^2 \quad ;$$

$$F_{24} = (a\gamma_0 + 1) \cdot \frac{e\hat{V}}{E_0} \sin\Phi \cdot \sum_\nu \delta(s - s_\nu) \quad ;$$

$$F_{26} = - K_z \quad ;$$

$$F_{31} = - (1 + a\gamma_0)(K_x^2 + g) - a\gamma_0 \cdot 2H^2 \quad ;$$

$$F_{32} = - F_{24} \quad ;$$

$$F_{33} = (1 + a\gamma_0)(N + H') \quad ;$$

$$F_{34} = F_{22} \quad ;$$

$$F_{36} = K_x \quad ;$$

$$F_{\mu\nu} = 0 \quad \text{otherwise.} \qquad (2.11)$$

From (2.5) and (2.6) it follows that

$$\frac{d}{ds} \vec{\xi} = (\underline{\Omega}^{(o)} + \underline{\omega}) \cdot \vec{\xi} \ , \qquad (2.12)$$

where we assume that $\underline{\omega}$ can be treated as a small perturbation.

If as in SLIM we make the ansatz

$$\vec{\xi} = \vec{\xi}^{(o)} + \vec{\xi}^{(1)} \tag{2.13}$$

we obtain from (2.12) in first order approximation

$$\frac{d}{ds} \vec{\xi}^{(o)} = \underline{\Omega}^{(o)} \cdot \vec{\xi}^{(o)} \quad ; \tag{2.14a}$$

$$\begin{aligned} \frac{d}{ds} \vec{\xi}^{(1)} &= \underline{\Omega}^{(o)} \cdot \vec{\xi}^{(1)} + \underline{\omega} \cdot \vec{\xi}^{(o)} \\ &= \vec{\Omega}^{(o)} \times \vec{\xi}^{(1)} + \vec{\omega} \times \vec{\xi}^{(o)} \quad ; \end{aligned} \tag{2.14b}$$

$$\vec{\Omega}^{(o)} = \begin{pmatrix} \Omega_\tau^{(o)} \\ \Omega_x^{(o)} \\ \Omega_z^{(o)} \end{pmatrix} \quad ; \quad \vec{\omega} = \begin{pmatrix} \omega_\tau \\ \omega_x \\ \omega_z \end{pmatrix} \quad .$$

3. The $(\vec{n}, \vec{m}, \vec{\ell})$ spin coordinate system

The matrix $\underline{\Omega}^{(o)}$ can now also serve to define a new orthogonal coordinate system $(\vec{n}, \vec{m}, \vec{\ell})$ for describing the spin motion, and we thus consider the 3x3 transfer matrix $\underline{N}(s,s_0)$ of the precession equation (2.14a):

$$\xi^{(o)}(s) = \underline{N}(s,s_0) \cdot \xi^{(o)}(s_0) \tag{3.1}$$

and investigate the eigenvalue spectrum of the one turn matrix $\underline{N}(s_0+L,s_0)$ to obtain:

$$\underline{N}(s_0+L,s_0)\,\vec{r}_\mu(s_0) = \alpha_\mu \cdot \vec{r}_\mu(s_0) \quad ; \tag{3.2a}$$

$$\alpha_1 = 1 \quad ; \quad \vec{r}_1(s_0) = \vec{n}(s_0) \quad ;$$

$$\alpha_2 = e^{+i\cdot 2\pi\nu} \; ; \quad \vec{r}_2(s_0) = \vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0) \quad ; \tag{3.2b}$$

$$\alpha_3 = e^{-i\cdot 2\pi\nu} \; ; \quad \vec{r}_3(s_0) = \vec{m}_0(s_0) - i\cdot\vec{\ell}_0(s_0)$$

where L is the length of the orbit and where the "spin tune" ν can be separated into an arbitrary integer part $\varkappa$ and a fractional part $\tilde{\nu}$:

$$\nu = \varkappa + \tilde{\nu} \quad ; \tag{3.2c}$$

$$0 \leqq \nu < 1$$

and where

$$\left\{ \begin{array}{l} \vec{n}_0(s_0) = \vec{m}_0(s_0) \times \vec{\ell}_0(s_0) \quad ; \\ \vec{m}_0(s_0) \perp \vec{\ell}_0(s_0) \quad ; \\ |\vec{n}_0(s_0)| = |\vec{m}_0(s_0)| = |\vec{\ell}_0(s_0)| = 1 \quad . \end{array} \right. \tag{3.2d}$$

Using, as usual [3] the spin phase function $\Psi(s)$ with the property

$$\Psi(s_0+L) - \Psi(s_0) = 2\pi\nu \tag{3.3}$$

we now introduce new vectors $(\vec{n}, \vec{m}, \vec{\ell})$ defined by the relations:

$$\vec{n}(s) = \underline{N}(s,s_0)\,\vec{n}(s_0) \tag{3.4}$$

and

$$\vec{m}(s) + i\cdot\vec{\ell}(s) = e^{-i\cdot[\Psi(s) - \Psi(s_0)]}\cdot \underline{N}(s,s_0)\cdot[\vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0)] \quad (3.5)$$

and we find

$$\begin{cases} \vec{n}(s) = \vec{m}(s) \times \vec{\ell}(s) \;; \\ \vec{m}(s) \perp \vec{\ell}(s) \;; \\ |\vec{n}(s)| = |\vec{m}(s)| = |\vec{\ell}(s)| = 1 \;; \end{cases} \quad (3.6)$$

$$(\vec{n}, \vec{m}, \vec{\ell})_{s=s_0+L} = (\vec{n}, \vec{m}, \vec{\ell})_{s=s_0} \quad (3.7)$$

so that the vectors $\vec{n}$, $\vec{m}$, $\vec{\ell}$ comprise an orthogonal system which transforms into itself after one turn.

In addition, from (3.3) and (3.4) and by using (3.1), it follows that:

$$\frac{d}{ds}[\vec{m}(s) + i\cdot\vec{\ell}(s)] = e^{-i\cdot[\Psi(s) - \Psi(s_0)]}\cdot\underbrace{\frac{d}{ds}\underline{N}(s,s_0)}_{\underline{\Omega}^{(o)}\,\underline{N}(s,s_0)}[\vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0)]$$

$$- i\cdot\Psi'(s)\cdot e^{-i\cdot[\Psi(s) - \Psi(s_0)]}\,\underline{N}(s,s_0)\cdot[\vec{m}_0(s_0) + i\cdot\vec{\ell}_0(s_0)]$$

$$= \underline{\Omega}^{(o)}\cdot[\vec{m}(s) + i\cdot\vec{\ell}(s)] - i\cdot\Psi'(s)\,[\vec{m}(s) + i\cdot\vec{\ell}(s)]$$

so that

$$\begin{aligned} \frac{d}{ds}\vec{m}(s) &= \underline{\Omega}^{(o)}\cdot\vec{m}(s) + \Psi'(s)\cdot\vec{\ell}(s) \;; \\ \frac{d}{ds}\vec{\ell}(s) &= \underline{\Omega}^{(o)}\cdot\vec{\ell}(s) - \Psi'(s)\cdot\vec{m}(s) \;; \end{aligned} \quad (3.8)$$

and

$$\frac{d}{ds}\vec{n}(s) = \underline{\Omega}^{(o)}\cdot\vec{n}(s) \quad (3.9)$$

We would like to emphasize here that apart from the restriction in equ. (3.3) the spin phase function $\Psi(s)$ can be otherwise quite arbitrary and can be taylored so as to lead to the choice of spin basis vectors best suited to the problem in hand. As is clear from (3.4) and the discussion to follow each choice corresponds to a particular choice of the rotating coordinate spin system from which spin perturbations are viewed.

4. Solutions of the equation of spin motion

In order to solve the spin perturbation equations (2.14) we use the following ansatz:

$$\vec{\xi}^{(o)} = \xi_o \cdot \vec{n}(s) \quad ; \tag{4.1a}$$

$$\vec{\xi}^{(1)} = \xi_o \cdot [\alpha(s) \cdot \vec{m}(s) + \beta(s) \cdot \vec{\ell}(s)] \quad . \tag{4.1b}$$

Thus, equation (2.14a) is, according to (4.1a) and (3.9) already fulfilled whereas from (2.14b) we obtain

$$\frac{d}{ds}[\alpha \cdot \vec{m} + \beta \cdot \vec{\ell}] = \underline{\Omega}^{(o)} \cdot [\alpha \cdot \vec{m} + \beta \cdot \vec{\ell}] + \vec{\omega} \times \vec{n}$$

or, using (3.8):

$$\alpha' \cdot \vec{m} + \beta' \cdot \vec{\ell} + \alpha \Psi' \cdot \vec{\ell} - \beta \Psi' \cdot \vec{m} = \vec{\omega} \times \vec{n} \quad .$$

Thus

$$\begin{aligned} \alpha' &= \beta \Psi' + \vec{m}^T \cdot (\vec{\omega} \times \vec{n}) \\ &= \beta \Psi' + \vec{\omega}^T \cdot (\vec{n} \times \vec{m}) \\ &= \beta \Psi' + \vec{\omega}^T \cdot \vec{\ell} \\ &= \beta \Psi' + \vec{\ell}^T \cdot \vec{\omega} \quad ; \end{aligned}$$

$$\begin{aligned} \beta' &= - \alpha \cdot \Psi' + \vec{\ell}^T \cdot (\vec{\omega} \times \vec{n}) \\ &= - \alpha \cdot \Psi' + \vec{\omega}^T \cdot (\vec{n} \times \vec{\ell}) \\ &= - \alpha \cdot \Psi' - \vec{\omega}^T \cdot \vec{m} \\ &= - \alpha \cdot \Psi' - \vec{m}^T \cdot \vec{\omega} \end{aligned}$$

so that with (2.9) we get:

$$\frac{d}{ds} \vec{\xi} = \underline{D}_o \cdot \vec{\xi} + \underline{R} \; [F \cdot \vec{y} + \vec{c}] \tag{4.2}$$

with

$$\vec{\xi} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad ; \tag{4.2a}$$

$$\underline{R} = \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \quad ; \tag{4.2b}$$

$$\underline{D}_0 = \Psi' \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \tag{4.2c}$$

The solution of (4.2) can be constructed in closed form as:

$$\vec{\xi}(s) = \underline{D}(s,s_0) \cdot \{\vec{\xi}(s_0) + \int_{s_0}^{s} d\tilde{s} \cdot \underline{D}(s_0,\tilde{s}) \cdot \underline{R}(\tilde{s}) \cdot [\underline{F}(\tilde{s})\vec{y} + \vec{c}(\tilde{s})] \tag{4.3}$$

where we introduce the rotation matrix

$$\underline{D}(s,s_0) = \begin{pmatrix} \cos[\Psi(s) - \Psi(s_0)] & \sin[\Psi(s) - \Psi(s_0)] \\ -\sin[\Psi(s) - \Psi(s_0)] & \cos[\Psi(s) - \Psi(s_0)] \end{pmatrix} \tag{4.4}$$

5. Calculation of the perturbed $\vec{n}$-axis

Equation (4.3) describes the spin motion on an arbitrary particle orbit $\vec{y}(s)$. However, (4.3) can also be immediately applied to the case where the particle is moving on the closed orbit $\hat{\vec{y}}(s)$ defined by

$$\frac{d}{ds}\,\hat{\vec{y}}(s) = \underline{A}\,\hat{\vec{y}}(s) + \vec{c}_0(s) + \vec{c}_1(s) \quad ;$$
$$\hat{\vec{y}}(s_0+L) = \hat{\vec{y}}(s_0) \; .$$

If we also require that the resulting spin motion is periodic:

$$\vec{\xi}(s_0+L) = \vec{\xi}(s_0) \; ,$$

then $\vec{\xi}(s)$ gives the perturbation $\delta\vec{n}(s)$ of the $\vec{n}$-axis caused by the error fields $\Delta B_x(s)$, $\Delta B_z(s)$ and $\Delta B_\tau(s)$ as well as the effect of energy variation on the closed orbit due to $\vec{c}_0$. Thus:

$$\delta\vec{n}(s) = \underline{D}(s,s_0)\cdot\{\delta\vec{n}(s_0) + \int_{s_0}^{s} d\tilde{s}\;\; \underline{D}(s_0,\tilde{s})\cdot\underline{R}(\tilde{s})\cdot\hat{\vec{c}}(\tilde{s})\} \tag{5.1}$$

with

$$\hat{\vec{c}}(s) = \underline{F}(s)\cdot\hat{\vec{y}}(s) + \vec{c}(s) \quad , \tag{5.2}$$

and the periodicity condition takes the form

$$\begin{aligned}\delta\vec{n}(s_0) &= \delta\vec{n}(s_0+L) \\ &= \underline{D}(s_0+L,s_0)\cdot\{\delta\vec{n}(s_0) + \int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\underline{R}(\tilde{s})\cdot\hat{\vec{c}}(\tilde{s})\} \; .\end{aligned} \tag{5.3}$$

From (5.3) one then obtains $\delta\vec{n}$ in the form

$$\delta\vec{n}(s_0) = [\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot\underline{D}(s_0+L,s_0) \quad \times$$
$$\times \quad \int_{s_0}^{s_0+L} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\underline{R}(\tilde{s})\cdot\hat{\vec{c}}(\tilde{s}) \quad . \tag{5.4}$$

By substituting (5.4) in (5.1), $\delta\vec{n}$ at arbitrary s becomes:

$$\delta\vec{n}(s) = \underline{D}(s,s_0)\cdot\{[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot\underline{D}(s_0+L,s_0) \times$$

$$\times \int_{s_0}^{s_0+L} d\tilde{s}\cdot D(s_0,\tilde{s})\cdot R(\tilde{s})\cdot\hat{c}(\tilde{s}) +$$

$$+ \int_{s_0}^{s} d\tilde{s}\cdot D(s_0,\tilde{s})\cdot R(\tilde{s})\cdot\vec{\hat{c}}(\tilde{s})\}$$

$$= \underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1} \times$$

$$\times \{\underline{D}(s_0+L,s_0) + [\underline{1} - \underline{D}(s_0+L,s_0)]\} \times$$

$$\times \int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot R(\tilde{s})\cdot\vec{\hat{c}}(\tilde{s}) +$$

$$+ \underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot\underline{D}(s_0+L,s_0) \times$$

$$\times \int_{s}^{s_0+L} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot R(\tilde{s})\cdot\vec{\hat{c}}(\tilde{s}) .$$

Since, by changing variables

$$\int_{s}^{s_0+L} d\tilde{s}\cdot\underline{D}(s_0+L,\tilde{s})\cdot\underline{R}(\tilde{s})\cdot\vec{\hat{c}}(\tilde{s}) =$$

$$= \int_{s-L}^{s_0} ds'\cdot\underline{D}(s_0+L,s'+L)\cdot\underline{R}(s'+L)\cdot\vec{\hat{c}}(s'+L) =$$

$$= \int_{s-L}^{s_0} ds'\cdot\underline{D}(s_0,s')\cdot\underline{R}(s')\cdot\vec{\hat{c}}(s') \quad (5.5a)$$

we can write

$$\delta\vec{n}(s) = \underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1} \quad \times$$

$$\times \quad \{ \int_{s_0}^{s} d\tilde{s}\cdot \underline{D}(s_0,\tilde{s})\cdot \underline{R}(\tilde{s})\cdot \vec{\hat{c}}(\tilde{s}) \quad +$$

$$+ \quad \underbrace{\int_{s}^{s_0+L} d\tilde{s}\cdot \underline{D}(s_0+L,\tilde{s})\cdot \underline{R}(\tilde{s})\cdot \vec{\hat{c}}(\tilde{s})\}}_{\int_{s-L}^{s_0} ds' \cdot \underbrace{\underline{D}(s_0+L,s'+L)}_{D(s_0,s')}\ \underbrace{\underline{R}(s'+L)}_{\underline{R}(s')}\ \underbrace{\vec{\hat{c}}(s'+L)}_{\vec{\hat{c}}(s')}}$$

(due to equ. (5.5a))

$$= \underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot \int_{s-L}^{s} d\tilde{s}\cdot \underline{D}(s_0,\tilde{s})\cdot \underline{R}(\tilde{s})\cdot \vec{\hat{c}}(\tilde{s}) \ . \qquad (5.5)$$

Since

$$\underline{D}(\alpha)\cdot \underline{D}(\beta) = \underline{D}(\alpha + \beta) = \underline{D}(\beta)\cdot \underline{D}(\alpha)$$

where

$$D(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

then

$$\underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)] = [\underline{1} - \underline{D}(s_0+L,s_0)]\cdot \underline{D}(s,s_0)$$

so that

$$[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot \underline{D}(s,s_0) = \underline{D}(s,s_0)\cdot[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}. \qquad (5.6)$$

Thus (5.5) becomes

$$\delta\vec{n}(s) = [\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}\cdot \int_{s-L}^{s} d\tilde{s}\cdot \underline{D}(s,\tilde{s})\cdot \underline{R}(\tilde{s})\cdot \vec{\hat{c}}(\tilde{s}) \ . \qquad (5.7)$$

Furthermore, the rotation matrix $\underline{D}(s_0+L,s_0)$ in (5.7) (see (3.3), (4.4))

$$\underline{D}(s_0+L,s_0) = \begin{pmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{pmatrix}$$

can be diagonalised as follows:

$$\underline{D}(s_0+L,s_0) = \underline{U}\ \underline{J}\ \underline{U}^{-1}$$

with

$$\underline{U} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \ ; \quad \underline{U}^{-1} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} ; \qquad (5.8a)$$

$$\underline{J} = \begin{pmatrix} e^{i\cdot 2\pi\nu} & 0 \\ 0 & e^{-i\cdot 2\pi\nu} \end{pmatrix} \ ; \qquad (5.8b)$$

so that the factor

$$[\underline{1} - \underline{D}(s_0+L,s_0)]^{-1}$$

on the right hand side of (5.7) becomes

$$\begin{aligned} [\underline{1} - \underline{D}(s_0+L,s_0)]^{-1} &= [\underline{U}\ \underline{U}^{-1} - \underline{U}\ \underline{J}\ \underline{U}^{-1}]^{-1} \\ &= [\underline{U} \cdot (\underline{1} - \underline{J})\ \ \underline{U}^{-1}]^{-1} \\ &= U \cdot (\underline{1} - \underline{J})^{-1} \cdot U^{-1}\ . \end{aligned} \qquad (5.9)$$

Equation (5.7) can be reexpressed as

$$\underline{U}^{-1} \cdot \delta\vec{n}(s) = (\underline{1} - \underline{J})^{-1} \cdot \int_{s-L}^{s} d\tilde{s}\ \ \underline{U}^{-1} \cdot \underline{D}(s,\tilde{s}) \cdot \underline{R}(\tilde{s}) \cdot \vec{\tilde{c}}(\tilde{s}) \qquad (5.10)$$

and by writing

$$\underline{R}(\tilde{s}) \cdot \vec{\tilde{c}}(\tilde{s}) = \vec{d}(\tilde{s}) \equiv \begin{pmatrix} d_1(\tilde{s}) \\ d_2(\tilde{s}) \end{pmatrix} \qquad (5.11)$$

(5.10) can finally be written (see (4.4) and (5.8)) as [23]:

$$\begin{pmatrix} \delta n_1(s) - i \cdot \delta n_2(s) \\ \delta n_1(s) + i \cdot \delta n_2(s) \end{pmatrix} = \frac{i}{2} \cdot \frac{1}{\sin \pi\nu} \cdot \begin{pmatrix} e^{-i\pi\nu} & 0 \\ 0 & -e^{+i\pi\nu} \end{pmatrix} \times$$

$$\times \int_{s-L}^{s} d\tilde{s} \cdot \begin{pmatrix} e^{i\cdot[\Psi(s) - \Psi(\tilde{s})]} \cdot [d_1(\tilde{s}) - i \cdot d_2(\tilde{s})] \\ e^{-i\cdot[\Psi(s) - \Psi(\tilde{s})]} \cdot [d_1(\tilde{s}) + i \cdot d_2(\tilde{s})] \end{pmatrix} .$$

Since, in this vector equation, the two components are just complex conjugates of each other, it suffices to use just one component:

$$\delta n_1(s) - i \cdot \delta n_2(s) = \frac{i}{2} \frac{1}{\sin \pi\nu} \cdot e^{i\cdot[\Psi(s) - \pi\nu]} \times$$

$$\times \int_{s-L}^{s} d\tilde{s} \cdot e^{-i\cdot\Psi(\tilde{s})} \cdot [d_1(\tilde{s}) - i \cdot d_2(\tilde{s})] \quad . \tag{5.12}$$

We also note that for $\delta\vec{n}$ to remain small so that the perturbation theory remains valid, ν must not be too close to an integer (equivalently, det $[\underline{D} - \underline{1}]$ must not be close to zero). It is also clear that the components δn_1 and δn_2 depend on the choice of phase function. However, $(\delta n_1)^2 + (\delta n_2)^2$ is of course independent of the choice of $\Psi(s)$.

Because $\vec{m}$ and $\vec{\ell}$ are periodic we find that the expression

$$[d_1(\tilde{s}) - i \cdot d_2(\tilde{s})]$$

in (5.12) is periodic and can thus be expanded in a Fourier series:

$$[d_1(\tilde{s}) - i \cdot d_2(\tilde{s})] = \sum_{k=-\infty}^{+\infty} f_k \cdot e^{i\cdot k\cdot 2\pi \frac{\tilde{s}}{L}} \tag{5.13a}$$

where

$$f_k = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot [d_1(\tilde{s}) - i \cdot d_2(\tilde{s})] \cdot e^{-i\cdot k\cdot 2\pi \frac{\tilde{s}}{L}} \tag{5.13b}$$

If we now choose a spin phase function which increases uniformly with s according to

$$\Psi(s) = \Psi(s_0) + 2\pi\nu \cdot \frac{s - s_0}{L} \quad . \tag{5.14}$$

so that (3.3) is satisfied, then (5.12) takes an especially simple form:

$$[\delta n_1(s) - i \cdot \delta n_2(s)] = -i \cdot \frac{L}{2\pi} \cdot \sum_k f_k \cdot \frac{e^{i\cdot 2\pi k \frac{s}{L}}}{k - \nu} \tag{5.15}$$

This equation describes the connection between the perturbation $\delta\vec{n}$ of the $\vec{n}$-axis and the Fourier harmonics f_k of the scalar function $d_1(s) - i \cdot d_2(s)$ (5.11) (see also equ. (5.2) and (2.10))

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} \cdot \left[\underline{F} \cdot \hat{y} - \frac{e}{E_0} \cdot \begin{pmatrix} \Delta B_\tau \; (1+a) \\ \Delta B_x \; (1+a\gamma_0) \\ \Delta B_z \; (1+a\gamma_0) \end{pmatrix} \right] \qquad (5.16)$$

which in turn is determined by the shape of the closed orbit and by the magnitude and the position of the field errors ΔB_x, ΔB_z, ΔB_τ.
This equation will serve as the starting point for the investigation of the optimization method but before proceeding, we will make a number of comments on the content of the last few equations.

Firstly, since in (5.16) d_1 and d_2 depend on the relative orientation of the vectors $\vec{m}$ and $\vec{\ell}$ and the closed orbit distortion, two machines with the same closed orbit deviation but with different orientations of the $\vec{n}$-axis in the interaction region (say) will have different distributions for the harmonic strengths f_k. Thus, in a machine like HERA[15] the strength of the correctors which would be applied for adjusting the f_k (see below) would, even if the closed orbit were to remain unchanged, depend on whether the spin rotators were switched on.

Secondly, it is clear that two totally different vectors $\vec{c}$ (defined in equ. (2.10)) generated by different sources of field error can result in the same strength for a selected harmonic f_k. Thus, an f_k generated by one type of error (e.g. a closed orbit deviation in a solenoid spin rotator[9] or a ΔB_τ due to an incorrectly compensated solenoid) could be cancelled by applying in addition a different kind of error such as a ΔB_x distribution. The second example would be a generalization of the use of local beam bumps to correct for solenoid effects already suggested for PETRA[16].

The phase function chosen in (5.14) differs from that used in Ref. 10) where the quantity representing the phase function only advances in the bending magnets[17]. However although, as mentioned above, $(\delta\vec{n})^2$ does not depend on the phase function, the advantage of the present choice is that it enables a simple Fourier expansion of $\delta\vec{n}$ to be made (5.15) so that the relationship between $\delta\vec{n}$ and the harmonics of the closed orbit is particularly clear.

Furthermore, once the f_k have been calculated, equ. (5.15) enables $\delta\vec{n}$ to be specified in a very simple manner at all points in the ring and at all energies. As we will see below, the latter possibility then allows $\delta\vec{n}$ to be minimized (with the aid of correction coils) at all points in the ring and not only near to horizontal bending magnets [10].

Although constructed from a different point of view (Equ. (3.3)) the spin reference frame generated by the phase function $\Psi(s)$ given in (5.14) is in fact identical to the frame used by Ya. Derbenev et. al.[18] and J. Buon[19]. Thus Eqs. (5.11) - (5.16) are also closely related to equations for $\delta\vec{n}$ in Refs. 18, 19). However, in the present treatment the dependence of $\delta\vec{n}$ on the complete 6-dimensional closed orbit is given so that energy variation on the closed orbit is included. The latter can be particularly important when solenoid spin rotators are used. Furthermore, as we will see below, with the form for the f_k given by (5.11) and (5.13b) we are already in a position to invent correction schemes for $\delta\vec{n}$ even for the exotic rings mentioned in the introduction.

Finally, for later considerations, we return again to equ. (3.2c) and note that the integer part $\varkappa$ of the tune ν and the phase function Ψ can be chosen so that the vectors $\vec{n}$, $\vec{m}$, $\vec{\ell}$ reflect the periodicity of the machine structure. For example, with a fourfold symmetric machine (see fig. 1) we can arrange that [20]

$$\begin{aligned} \vec{n}(s + \tfrac{L}{4}) &= \vec{n}(s) \quad ; \\ \vec{m}(s + \tfrac{L}{4}) &= \vec{m}(s) \quad ; \\ \vec{\ell}(s + \tfrac{L}{4}) &= \vec{\ell}(s) \quad . \end{aligned} \tag{5.17}$$

6. Correction schemes

As will be recalled[10], if in a flat storage ring the $\vec{n}$-axis on the closed orbit is tilted from the vertical in the arcs of the ring, strong depolarization can occur as a result of horizontal betatron and horizontal dispersion motion. The remedy is then to reduce the tilt, $\delta\vec{n}$.

In more complicated rings such as those containing spin rotators the "design" $\vec{n}$-axis may not be vertical everywhere and the special optical design strategies (spin matching) adopted to ensure that at least in linear approximation, the depolarizing effects in the perfectly aligned machine are zero, become more involved. Moreover, it is again necessary to consider the effects of closed orbit errors and in these cases, a non-zero $\delta\vec{n}$ represents not only a tilt of the $\vec{n}$-axis from the vertical in the arc but could also represent, for example, a tilt out of the horizontal plane in the interaction region. Nevertheless, at the level of linear theory, the main depolarizing effect is expected to arise from the tilt $\delta\vec{n}$ of the $\vec{n}$-axis from the vertical in the arcs. The purpose of this section is then to investigate how equ. (5.15) can be exploited so that $\delta\vec{n}$ can be made small even in the presence of exotic elements such as experimental solenoids, skew quadrupoles and vertical bends[12,13] or solenoid type spin rotators[9].

From (5.15) it is clear that the largest contributions to $\delta\vec{n}$ tend to come from the harmonics for which $k \approx \nu$ and that $\delta\vec{n}$ could be reduced by adjusting the corresponding f_k's to zero. This can be achieved with the aid of suitable closed orbit corrections.

Thus, we begin by separating the coefficients f_k into two parts:

$$f_k = \tilde{f}_k + f_k^{(o)}$$

where $\tilde{f}_k$ results from the closed orbit distortions caused by field errors $\Delta\tilde{B}_x$, $\Delta\tilde{B}_z$ and $\Delta\tilde{B}_\tau$ and from closed orbit energy variations (due to the vector $\vec{c}_0$) and $f_k^{(o)}$ results from correction fields $\Delta B^{(o)}$. With this description f_k will be zero when $f_k^{(o)}$ is adjusted to be equal to $-\tilde{f}_k$. As mentioned above, there is some freedom as to how the $f_k^{(o)}$ should be generated and in the spirit of the scheme of Ref. 10) we will, in the following, only consider the use of vertical orbit correction coils. These are always able to influence the tilt of the n-axis. The task is then to discover what distribution of coil strengths is required for generating a particular $f_k^{(o)}$.

Since in practice we cannot measure the closed orbit with sufficient accuracy, we do not know the $\tilde{f}_k$. Thus we calculate the ΔB_μ to within a scale factor and would adjust the overall strengths of the coils empirically so as to maximize the polarization.

By (5.13b) the $f_k^{(o)}$ are given as:

$$f_k^{(o)} = \frac{1}{L} \cdot \int_{s_o}^{s_o+L} d\tilde{s} \cdot [d_1^{(o)}(\tilde{s}) - i \cdot d_2^{(o)}(\tilde{s})] \cdot e^{-i \cdot k \cdot 2\pi \frac{\tilde{s}}{L}} =$$

$$= a_k - i \cdot b_k; \tag{6.1}$$

$$a_k = \frac{1}{L} \cdot \int_{s_o}^{s_o+L} d\tilde{s} \cdot \{d_1^{(o)}(\tilde{s}) \cdot \cos(2\pi k \frac{\tilde{s}}{L}) - d_2^{(o)}(\tilde{s}) \cdot \sin(2\pi k \frac{\tilde{s}}{L})\} ; \tag{6.2a}$$

$$b_k = \frac{1}{L} \cdot \int_{s_o}^{s_o+L} d\tilde{s} \cdot \{d_1^{(o)}(\tilde{s}) \cdot \sin(2\pi k \frac{\tilde{s}}{L}) + d_2^{(o)}(\tilde{s}) \cdot \cos(2\pi k \frac{\tilde{s}}{L})\} \tag{6.2b}$$

where we use

$$\vec{d}^{(o)}(s) = \underline{R}(s) \cdot [\underline{F}(s) \cdot \vec{y}^{(o)}(s) + \vec{c}^{(o)}(s)] \equiv \begin{pmatrix} d_1^{(o)}(s) \\ d_2^{(o)}(s) \end{pmatrix} \tag{6.3}$$

and where, by our restriction to vertical correction coils and by (2.10)

$$\vec{c}^{(o)}(s) = -\frac{e}{E_o} \begin{pmatrix} 0 \\ \Delta B_x^{(o)}(s) \cdot (1 + a\gamma_o) \\ 0 \end{pmatrix}$$

so that

$$\underline{R}(s) \cdot \vec{c}^{(o)}(s) = -(1 + a\gamma_o) \cdot \frac{e}{E_o} \cdot \Delta B_x^{(o)}(s) \begin{pmatrix} \ell_x(s) \\ -m_x(s) \end{pmatrix} . \tag{6.4}$$

The closed orbit $\vec{y}^{(o)}(s)$ in (6.3) resulting from $\Delta B_x^{(o)}$ obeys the equations:

$$\frac{d}{ds} \vec{y}^{(o)} = \underline{A}\, \vec{y}^{(o)} + \frac{e}{E_o} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \Delta B_x^{(o)} \\ 0 \\ 0 \end{pmatrix} ; \tag{6.5a}$$

$$\vec{y}^{(o)}(s_o + L) = \vec{y}^{(o)}(s_o) . \tag{6.5b}$$

By approximating the correction fields using delta functions so that

$$\Delta B_x^{(o)}(s) = \sum_{\mu} \Delta\hat{B}_{\mu} \cdot \delta(s - s_{\mu}) \quad . \tag{6.6}$$

where s_μ is the position of the μth correction coil, the closed orbit generated by the correction coils in collaboration with the other (arbitrarily complicated) linear machine elements[21] can be written in the form

$$\vec{y}^{(o)}(s) = \sum_{\mu} \Delta\hat{B}_{\mu} \cdot \vec{y}_{\mu}(s) \tag{6.7}$$

where

$$\frac{d}{ds} \vec{y}_{\mu} = A \, \vec{y}_{\mu} + \frac{e}{E_o} \cdot \delta(s - s_{\mu}) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad ; \tag{6.8a}$$

and

$$\vec{y}_{\mu}(s_o + L) = \vec{y}_{\mu}(s_o) \quad . \tag{6.8b}$$

Thus, for the vector $\vec{d}_o$ in (6.3) we obtain

$$\vec{d}^{(o)}(s) = \sum_{\mu} \Delta\hat{B}_{\mu} \cdot \{\underline{R}(s) \cdot \underline{F}(s) \cdot \vec{y}_{\mu}(s) \; -$$

$$- \; \delta(s - s_{\mu}) \cdot \frac{e}{E_o} \cdot (1 + a\gamma_o) \begin{pmatrix} \ell_x(s) \\ -m_x(s) \end{pmatrix} \} \equiv \begin{pmatrix} d^{(o)}(s) \\ d^{(o)}(s) \end{pmatrix}$$

or alternatively, in components

$$\begin{aligned} d_1^{(o)}(s) &= \sum_{\mu} C_{1\mu}(s) \cdot \Delta\hat{B}_{\mu} \quad ; \\ d_2^{(o)}(s) &= \sum_{\mu} C_{2\mu}(s) \cdot \Delta\hat{B}_{\mu} \end{aligned} \tag{6.9}$$

with

$$C_{1\mu}(s) = (\ell_{\tau}(s) \quad \ell_x(s) \quad \ell_z(s)) \;\; \underline{F}(s) \cdot \vec{y}_{\mu}(s) \; -$$

$$- \; \delta(s - s_{\mu}) \cdot \frac{e}{E_o} \;\; (1 + a\gamma_o) \cdot \ell_x(s) \quad ; \tag{6.10a}$$

$$C_{2\mu}(s) = - \; (m_{\tau}(s) \quad m_x(s) \quad m_z(s)) \cdot \underline{F}(s) \cdot \vec{y}_{\mu}(s) \; +$$

$$+ \; \delta(s - s_{\mu}) \;\; \frac{e}{E_o} \;\; (1 + a\gamma_o) \;\; m_x(s) \quad . \tag{6.10b}$$

Finally, by substituting (6.10) in (6.2) the quantities a_k, b_k are given as

$$a_k = \sum_\mu A_{k\mu} \cdot \Delta\hat{B}_\mu \quad ; \qquad b_k = \sum_\mu B_{k\mu} \cdot \Delta\hat{B}_\mu \quad . \tag{6.11}$$

where

$$A_{k\mu} = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \{C_{1\mu}(\tilde{s}) \cdot \cos[2\pi k \frac{\tilde{s}}{L}] - C_{2\mu}(\tilde{s}) \cdot \sin[2\pi k \frac{\tilde{s}}{L}] \quad ; \tag{6.11a}$$

$$B_{k\mu} = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \{C_{1\mu}(\tilde{s}) \cdot \sin[2\pi k \frac{\tilde{s}}{L}] + C_{2\mu}(\tilde{s}) \cdot \cos[2\pi k \frac{\tilde{s}}{L}] \quad . \tag{6.11b}$$

We now have expressions specifying how the effects of quite arbitrary machine errors in an arbitrarily complicated linear machine can be minimized just by the use of vertical orbit correction coils.

As an example we consider a family of 8 correction coils with fields $\Delta\hat{B}_1$, $\Delta\hat{B}_2$, ..., $\Delta\hat{B}_8$. Since we have 8 free parameters - the strengths, we expect that it should be possible to set a total of 4 different f_k's to zero (each f_k has an a_k and b_k part). Naturally these f_k's are chosen to correspond to k's close to the spin tune ν.

By considering (6.11) for k = r, r+1, r+2, r+3, we may rewrite it in matrix form as

$$\begin{pmatrix} a_r \\ b_r \\ a_{r+1} \\ b_{r+1} \\ a_{r+2} \\ b_{r+2} \\ a_{r+3} \\ b_{r+3} \end{pmatrix} = \underline{K} \cdot \begin{pmatrix} \Delta\hat{B}_1 \\ \Delta\hat{B}_2 \\ \Delta\hat{B}_3 \\ \Delta\hat{B}_4 \\ \Delta\hat{B}_5 \\ \Delta\hat{B}_6 \\ \Delta\hat{B}_7 \\ \Delta\hat{B}_8 \end{pmatrix} \tag{6.12}$$

where

$$\underline{K} = \begin{pmatrix} A_{r1} & A_{r2} & A_{r3} & A_{r4} & A_{r5} & A_{r6} & A_{r7} & A_{r8} \\ B_{r1} & B_{r2} & B_{r3} & B_{r4} & B_{r5} & B_{r6} & B_{r7} & B_{r8} \\ A_{r+1,1} & A_{r+1,2} & A_{r+1,3} & A_{r+1,4} & A_{r+1,5} & A_{r+1,6} & A_{r+1,7} & A_{r+1,8} \\ B_{r+1,1} & B_{r+1,2} & B_{r+1,3} & B_{r+1,4} & B_{r+1,5} & B_{r+1,6} & B_{r+1,7} & B_{r+1,8} \\ A_{r+2,1} & A_{r+2,2} & A_{r+2,3} & A_{r+2,4} & A_{r+2,5} & A_{r+2,6} & A_{r+2,7} & A_{r+2,8} \\ B_{r+2,1} & B_{r+2,2} & B_{r+2,3} & B_{r+2,4} & B_{r+2,5} & B_{r+2,6} & B_{r+2,7} & B_{r+2,8} \\ A_{r+3,1} & A_{r+3,2} & A_{r+3,3} & A_{r+3,4} & A_{r+3,5} & A_{r+3,6} & A_{r+3,7} & A_{r+3,8} \\ B_{r+3,1} & B_{r+3,2} & B_{r+3,3} & B_{r+3,4} & B_{r+3,5} & B_{r+3,6} & B_{r+3,7} & B_{r+3,8} \end{pmatrix}$$

and depends on the optical and spin state of the ring. We are then immediately in the position to calculate the $\Delta\hat{B}_\mu$ that are required for varying the quantities

$$a_i = \mathrm{Re} \cdot f_i \quad ; \qquad (i = r, r+1, r+2, r+3) \qquad (6.14)$$
$$b_i = -\mathrm{Im} \cdot f_i \quad ;$$

independently of each other:

$$\begin{pmatrix} \Delta\hat{B}_1 \\ \Delta\hat{B}_2 \\ \Delta\hat{B}_3 \\ \Delta\hat{B}_4 \\ \Delta\hat{B}_5 \\ \Delta\hat{B}_6 \\ \Delta\hat{B}_7 \\ \Delta\hat{B}_8 \end{pmatrix} = \underline{K}^{-1} \begin{pmatrix} a_r \\ b_r \\ a_{r+1} \\ b_{r+1} \\ a_{r+2} \\ b_{r+2} \\ a_{r+3} \\ b_{r+3} \end{pmatrix} \qquad (6.15)$$

If, for example only a_r is to be corrected, then in (6.15) we use:

$$\begin{pmatrix} a_r \\ b_r \\ a_{r+1} \\ b_{r+1} \\ a_{r+2} \\ b_{r+2} \\ a_{r+3} \\ b_{r+3} \end{pmatrix} = \rho \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (6.16)$$

where ρ is a scale factor reflecting the fact mentioned above that in general the exact amount of a_r to be corrected must be discovered empirically by varying all the ΔB_μ with the same common factor.

The treatment so far was quite general and made no assumptions about symmetries in the ring structure. Thus, in this formalism there is no reason why the family of coils should not be expanded so that a larger number of harmonics $f_k^{(o)}$ could be controlled together. However, care would be needed in the handling of the inversion of the large $\underline{K}$-matrices. Also, it is certainly inadvisible to try to use all available coils simultaniously since, in practice, not all power supplies will be in operation.

Finally, we note that in a typical machine there will be many distinct families of 8 coils so that in principle a particular group of 8 harmonics can be corrected in many different ways. Thus, if the number of coils to be used is kept to a minimum we retain the flexibility to choose that combination which has the smallest effect on the closed orbit but which at the same time has the largest effect on the harmonics [22].

In practice, the optics and the orientation of the n-axis often exhibit symmetries and thus can lead to further simplification. Consider a fourfold symmetric arrangement (Fig. 1) with 8 coils positioned as shown (see also equ. (5.17)), then with (6.10):

$$\left.\begin{aligned} C_{1\mu}(s) &= C_{11}(s - \frac{\mu - 1}{8} \cdot L) \\ C_{2\mu}(s) &= C_{21}(s - \frac{\mu - 1}{8} \cdot L) \end{aligned}\right\} \quad \text{für } \mu = 3, 5, 7 \tag{6.17}$$

and

$$\left.\begin{aligned} C_{1\mu}(s) &= C_{12}(s - \frac{\mu - 2}{8} \cdot L) \\ C_{2\mu}(s) &= C_{22}(s - \frac{\mu - 2}{8} \cdot L) \end{aligned}\right\} \quad \text{für } \mu = 4, 6, 8\ . \tag{6.18}$$

Thus, the matrix elements $A_{k\mu}$ and $B_{k\mu}$ for $\mu = 3, 5, 7$ $(\mu = 4, 6, 8)$ can be written in terms of A_{k1} and B_{k1} (A_{k2} and B_{k2}):

a) $\mu = 3, 5, 7$:

$$A_{k\mu} = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \{C_{11}(\tilde{s} - \frac{\mu - 1}{8} \cdot L) \cdot \cos[2\pi k \frac{\tilde{s}}{L}] - C_{21}(\tilde{s} - \frac{\mu - 1}{8} \cdot L) \cdot \sin[2\pi k \frac{\tilde{s}}{L}]\}$$

$$(s' = s - \frac{\mu - 1}{8} \cdot L\ ;\ ds' = ds\ ;\ \frac{\tilde{s}}{L} = \frac{s'}{L} + \frac{\mu - 1}{8})$$

$$= \frac{1}{L} \cdot \int_{s_0}^{s_0+L} ds' \cdot \{C_{11}(s') \cdot \cos[2\pi k \frac{s'}{L} + 2\pi k \cdot \frac{\mu - 1}{8}] - C_{21}(s') \cdot \sin[2\pi k \frac{s'}{L} + 2\pi k \cdot \frac{\mu - 1}{8}]\}$$

$$= \frac{1}{L} \cdot \int_{s_0}^{s_0+L} ds' \cdot \{C_{11}(s') \cdot [\cos(2\pi k \frac{s'}{L}) \cdot \cos(2\pi k \cdot \frac{\mu - 1}{8}) - \sin(2\pi k \frac{s'}{L}) \cdot \sin(2\pi k \cdot \frac{\mu - 1}{8})] - C_{21}(s') \cdot [\sin(2\pi k \frac{s'}{L}) \cdot \cos(2\pi k \cdot \frac{\mu - 1}{8}) + \cos(2\pi k \frac{s'}{L}) \cdot \sin(2\pi k \cdot \frac{\mu - 1}{8})]\}$$

$$= A_{k1} \cdot \cos(2\pi k \cdot \frac{\mu - 1}{8}) - B_{k1} \cdot \sin(2\pi k \cdot \frac{\mu - 1}{8}) \quad ; \tag{6.19a}$$

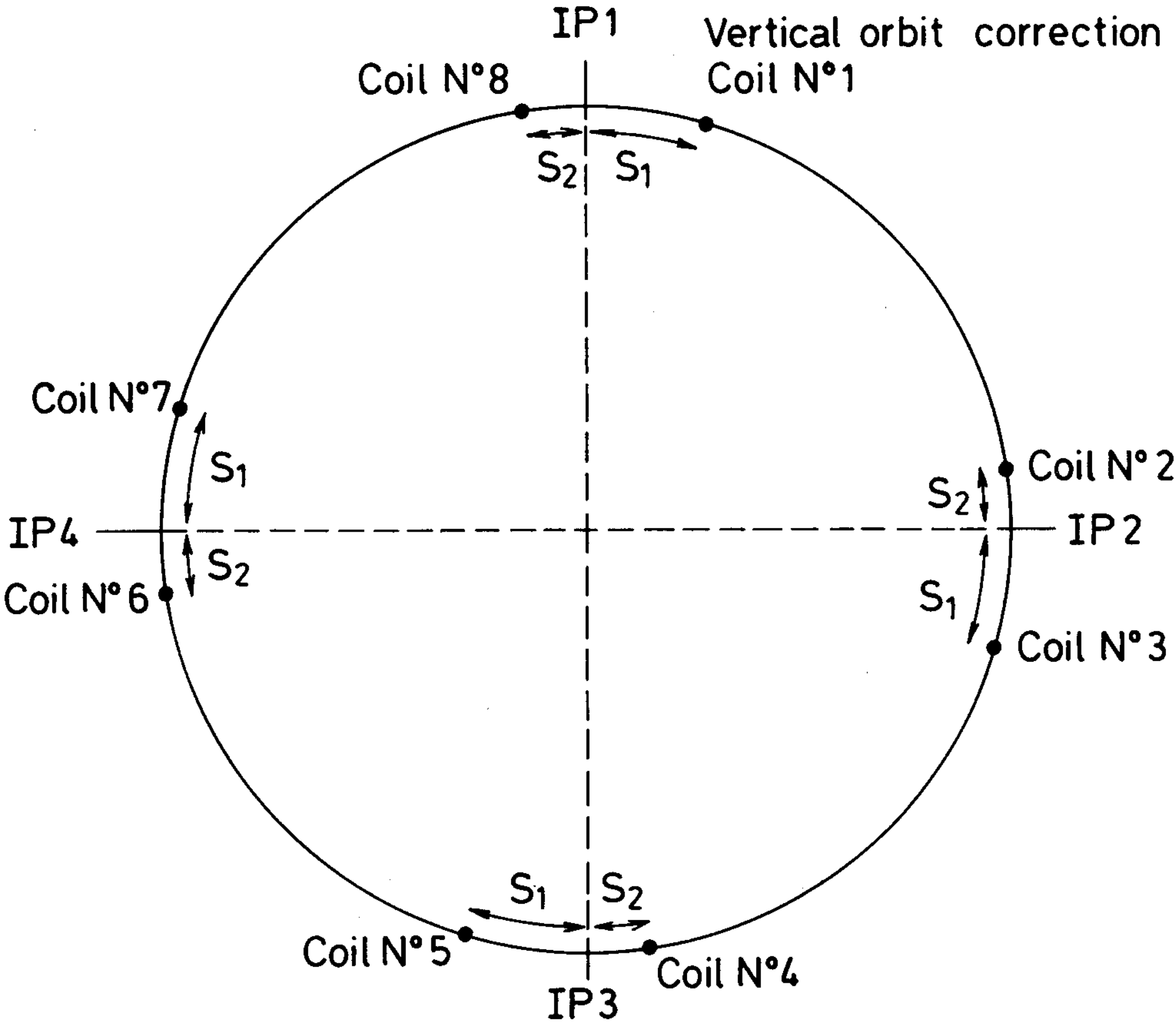

Fig.1

Layout of the ring with 4 equally separated interaction points (I.P.).
The positions of a family of 8 vertical orbit correction coils are indicated.

$$B_{k\mu} = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot \{C_{11}(\tilde{s} - \frac{\mu - 1}{8} \cdot L) \cdot \sin[2\pi k \frac{\tilde{s}}{L}] + C_{21}(\tilde{s} - \frac{\mu - 1}{8} \cdot L) \cdot \sin[2\pi k \frac{\tilde{s}}{L}]\}$$

$$= A_{k1} \cdot \sin(2\pi k \cdot \frac{\mu - 1}{8}) + B_{k1} \cdot \cos(2\pi k \cdot \frac{\mu - 1}{8}) \quad ; \qquad (6.19b)$$

b) $\mu = 4, 6, 8$:

$$A_{k\mu} = A_{k2} \cdot \cos(2\pi k \cdot \frac{\mu - 2}{8}) - B_{k2} \cdot \sin(2\pi k \cdot \frac{\mu - 2}{8}) \quad ; \qquad (6.20a)$$

$$B_{k\mu} = A_{k2} \cdot \sin(2\pi k \cdot \frac{\mu - 2}{8}) + B_{k2} \cdot \cos(2\pi k \cdot \frac{\mu - 2}{8}) \quad . \qquad (6.20b)$$

If in addition, by a suitable choice of $\psi(s_0)$ in (3.5) and by setting $s_1 = s_2$, the following conditions are satisfied (reflecting the mirror symmetry of the guide field and $\vec{n}$, $\vec{m}$, $\vec{\ell}$ axes with respect to the interaction point):

$$\left.\begin{aligned} C_{1\mu}(s) &= + C_{11}(L - s - \frac{8 - \mu}{8} \cdot L) \\ C_{2\mu}(s) &= - C_{21}(L - s - \frac{8 - \mu}{8} \cdot L) \end{aligned}\right\} \mu = 2, 4, 6, 8 \qquad (6.21)$$

then

$$A_{k\mu} = A_{k1} \cdot \cos(2\pi k \ \frac{8 - \mu}{8}) - B_{k1} \cdot \sin(2\pi k \ \frac{8 - \mu}{8}) \quad ; \qquad (6.22a)$$

$$B_{k\mu} = - A_{k1} \cdot \sin(2\pi k \ \frac{8 - \mu}{8}) - B_{k1} \cdot \cos(2\pi k \ \frac{8 - \mu}{8}) \quad . \qquad (6.22b)$$

for $\mu = 2, 4, 6, 8$

so that to calculate $\underline{K}$ it suffices only to know A_{k1}, B_{k1} for $k = r, r+1, r+2, r+3$. In this case, it is even possible to solve the equation system (6.12) for the harmonics f_r to f_{r+3} directly.

To do this, we expand (6.19) and (6.20) for $k = 4n, 4n+1, 4n+2, 4n+3$ (i.e. we put $r = 4n$):

1) $\mu = 3, 5, 7$:

a) $k = 4n$

$A_{k\mu} = A_{k1}$;

$B_{k\mu} = B_{k1}$;

b) $k = 4n+1$

$$A_{k\mu} = A_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{\pi}{4}] - B_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{\pi}{4}] \quad ;$$

$$B_{k\mu} = A_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{\pi}{4}] + B_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{\pi}{4}] \quad ;$$

c) $k = 4n+2$

$$A_{k\mu} = A_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{\pi}{2}] + B_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{\pi}{2}] \quad ;$$

$$B_{k\mu} = A_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{\pi}{2}] + B_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{\pi}{2}] \quad ;$$

d) $k = 4n+3$

$$A_{k\mu} = A_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{3\pi}{4}] - B_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{3\pi}{4}] \quad ;$$

$$B_{k\mu} = A_{k_1} \cdot \sin[(\mu - 1) \cdot \frac{3\pi}{4}] + B_{k_1} \cdot \cos[(\mu - 1) \cdot \frac{3\pi}{4}] \quad ;$$

2) $\mu = 4, 6, 8$:

a) $k = 4n$

$$A_{k\mu} = A_{k_2} \quad ;$$

$$B_{k\mu} = B_{k_2} \quad ;$$

b) $k = 4n+1$

$$A_{k\mu} = A_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{\pi}{4}] - B_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{\pi}{4}] \quad ;$$

$$B_{k\mu} = A_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{\pi}{4}] + B_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{\pi}{4}] \quad ;$$

c) $k = 4n+2$

$$A_{k\mu} = A_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{\pi}{2}] - B_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{\pi}{2}] \quad ;$$

$$B_{k\mu} = A_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{\pi}{2}] + B_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{\pi}{2}] \quad ;$$

d) $k = 4n+3$

$$A_{k\mu} = A_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{3\pi}{4}] - B_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{3\pi}{4}] \quad ;$$

$$B_{k\mu} = A_{k_2} \cdot \sin[(\mu - 2) \cdot \frac{3\pi}{4}] + B_{k_2} \cdot \cos[(\mu - 2) \cdot \frac{3\pi}{4}] \quad .$$

Then, recalling the condition (6.22)

$$\begin{cases} A_{k8} = A_{k1} \quad ; \\ B_{k8} = -B_{k1} \, ; \end{cases}$$

and making the abbreviations

$$\left.\begin{aligned} p_0 &= A_{k1} \\ q_0 &= B_{k1} \end{aligned}\right\} \text{ for } k = 4n \tag{6.23a}$$

$$\left.\begin{aligned} p_1 &= A_{k1} \\ q_1 &= B_{k1} \end{aligned}\right\} \text{ for } k = 4n+1 \tag{6.23b}$$

$$\left.\begin{aligned} p_2 &= A_{k1} \\ q_2 &= B_{k1} \end{aligned}\right\} \text{ for } k = 4n+2 \tag{6.23c}$$

$$\left.\begin{aligned} p_3 &= A_{k1} \\ q_3 &= B_{k1} \end{aligned}\right\} \text{ for } k = 4n+3 \tag{6.23d}$$

where A_{k1}, B_{k1} (k = 4n, 4n+1, 4n+2, 4n+3) are obtained from (6.11), the matrix $\underline{K}$ in (6.13) takes the form

$$\underline{K} = \begin{pmatrix} p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 & p_0 \\ q_0 & -q_0 & q_0 & -q_0 & q_0 & -q_0 & q_0 & -q_0 \\ p_1 & q_1 & -q_1 & -p_1 & -p_1 & -q_1 & q_1 & p_1 \\ q_1 & p_1 & p_1 & q_1 & -q_1 & -p_1 & -p_1 & -q_1 \\ p_2 & -p_2 & -p_2 & p_2 & p_2 & -p_2 & -p_2 & p_2 \\ q_2 & q_2 & -q_2 & -q_2 & q_2 & q_2 & -q_2 & -q_2 \\ p_3 & -q_3 & q_3 & -p_3 & -p_3 & q_3 & -q_3 & p_3 \\ q_3 & -p_3 & -p_3 & q_3 & -q_3 & p_3 & p_3 & -q_3 \end{pmatrix} \tag{6.24}$$

The inverse matrix $\underline{K}^{-1}$ can now be written in the form

$$\underline{K}^{-1} = \lambda \cdot \begin{pmatrix} 1 & 1\cdot\lambda_2 & 1\cdot\lambda_3 & 1\cdot\lambda_4 & 1\cdot\lambda_5 & 1\cdot\lambda_6 & 1\cdot\lambda_7 & 1\cdot\lambda_8 \\ 1 & -1\cdot\lambda_2 & \frac{p_3}{q_3}\cdot\lambda_3 & \frac{q_3}{p_3}\cdot\lambda_4 & -1\cdot\lambda_5 & 1\cdot\lambda_6 & -\frac{p_1}{q_1}\cdot\lambda_7 & -\frac{q_1}{p_1}\cdot\lambda_8 \\ 1 & 1\cdot\lambda_2 & -\frac{p_3}{q_3}\cdot\lambda_3 & \frac{q_3}{p_3}\cdot\lambda_4 & -1\cdot\lambda_5 & -1\cdot\lambda_6 & \frac{p_1}{q_1}\cdot\lambda_7 & -\frac{q_1}{p_1}\cdot\lambda_8 \\ 1 & -1\cdot\lambda_2 & -1\cdot\lambda_3 & 1\cdot\lambda_4 & 1\cdot\lambda_5 & -1\cdot\lambda_6 & -1\cdot\lambda_7 & 1\cdot\lambda_8 \\ 1 & 1\cdot\lambda_2 & -1\cdot\lambda_3 & -1\cdot\lambda_4 & 1\cdot\lambda_5 & 1\cdot\lambda_6 & -1\cdot\lambda_7 & -1\cdot\lambda_8 \\ 1 & -1\cdot\lambda_2 & -\frac{p_3}{q_3}\cdot\lambda_3 & -\frac{q_3}{p_3}\cdot\lambda_4 & -1\cdot\lambda_5 & 1\cdot\lambda_6 & \frac{p_1}{q_1}\cdot\lambda_7 & \frac{q_1}{p_1}\cdot\lambda_8 \\ 1 & 1\cdot\lambda_2 & \frac{p_3}{q_3}\cdot\lambda_3 & -\frac{q_3}{p_3}\cdot\lambda_4 & -1\cdot\lambda_5 & -1\cdot\lambda_6 & -\frac{p_1}{q_1}\cdot\lambda_7 & \frac{q_1}{p_1}\cdot\lambda_8 \\ 1 & -1\cdot\lambda_2 & 1\cdot\lambda_3 & -1\cdot\lambda_4 & 1\cdot\lambda_5 & -1\cdot\lambda_6 & 1\cdot\lambda_7 & -1\cdot\lambda_8 \end{pmatrix}$$

where λ and λ_i are suitable constants.

This matrix has a structure identical to the table of excitation currents given in Ref. 10) and we see that although the detailed forms of the p_i, q_i will differ from those in Ref. 10) the structure of $\underline{K}^{-1}$ only depends on the assumption of the fourfold mirror symmetry. This is the case even when the machine is otherwise arbitrarily complicated and not just flat.

Summary

We have shown how the SLIM [1,3] formalism can be extended to provide a systematic means of correcting perturbations to the equilibrium spin axis in electron storage rings with magnet misalignments and field errors. This 6x6 fully coupled formalism is straight forward to implement even in rings with complicated spin rotator systems and includes the effects of energy variations on the closed orbit.

Naturally, in addition to the closed orbit effects treated here correction schemes for dealing with the effects of gradient errors and the depolarizing effects of spurious vertical dispersion are also needed [10,7].

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20) In this formalism, the spin tune, ν, defined in (3.2c) is, as stated, arbitrary up to an integer $\varkappa$. Thus, even in a flat machine, in this formalism, the tune ν can be chosen, so that it is not identical to the conventionally used tune value of $a\gamma$ which would give the number of complete rotations around $\vec{n}_0$ of $\vec{m}_0$ and $\vec{\ell}_0$ as generated by the matrix $\underline{N}$ during one circuit of the machine.

21) The contribution of the sextupoles to the closed "correction orbit" $\vec{y}^{(o)}$ can be neglected since the coil set used will be chosen so that $\vec{y}^{(o)}$ (i = 1, 3) is small.

22) J. Kewisch and R. Schmidt, Private Communications. See also Ref. 10 and 7.

23) Just as SLIM uses 8-dimensional eigenvectors composed of 6 orbital components and 2 spin components, here we can consider that δn_1, δn_2 and the 6 closed orbit components form an 8-dimensional closed orbit.