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SPIN DYNAMICS IN MODERN ELECTRON STORAGE RINGS: COMPUTATIONAL AND THEORETICAL ASPECTS

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INTRODUCTION

In this presentation we describe some numerical and analytical results from our work on the spin polarization in high energy electron storage rings aimed towards the proposed Future Circular Collider (FCC-ee) and the proposed Circular Electron Positron Collider (CEPC). Photon emission in synchrotron radiation imparts a stochastic element ("noise") into particle motion and there are also damping effects. However, instead of considering single particles it is often convenient to model the stochastic photon emission as a Gaussian white noise process and to then study the evolution of the particle density in phase space with a Fokker-Planck equation.

The noise in trajectories together with the spin-orbit coupling embodied in the Thomas-BMT equation of spin precession [1], can cause spin diffusion and thus depolarization. On the other hand photon emission can lead to a build up of polarization via spin flip. This is the Sokolov-Ternov process [2]. The attainable polarization is the outcome of the balance of the two effects.

So far, analytical estimates of the attainable polarization have been based on the so-called *Derbenev-Kondratenko formulas* [3, 4]. In analogy with studies of the trajectories of single particles, that approach leans towards the study of single spins and relies in part on plausible assumptions grounded in deep physical intuition. However, just as with particle motion it would be convenient to have a treatment of the Fokker-Planck (F-P) kind and thereby minimize the reliance on assumptions. But the polarization at a point in phase space cannot be handled in that way since polarization is not a density. Nevertheless a density is available, namely the density in phase space of the spin angular momentum and with this there is a generalization of the F-P equation which we call the Bloch equation. We use that name to reflect the analogy with equations for magnetization in condensed matter [5]. In fact the Bloch equation works with the so-called *polarization density*. This is proportional to the spin angular momentum density per particle in phase space. With this we can calculate the polarization vector of the bunch.

Thus we study the initial value problem of what we call the full Bloch equation (FBE). The FBE takes into account non spin-flip and spin-flip effects due to synchrotron radiation including the spin-diffusion effects and the Sokolov-Ternov effect with its Baier-Katkov generalization. The FBE was introduced by Derbenev and Kondratenko in 1975 [6] as a generalization to the whole phase space (with its noisy trajectories) of the Baier-Katkov-Strakhovenko (BKS) equation which just describes the evolution of polarization by spin flip along a single trajectory [7]. The FBE is a system of three F-P equations coupled by a Thomas-BMT term and the BKS terms but uncoupled within the F-P terms. By neglecting the spin flip terms in the FBE we obtain what we call the reduced Bloch equation (RBE). The RBE approximation is sufficient for computing the physically interesting depolarization time and it shares the terms with the FBE that are challenging to discretize. Thus, here we only consider the discretization of the RBE.

Our approach has three parts. First we approximate the RBE analytically using the method of averaging, resulting in an average RBE which allows us to use large time steps. The minimum length of the time interval of interest is of the order of the orbital damping times. Secondly, the phase space coordinates of the average RBE come in $d = \{1, 2, 3\}$ pairs of polar-radial coordinates that we discretize using a Fourier-Chebyshev pseudospectral approach. The averaging decouples the parabolic and mode coupling terms allowing for a parallel implementation with only local communication. Thirdly, we further exploit the decoupling by evolving the resulting system of ODEs by an implicit-explicit (ARK) method. Parabolic operators are treated implicitly and can be inverted rapidly due to the decoupling. If each of the d angle variables is discretized on a grid of M grid points and if each of the d radial variables is discretized on a grid of N grid points then the total number of operations for each time step scales, to leading order, as $O(N^{dq}M^d)$ where $1 \leq q \leq 3$, depending on the algorithms used for the linear solve. For Gaussian elimination $q = 3$. Details and more results have been presented in this meeting by O.Beznosov, see [8].

The main issues for very high energy rings like the FCC-ee and CEPC are: (i) Can one get polarization, (ii) what are the theoretical limits of the polarization? We believe that the FBE offers a more complete starting point for very high energy rings than the Derbenev-Kondratenko formulas. See [9] for a recent review of polarization history and phenomenology.

RBE IN LAB FRAME

In a semiclassical probabilistic description of an electron bunch the spin-orbit dynamics is described by the *spin*-1/2 *Wigner function* (also called the *Stratonovich function*) ρ

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written as

$$\rho(t,z) = \frac{1}{2}\left(f(t,z)I_{2\times 2} + \vec{\sigma}\cdot\vec{\eta}(t,z)\right), \tag{1}$$

where f is the classical phase-space density normalized by $\int f(t,z)dz = 1$ and $\vec{\eta}$ is the polarization density of the bunch and thus proportional to the spin angular momentum density. Here $z = (r,p)$ where r and p are the position and momentum vectors of the phase space and t is the time. Also, $\vec{\sigma}$ is the vector of the three Pauli matrices. Thus $f = Tr[\rho]$ and $\vec{\eta} = Tr[\rho\vec{\sigma}]$. The polarization vector $\vec{P}(t)$ of the bunch is $\vec{P}(t) = \int \vec{\eta}(t,z)dz$. Here and in the following we use arrows on spin-related quantities and no arrows on other quantities. Moreover the spin-related quantities will be represented by column matrices. When the particle motion is governed just by a Hamiltonian, as in the case of protons, the phase-space density is conserved along a trajectory so that the polarization density obeys the Thomas-BMT equation along each trajectory. However, if the particles are subject to noise and damping due to synchrotron radiation, the evolution of the density of particles in phase space is more complicated. But as advertised above it can be handled with a F-P formalism. Then by neglecting collective effects and after several other approximations, ρ evolves via

$$\partial_t f = L_{FP}(t,z)f, \tag{2}$$

$$\partial_t\vec{\eta} = L_{FP}(t,z)\vec{\eta} + \Omega(t,z)\vec{\eta} + G(t,z)\vec{\eta} + \vec{g}(t,z)f + \vec{L}(t,z)f \tag{3}$$

where (2) is the F-P equation for the orbital density and (3) is the FBE mentioned above, both in the lab frame, i.e., in cartesian coordinates. The F-P operator L_{FP} is the linear second-order partial differential operator commonly used for electron synchrotrons and storage rings [10, Section 2.5.4], [11, 12]. The skew-symmetric matrix $\Omega(t,z)$ in the FBE takes into account the Thomas-BMT spin-precession effect. The terms $G\vec{\eta}$, $\vec{g}f$ and $\vec{L}f$ take into account spin flips due to synchrotron radiation. In particular they include the Sokolov-Ternov effect and its Baier-Katkov correction the latter belonging to $G\vec{\eta}$. As usual, since it is minuscule compared to all other forces, the Stern-Gerlach effect from the spin onto the orbit is neglected in (2). The explicit forms of L_{FP}, Ω, G, $\vec{g}$ and $\vec{L}$ are given in [6].

If we neglect the spin flip terms in the FBE then (3) simplifies to

$$\partial_t\vec{\eta} = L_{FP}(t,z)\vec{\eta} + \Omega(t,z)\vec{\eta} \tag{4}$$

The RBE (4) just takes care of spin diffusion due to the orbital motion.

The Equations (2) and (3) can be derived from quantum electrodynamics, followed by making the semiclassical approximation of the Foldy-Wouthuysen transformation of the Dirac Hamiltonian and finally by making a Markov approximation [13]. We stress however, that the RBE (4) can be derived purely classically as in [14]. In fact, we show again how to do this at the end of the next section.

RBE IN THE BEAM FRAME

In the beam frame, i.e., in accelerator coordinates, the RBE (4) becomes

$$\partial_\theta\vec{\eta}_Y = (L_Y + L_{Y,TBMT})\vec{\eta}_Y \tag{5}$$

where θ is the accelerator azimuth,

$$L_Y = -\sum_{j=1}^{6}\partial_{y_j}\left(\mathcal{A}(\theta)y\right)_j + \frac{1}{2}\omega_Y(\theta)\partial_{y_6}^2\,,$$

$$L_{Y,TBMT}\vec{\eta}_Y = \Omega_Y(\theta,y)\vec{\eta}_Y$$

and where $\mathcal{A}(\theta)$ is a 6×6 matrix encapsulating radiationless motion and the deterministic effects of synchrotron radiation. Also $\Omega_Y(\theta,y)$ is the Thomas-BMT term and it is a skew-symmetric 3×3 matrix linear in y and $\omega_Y(\theta)$ is the magnitude of the noise. Note that $\mathcal{A}(\theta)$, $\Omega_Y(\theta,y)$ and $\omega_Y(\theta)$ are 2π-periodic in θ. Given the beam frame polarization density $\vec{\eta}_Y$ the beam frame polarization vector $\vec{P}(\theta)$ of the bunch at azimuth θ is

$$\vec{P}(\theta) = \int dy\ \vec{\eta}_Y(\theta,y) \tag{6}$$

Our central computational focus in this paper is the RBE (5) with $\vec{P}(\theta)$ being a quantity of interest. To proceed with this it is important that (5) has an underlying system of Langevin equations and thus an underlying F-P equation. In fact the system of Langevin equations is

$$Y' = \mathcal{A}(\theta)Y + \sqrt{\omega_Y(\theta)}e_6\xi(\theta)\,, \tag{7}$$

$$\vec{S}' = \Omega_Y(\theta,Y)\vec{S} \tag{8}$$

where ξ is a version of the white noise process, $e_6 = (0,0,0,0,0,1)^T$ and $\vec{S}$ is the single-particle spin expectation value. Note that (7) can be written as the Ito stochastic differential equation: $dY = \mathcal{A}(\theta)Y d\theta + \sqrt{\omega_Y(\theta)}e_6 dW$ which is linear in the narrow sense and thus defines a Gaussian process Y if $Y(0)$ is Gaussian. In principle (5) could be obtained by transforming (4) and the coefficients $\mathcal{A}$, Ω_Y and ω from the lab frame to the beam frame, However this is not necessary since (7) and (8) and the $\mathcal{A}, \Omega_Y$ and ω can be found in virtually every exposition on spin in high-energy electron storage rings, e.g., [15]. Note that these expositions make some approximations. We use [15] which involves linearizing w.r.t. y as can be seen in (7) and (8). For (5) see also [14].

The F-P equation for the Gaussian process Y is

$$\partial_\theta\mathcal{P}_Y = L_Y\mathcal{P}_Y \tag{9}$$

For getting (9) from (7) see [16–18]. With (7) and (8) the evolution equation for the spin-orbit joint probability density $\mathcal{P}_{YS} = \mathcal{P}_{YS}(\theta,y,\vec{s})$ is the following F-P equation:

$$\partial_\theta\mathcal{P}_{YS} = L_Y\mathcal{P}_{YS} - \sum_{j=1}^{3}\partial_{s_j}\left(\left(\Omega_Y(\theta,y)\vec{s}\right)_j\mathcal{P}_{YS}\right) \tag{10}$$

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Note that $\mathcal{P}_Y$ is related to $\mathcal{P}_{YS}$ by

$$\mathcal{P}_Y(\theta, y) = \int ds\, \mathcal{P}_{YS}(\theta, y, \vec{s}) \tag{11}$$

where the integral is over $\mathbb{R}^3$. Note also that since the spin variable $\vec{S}$ is normalized, $\mathcal{P}_{YS}$ is supported on the 2-sphere, i.e., where $|\vec{s}| = 1$. Hence $\mathcal{P}_{YS}(\theta, y, \vec{s})$ is proportional to $\delta(|\vec{s}| - 1)$. By integrating (10) over $\vec{s}$ one recovers (9). The polarization density $\vec{\eta}_Y$ corresponding to $\mathcal{P}_{YS}$ is defined by

$$\vec{\eta}_Y(\theta, y) = \int ds\vec{s}\, \mathcal{P}_{YS}(\theta, y, \vec{s}) \tag{12}$$

The RBE (5) follows from (10) by differentiating (12) w.r.t. θ.

APPROXIMATING THE BEAM FRAME RBE BY THE METHOD OF AVERAGING

Because the coefficients of L_Y are θ-dependent, the RBE (5) is numerically quite complex. So we first approximate it analytically in order to solve it numerically. We will find this approximate RBE by refining the averaging technique presented by Ellison, Mais and Ripken in the Accelerator Handbook [19, Section 2.1.4]. This refinement allows us to use that method of averaging to approximate the system of Langevin Equations (7). We just give a sketch here (a detailed account will be published elsewhere [20]). Note that both [19, Section 2.1.4] and our refinement are restricted to first-order averaging. We first rewrite (7) as

$$Y' = (A(\theta) + \epsilon\delta A(\theta))Y + \sqrt{\epsilon}\sqrt{\omega(\theta)}e_6\xi(\theta) \tag{13}$$

with $\sqrt{\epsilon}\sqrt{\omega(\theta)} = \sqrt{\omega_Y(\theta)}$, where $A(\theta)$ is the Hamiltonian part of $\mathcal{A}(\theta)$ and ϵ is a perturbation parameter, and where $\epsilon\delta A(\theta)$ represents the part of $\mathcal{A}(\theta)$ associated with damping effects due to synchrotron radiation and cavities (see, e.g., [15, eq. 5.3]). The mean m_Y and covariance matrix K_Y of Y satisfy the ODEs

$$m_Y' = (A(\theta) + \epsilon\delta A(\theta))m_Y\,, \tag{14}$$

$$K_Y' = (A(\theta) + \epsilon\delta A(\theta))K_Y + K_Y(A(\theta) + \epsilon\delta A(\theta))^T + \epsilon\omega(\theta)e_6e_6^T \tag{15}$$

In (15) the δA terms and the ω are balanced at $O(\epsilon)$ and so can be treated together in first order perturbation theory. This is the reason for the $\sqrt{\epsilon}$ in (13). However this balance is also physical as the damping and diffusion come from the same source and the cavities replenish the energy loss.

To apply the method of averaging to (14) and (15) we must transform them to a standard form for averaging. We do this by using a fundamental solution matrix X of the unperturbed $\epsilon = 0$ part of (13) and (14), i.e.,

$$X' = A(\theta)X \tag{16}$$

We thus transform Y, m_Y and K_Y into U, m_U and K_U via

$$Y = X(\theta)U,\ m_Y = X(\theta)m_U, K_Y = X(\theta)K_UX^T(\theta) \tag{17}$$

and (13), (14) and (15) are transformed to

$$U' = \epsilon\mathcal{D}(\theta)U + \sqrt{\epsilon}\sqrt{\omega(\theta)}X^{-1}(\theta)e_6\xi(\theta) \tag{18}$$

$$m_U' = \epsilon\mathcal{D}(\theta)m_U\,, \tag{19}$$

$$K_U' = \epsilon(\mathcal{D}(\theta)K_U + K_U\mathcal{D}^T(\theta)) + \epsilon\mathcal{E}(\theta) \tag{20}$$

Here $\mathcal{D}(\theta)$ and $\mathcal{E}(\theta)$ are defined by

$$\mathcal{D}(\theta) = X^{-1}(\theta)\delta A(\theta)X(\theta)\,, \tag{21}$$

$$\mathcal{E}(\theta) = \omega(\theta)X^{-1}(\theta)e_6e_6^TX^{-T}(\theta) \tag{22}$$

Of course, (18)–(20) carry the same information as (13)–(15).

Now, applying the method of averaging to (19) and (20), we obtain

$$m_V' = \epsilon\bar{\mathcal{D}}m_V\,, \tag{23}$$

$$K_V' = \epsilon(\bar{\mathcal{D}}K_V + K_V\bar{\mathcal{D}}^T) + \epsilon\bar{\mathcal{E}} \tag{24}$$

where the bar denotes θ-averaging, i.e., the operation $\lim_{T\to\infty}(1/T)\int_0^T d\theta\cdots$. For physically reasonable A each fundamental matrix X is a quasiperiodic function whence $\mathcal{D}$ and $\mathcal{E}$ are quasiperiodic functions so that their time averages $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$ exist. By averaging theory $|m_U(\theta) - m_V(\theta)| \le C_1(T)\epsilon$ and $|K_U(\theta) - K_V(\theta)| \le C_2(T)\epsilon$ for $0 \le \theta \le T/\epsilon$ where T is a constant (see also [21–24]) and ϵ small. However, we expect to be able to show that these estimates are uniformly valid on $[0, \infty)$, since the long time behavior is exact.

The key point now is that every Gaussian process V, whose mean m_V and covariance matrix K_V satisfy the ODEs (23) and (24), satisfies the system of Langevin equations

$$V' = \epsilon\bar{\mathcal{D}}V + \sqrt{\epsilon}\mathcal{B}(\xi_1, ..., \xi_k)^T \tag{25}$$

Here $\xi_1, ..., \xi_k$ are statistically independent versions of the white noise process and $\mathcal{B}$ is a $6 \times k$ matrix which satisfies

$$\mathcal{B}\mathcal{B}^T = \bar{\mathcal{E}} \tag{26}$$

with $k = rank(\bar{\mathcal{E}})$. Since $m_U(\theta) = m_V(\theta) + O(\epsilon)$ and $K_U(\theta) = K_V(\theta) + O(\epsilon)$ we get $U(\theta) \approx V(\theta)$. In particular $X^{-1}(\theta)V(\theta) \approx Y(\theta)$ (more details will be in [20]). Conversely, the mean vector m_V and covariance matrix K_V of every V in (25) satisfy the ODEs (23) and (24).

It's likely that stochastic averaging techniques [25, and references therein] can be applied directly to (18) giving (25) as an approximation and we are looking into that. However, because (18) is linear and defines a Gaussian process, the theory for getting to (25) from the ODEs for the moments could not be simpler, even though it is indirect.

To include the spin we extend (25) to the spin-orbit system of Langevin equations

$$V' = \epsilon\bar{\mathcal{D}}V + \sqrt{\epsilon}\mathcal{B}(\xi_1, ..., \xi_k)^T\,, \tag{27}$$

$$\vec{S}' = \Omega_Y(\theta, X(\theta)V)\vec{S} \tag{28}$$

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With (27) and (28) the evolution equation for the spin-orbit probability density $\mathcal{P}_{VS} = \mathcal{P}_{VS}(\theta, \mathrm{v}, \vec{s})$ is the following F-P equation:

$$\partial_\theta \mathcal{P}_{VS} = L_V \mathcal{P}_{VS} - \sum_{j=1}^{3} \partial_{s_j} \left(\left(\Omega_Y(\theta, X(\theta)\mathrm{v})\vec{s} \right)_j \mathcal{P}_{VS} \right) \tag{29}$$

where

$$L_V = -\epsilon \sum_{j=1}^{6} \partial_{\mathrm{v}_j} (\bar{\mathcal{D}}\mathrm{v})_j + \frac{\epsilon}{2} \sum_{i,j=1}^{6} \bar{\mathcal{E}}_{ij} \partial_{\mathrm{v}_i} \partial_{\mathrm{v}_j} \tag{30}$$

The polarization density $\vec{\eta}_V$ corresponding to $\mathcal{P}_{VS}$ is defined by

$$\vec{\eta}_V(\theta, \mathrm{v}) = \int ds \vec{s}\, \mathcal{P}_{VS}(\theta, \mathrm{v}, \vec{s}) \tag{31}$$

so that by (29), the RBE is

$$\partial_\theta \vec{\eta}_V = (L_V + L_{V,TBMT}) \vec{\eta}_V \tag{32}$$

where

$$L_{V,TBMT} \vec{\eta}_V = \Omega_Y(\theta, X(\theta)\mathrm{v}) \vec{\eta}_V \tag{33}$$

The coefficients of L_V are θ-independent for every choice of X and this is necessary for our numerical method. Note that the averaging which leads to (32) affects only the orbital variables. It was justified by using the fact that (27) is linear whence it defines a Gaussian process when the initial condition is Gaussian. This allowed us to apply the averaging approach to the first and second moments rather than the Langevin equation itself. We cannot apply this approach to the combined spin-orbit dynamics in (27)-(28) because (28) has a quadratic nonlinearity. In future work, we will pursue this using stochastic averaging as in [25].

We now need an appropriate X and we note that

$$X(\theta) = M(\theta) C \tag{34}$$

where C is an arbitrary invertible 6×6 matrix and M is the principal solution matrix, i.e., $M' = A(\theta)M, M(0) = I$. Thus choosing X boils down to choosing a good C. As is common for spin physics in electron storage rings we emulate Chao's approach [19, Section 2.1.4], [26,27] and use the eigenvectors of $M(2\pi)$. We assume that the unperturbed orbital motion is stable. Thus $M(2\pi)$ has a full set of linearly independent eigenvectors and the eigenvalues are on the unit circle in the complex plane [28]. We further assume a non-resonant condition on the orbital frequencies. We construct C as a real matrix using the real and imaginary parts of the eigenvectors in its columns and using the fact that $M(2\pi)$ is symplectic (since $A(\theta)$ is a Hamiltonian matrix). It follows that $\bar{\mathcal{D}}$ has block diagonal form and $\bar{\mathcal{E}}$ has diagonal form. Then the three degrees of freedom are uncoupled in the operator L_V in (30). Explicitly,

$$\bar{\mathcal{D}} = \begin{pmatrix} \mathcal{D}_I & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & \mathcal{D}_{II} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & \mathcal{D}_{III} \end{pmatrix}, \tag{35}$$

$$\mathcal{D}_\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ -b_\alpha & a_\alpha \end{pmatrix}, (\alpha = I, II, III) \tag{36}$$

and $\bar{\mathcal{E}} = diag(\mathcal{E}_I, \mathcal{E}_I, \mathcal{E}_{II}, \mathcal{E}_{II}, \mathcal{E}_{III}, \mathcal{E}_{III})$ with $a_\alpha \leq 0$ and $\mathcal{E}_I, \mathcal{E}_{II}, \mathcal{E}_{III} \geq 0$. Thus the three degrees of freedom are uncoupled in L_V since, by (30),

$$L_V = L_{V,I} + L_{V,II} + L_{V,III} \tag{37}$$

where each $L_{V,\alpha}$ is an operator in one degree of freedom and is determined by $\mathcal{D}_\alpha$ and $\mathcal{E}_\alpha$ via (30) ($\alpha = I, II, III$).

We now have $Y(\theta) = X(\theta)U(\theta) \approx Y_a(\theta) := X(\theta)V(\theta)$ and it follows that $\vec{\eta}_Y$ in (5) is given approximately by

$$\vec{\eta}_Y(\theta, y) \approx \vec{\eta}_{Y,a}(\theta, y) = \det(X^{-1}(0)) \vec{\eta}_V(\theta, X^{-1}(\theta)\mathrm{y}) \tag{38}$$

Now (32) and the RBE for $\vec{\eta}_{Y,a}$ carry the same information. However in general the RBE for $\vec{\eta}_{Y,a}$ does not have the nice features of (32), e.g., (35), (36) and L_V being θ-independent, which make the latter useful for our numerical method (see below). Hence we discretize (32) rather than the RBE for $\vec{\eta}_{Y,a}$.

We finally mention a feature of $\vec{\eta}_V$ which is helpful for finding an appropriate numerical phase space domain for $\vec{\eta}_V$. The orbital probability density $\mathcal{P}_V$ corresponding to $\mathcal{P}_{VS}$ is defined by

$$\mathcal{P}_V(\theta, \mathrm{v}) = \int ds \mathcal{P}_{VS}(\theta, \mathrm{v}, \vec{s}) \tag{39}$$

whence by (31),

$$|\vec{\eta}_V(\theta, \mathrm{v})| = \left| \int ds \vec{s} \mathcal{P}_{VS}(\theta, \mathrm{v}, s) \right| \leq \int ds |\vec{s}| \mathcal{P}_{VS}(\theta, \mathrm{v}, s)$$
$$= \int ds \mathcal{P}_{VS}(\theta, \mathrm{v}, s) = \mathcal{P}_V(\theta, \mathrm{v}) \tag{40}$$

so that the numerical phase space domain for $\vec{\eta}_V$ can be identified with the numerical phase space domain for $\mathcal{P}_V$. The latter is easy to find since we generally use exact expressions of $\mathcal{P}_V$, e.g., the one for orbital equilibrium.

TWO-DEGREE-OF-FREEDOM CASE

We now consider a case of two degrees of freedom in a flat ring just with FODO cells and cavities. The case of two degrees of freedom is a natural step towards three degrees of freedom. Moreover the case of a flat ring allows us to use a one-dimensional approach to spin, leading to a linear spin-orbit system, a system to which we can apply our averaging approach. The Gaussian nature of the associated process allows us to analytically solve the average RBE.

In our flat ring model Ω_Y has the simple form

$$\Omega_Y(\theta, Y) = -a_Y(\theta) Y \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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where $Y = (Y_1, Y_2, Y_3, Y_4)^T$ represents the horizontal and longitudinal motions which are uncoupled from the vertical motion in the flat ring model. It is convenient to use spherical coordinates as spin variables, i.e., $\vec{S} = (\cos(\Psi)\sin(\Phi), \sin(\Psi)\sin(\Phi), \cos(\Phi))^T$. The beam frame system of Langevin equations are then

$$Y' = (A(\theta) + \epsilon\delta A(\theta))Y + \sqrt{\epsilon}\sqrt{\omega(\theta)}(0,0,0,1)^T\xi(\theta)\,, \quad (41)$$
$$\Psi' = a_Y(\theta)Y\,, \quad (42)$$
$$\Phi' = 0 \quad (43)$$

where the row vector $a_Y(\theta)$ is 2π-periodic in θ. To apply the method of averaging to the system (41)-(43) we transform the system to a standard form for averaging. We do this by defining $\tilde{Y} := (Y_1, Y_2, Y_3, Y_4, \Psi, \Phi)^T$ and by using a fundamental solution matrix Z of the unperturbed $\epsilon = 0$ part of (41)-(43), i.e.,

$$Z' = \begin{pmatrix} A(\theta) & 0_{4\times2} \\ a_Y(\theta) & 0_{1\times2} \\ 0_{1\times4} & 0_{1\times2} \end{pmatrix} Z \quad (44)$$

By transforming $\tilde{Y}$ into Q via

$$\tilde{Y} = Z(\theta)Q \quad (45)$$

one gets the system of Langevin equations

$$Q' = \epsilon\mathcal{D}(\theta)Q + \sqrt{\epsilon}\sqrt{\omega(\theta)}Z^{-1}(\theta)e_4\xi(\theta) \quad (46)$$

where $e_4 = (0,0,0,1,0,0)^T$ and

$$\mathcal{D}(\theta) = Z^{-1}(\theta)\begin{pmatrix} \delta A(\theta) & 0_{4\times2} \\ 0_{2\times4} & 0_{2\times2} \end{pmatrix} Z(\theta)\,, \quad (47)$$

Thus the mean m_Q and covariance matrix K_Q of Q satisfy the ODEs

$$m_Q' = \epsilon\mathcal{D}(\theta)m_Q\,, \quad (48)$$
$$K_Q' = \epsilon(\mathcal{D}(\theta)K_Q + K_Q\mathcal{D}^T(\theta)) + \epsilon\mathcal{E}(\theta) \quad (49)$$

where

$$\mathcal{E}(\theta) = \omega(\theta)Z^{-1}(\theta)e_4e_4^TZ^{-T}(\theta) \quad (50)$$

By averaging (48) and (49) we get the ODEs

$$m_W' = \epsilon\bar{\mathcal{D}}m_W\,, \quad (51)$$
$$K_W' = \epsilon(\bar{\mathcal{D}}K_W + K_W\bar{\mathcal{D}}^T) + \epsilon\bar{\mathcal{E}} \quad (52)$$

where the bar denotes θ-averaging. Since the ODE system (51),(52) is autonomous it can be analytically solved. For physically reasonable choices of the parameters in (41)-(43) each fundamental matrix Z is a quasiperiodic function whence $\mathcal{D}$ and $\mathcal{E}$ are quasiperiodic functions so that their θ-averages $\bar{\mathcal{D}}$ and $\bar{\mathcal{E}}$ exist. By averaging theory $m_Q(\theta) = m_W(\theta) + O(\epsilon)$ and $K_Q(\theta) = K_W(\theta) + O(\epsilon)$ for $0 \le \theta \le T/\epsilon$ where T is a constant (see also [21–24]). Every Gaussian process W, whose mean m_W and covariance matrix K_W satisfy the ODEs (51) and (52), satisfies the system of Langevin equations

$$W' = \epsilon\bar{\mathcal{D}}W + \sqrt{\epsilon}\mathcal{B}(\xi_1, ..., \xi_k)^T \quad (53)$$

Here $\xi_1, ..., \xi_k$ are statistically independent versions of the white noise process and where $\mathcal{B}$ is a $6 \times k$ matrix which satisfies

$$\mathcal{B}\mathcal{B}^T = \bar{\mathcal{E}} \quad (54)$$

with $k = rank(\bar{\mathcal{E}})$. Since $m_Q(\theta) = m_W(\theta) + O(\epsilon)$ and $K_Q(\theta) = K_W(\theta) + O(\epsilon)$ we get $Q(\theta) \approx W(\theta)$. In particular $Z^{-1}(\theta)W(\theta) \approx \tilde{Y}(\theta)$. Clearly the third component of the spin does not evolve, the spins only evolve in the plane.

As in the case of three degrees of freedom we assume that the unperturbed orbital motion is stable and nonresonant. Thus, as in the case of three degrees of freedom, we can construct a fundamental matrix Z such that the orbital part of $\bar{\mathcal{D}}$ has block diagonal form and such that the orbital part of $\bar{\mathcal{E}}$ has diagonal form, i.e.,

$$\bar{\mathcal{D}} = \begin{pmatrix} \mathcal{D}_I & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & \mathcal{D}_{II} & 0_{2\times2} \\ \bar{\mathcal{D}}_{51}\ \bar{\mathcal{D}}_{52} & \bar{\mathcal{D}}_{53}\ \bar{\mathcal{D}}_{54} & 0_{1\times2} \\ 0_{1\times2} & 0_{1\times2} & 0_{1\times2} \end{pmatrix}, \quad (55)$$

$$\bar{\mathcal{E}} = \begin{pmatrix} \mathcal{E}_I & 0 & 0 & 0 & \bar{\mathcal{E}}_{15} & 0 \\ 0 & \mathcal{E}_I & 0 & 0 & \bar{\mathcal{E}}_{25} & 0 \\ 0 & 0 & \mathcal{E}_{II} & 0 & \bar{\mathcal{E}}_{35} & 0 \\ 0 & 0 & 0 & \mathcal{E}_{II} & \bar{\mathcal{E}}_{45} & 0 \\ \bar{\mathcal{E}}_{15} & \bar{\mathcal{E}}_{25} & \bar{\mathcal{E}}_{35} & \bar{\mathcal{E}}_{45} & \bar{\mathcal{E}}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (56)$$

where $\mathcal{D}_I, \mathcal{D}_{II}$ are 2×2 matrices of the form (36) and $\mathcal{E}_I, \mathcal{E}_{II}$ are nonnegative. If $\mathcal{P}_W = \mathcal{P}_W(\theta, w)$ is a probability density of a Gaussian process associated with (53) then the polarization density $\vec{\eta}_W$ corresponding to $\mathcal{P}_W$ is defined by

$$\vec{\eta}_W(\theta, w) = \int dw_5 dw_6 \begin{pmatrix} \cos(w_5)\sin(w_6) \\ \sin(w_5)\sin(w_6) \\ \cos(w_6) \end{pmatrix} \mathcal{P}_W(\theta, w) \quad (57)$$

and satisfies the RBE

$$\begin{aligned} \partial_\theta\vec{\eta}_W = &-\epsilon\sum_{j=1}^{2}\partial_{w_j}\left(\left(\mathcal{D}_I(w_1,w_2)^T\right)_j\vec{\eta}_W\right) \\ &-\epsilon\sum_{j=3}^{4}\partial_{w_j}\left(\left(\mathcal{D}_{II}(w_3,w_4)^T\right)_j\vec{\eta}_W\right) \\ &+\frac{\epsilon}{2}\mathcal{E}_I\left(\partial_{w_1}\partial_{w_1} + \partial_{w_2}\partial_{w_2}\right)\vec{\eta}_W \\ &+\frac{\epsilon}{2}\mathcal{E}_{II}\left(\partial_{w_3}\partial_{w_3} + \partial_{w_4}\partial_{w_4}\right)\vec{\eta}_W \\ &-\epsilon\sum_{j=1}^{4}\bar{\mathcal{D}}_{5j}w_j\mathcal{J}\vec{\eta}_W - \frac{\epsilon}{2}\bar{\mathcal{E}}_{55}\vec{\eta}_W + \epsilon\sum_{j=1}^{4}\bar{\mathcal{E}}_{j5}\mathcal{J}\vec{\eta}_W \end{aligned} \quad (58)$$

Since the ODE system (51),(52) can be analytically solved, $\mathcal{P}_W$ can be computed analytically for every Gaussian process. Then by (57), $\vec{\eta}_W$ can be computed analytically.

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ONE-DEGREE-OF-FREEDOM CASE

We now consider the case of one degree of freedom using the model studied in [29, 30], which involves only synchrotron motion. The case of one degree of freedom is the first step towards two and three degrees of freedom. The one-degree-of-freedom model here is obtained from the two-degrees-of-freedom flat-ring model of the previous section by setting, in (55) and (56),

$$0 = \mathcal{D}_{II} = \bar{\mathcal{D}}_{52} = \bar{\mathcal{D}}_{53} = \bar{\mathcal{D}}_{54} = \mathcal{E}_{II} = \bar{\mathcal{E}}_{25} = \bar{\mathcal{E}}_{35} = \bar{\mathcal{E}}_{45} \ ,$$
$$\mathcal{D}_I = -I_{2\times 2} \ , \ \mathcal{E}_I = 1 \ , \ \bar{\mathcal{E}}_{15} = -\bar{\mathcal{D}}_{51} \ , \ \bar{\mathcal{E}}_{55} = (\bar{\mathcal{E}}_{15})^2 \qquad (59)$$

One can justify the step from (55) and (56) to (59) as a good approximation by applying the betatron-dispersion formalism to the flat ring model [31]. With (59) the variables W_3, W_4, W_6 are uncoupled so that we are left with the following one-degree-of-freedom model resulting in the following system of Langevin equations for the orbital variables W_1, W_2 and the spin variable W_5:

$$\begin{pmatrix} W_1' \\ W_2' \\ W_5' \end{pmatrix} = \epsilon \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ g & 0 & 0 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \\ W_5 \end{pmatrix} + \sqrt{\frac{\epsilon}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -g & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

where $g = \bar{\mathcal{D}}_{51} = -\bar{\mathcal{E}}_{15}$ and ξ_1, ξ_2 are statistically independent versions of the white noise process. Denoting the polarization density for our one-degree-of-freedom model by $\vec{\eta}_{1D}$, one can show in analogy to the previous section that it satisfies the RBE

$$\partial_\theta \vec{\eta}_{1D} = \epsilon \Big(\partial_{w_1}(w_1 \vec{\eta}_{1D}) + \partial_{w_2}(w_2 \vec{\eta}_{1D}) \Big) + \frac{\epsilon}{4} \partial_{w_1} \partial_{w_1} \vec{\eta}_{1D} + \frac{\epsilon}{4} \partial_{w_2} \partial_{w_2} \vec{\eta}_{1D} - \epsilon g w_1 \mathcal{J} \vec{\eta}_{1D} - \frac{\epsilon}{2} g \mathcal{J} \partial_{w_1} \vec{\eta}_{1D} - \frac{\epsilon}{4} g^2 \vec{\eta}_{1D} \qquad (60)$$

Note that the analytic solutions of the RBE (60) give strong evidence for the validity of the averaging method since the analytic solutions of (60) can be compared with the solutions of the analytic solutions of the exact RBE in [29, 30].

SKETCH OF THE NUMERICAL APPROACH

We now briefly sketch our numerical approach to the RBEs (32), (58) and (60). For simplicity we here focus on (32). The numerical computations are performed by using 3 pairs $(r_\alpha, \varphi_\alpha)$ of polar coordinates, i.e., $\mathrm{v}_1 = r_I \cos\varphi_I, ..., \mathrm{v}_6 = r_{III} \sin\varphi_{III}$. The angle variables are Fourier transformed hence the Fourier coefficients are functions of time and the radial variables. We discretize the radial variables by using the pseudospectral method [32, 33] using a Chebychev grid for each radial variable. This results for each Fourier mode in a system of linear first-order ODEs in θ which we discretize by using an implicit/explicit θ-stepping scheme. Because of (30), (35) and (36) the Fourier modes are uncoupled in $L_V \vec{\eta}_V$ so that the only coupling of Fourier modes in (32) comes via $L_{V,TBMT} \vec{\eta}_V = \Omega_Y(\theta, X(\theta)\mathrm{v}) \vec{\eta}_V$ and this coupling is local since $\Omega_Y(\theta, X(\theta)\mathrm{v})$ is linear in v. Thus the parabolic terms are separated from the mode coupling terms, i.e., in the time stepping $L_V \vec{\eta}_V$ is treated implicitly and $L_{V,TBMT} \vec{\eta}_V$ is treated explicitly. The implicit time stepping involves a linear solver whose efficiency depends on L_V being θ-independent. Note that the pseudospectral method is a minimial-residue method by which the residual of a PDE is zero at the numerical grid points. Note also that the numerical boundary conditions are periodic in the angle variables and for each radial variable r_α we impose homogenous Dirichlet boundary conditions at $r_\alpha = r_{\max}$. The latter are justified by the inequality (40) and the fact we impose these boundary conditions also on the orbital probability density $\mathcal{P}_V$.

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