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A Modified Harmonic Closed Orbit Adjustment Formalism for the Optimization of Polarization in Electron Storage Rings

by

D.P. Barber and G. Ripken
Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

We extend the closed orbit correction formalism of Ref. 1 to take into account the notion that the polarization in electron storage rings may be more sensitive to tilts, $\delta\vec{n}$, of the equilibrium spin axis in some parts of the lattice than in others. This is achieved by means of a periodic weighting function. The formalism requires some modifications of the computer program already developed for implementing the ideas of Ref. 1.

Contents

1. Introduction

In an earlier article /1/ we described a general closed orbit correction scheme designed to minimize the detrimental effect of misalignments on the degree of polarization in electron storage rings. The technique consists of minimizing the tilt away from the design direction of the periodic spin solution, $\vec{n}_0$, on the closed orbit.

That article is the basis of the orbit optimization program FIDO /2/, written by S. Mane and which is used for simulation of the optimization of polarization in HERA.

In this note we show how the formulation can be extended to take into account of the fact that a certain deviation, $\delta\vec{n}$, of the periodic spin solution, $\vec{n}$, from the design solution, $\vec{n}_0$, may be more detrimental at some positions in the lattice than at others. As an example, it is clear, trivially, that a tilt, $\delta\vec{n}$, from the vertical in the arcs is only important at the quadrupoles /3/. However, it is also clear that the effect of the tilt, $\delta\vec{n}$, will tend to be most important at those quadrupoles where the beta functions and dispersions are particularly large or, equivalently, where the (periodic) absolute values of orbit eigenvector components are large. The relative orbit and spin phases are of course also important /3/ but it would nevertheless be of interest if, instead of minimizing $\delta\vec{n}$, we were able to minimize the product $\delta\vec{n}(s)\cdot g(s)$ where $g(s)$ is some general periodic weighting function such as a Twiss parameter.

This is the object of the formalism described below.

2. The change in the $\vec{n}$-axis $\delta\vec{n}$ caused by closed orbit shifts

2.1 The equation of spin orbit motion

As a starting point we summarize the basic equations of spin orbit coupling. More introductory material and details of notation may be found in Ref. 1. As before, we work within the framework of linear spin-orbit theory.

1) Orbit

The linearized equations of orbit motion are:

$$\frac{d}{ds}\vec{y} = \underline{A}\,\vec{y} + \vec{c}_0 + \vec{c}_1 \tag{2.1}$$

with

$$\underline{A} = \begin{pmatrix} 0 & 1 & H & 0 & 0 & 0 \\ -(G_1 + H^2) & 0 & N & H & 0 & K_x \\ -H & 0 & 0 & 1 & 0 & 0 \\ N & -H & -(G_2 + H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_0}\,k\cdot\frac{2\pi}{L}\cos\phi\cdot\sum_\mu \delta(s-s_\mu) & 0 \end{pmatrix}; \tag{2.1a}$$

$$\vec{c}_0^{\,T} = (0,0,0,0,0,\ \frac{e\hat{V}}{E_0}\sin\phi\cdot\sum_\mu \delta(s-s_\mu) - C_1(K_x^2 + K_z^2))\ ; \tag{2.1b}$$

$$\vec{c}_1^{\,T} = (0,\ -\frac{e}{E_0}\cdot\Delta B_z, 0,\ \frac{e}{E_0}\cdot\Delta B_x, 0, 0)\ ; \tag{2.1c}$$

$$H = \frac{1}{2}\frac{e}{E_0}\cdot B_\tau^{(o)}\ ; \tag{2.1d}$$

$$N = \frac{1}{2}\frac{e}{E_0}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_z}{\partial z}\right)_{x=z=0}\ ; \tag{2.1e}$$

$$g = \frac{e}{E_0}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}\ ; \tag{2.1f}$$

$$\begin{cases} G_1 = K_x^2 + g\ ; \\ G_2 = K_z^2 - g\ ; \end{cases} \tag{2.1g}$$

$$C_1 = \frac{2}{3} e^2 \frac{\gamma_0^4}{E_0} \quad ; \qquad (2.1h)$$

$$\vec{y}^T = (x,\ p_x,\ z,\ p_z,\ \sigma,\ \eta) \quad ; \qquad (2.1i)$$

$$\begin{cases} p_x = x' - H\cdot z \quad ; \\ p_z = z' + H\cdot x \quad . \end{cases} \qquad (2.1j)$$

The matrix $\underline{A}$ describes the influence of lenses and cavities on the particle motion, the vector $\vec{c}_1$ the action of the fields ΔB_x and ΔB_z (due to field errors and correction coils) while $\vec{c}_0$ is determined by local radiation of energy in bending magnets and energy uptake in cavities.

In detail we have:

a) $g \neq 0$; $N = H = \hat{V} = 0$; $K_x = K_z = 0$;
quadrupole ;

b) $N \neq 0$; $H = g = \hat{V} = 0$; $K_x = K_z = 0$;
skew quadrupole;

c) $G_1 = K_x^2 + g$; $G_2 = -g$ or $G_2 = g$; $G_2 = K_z^2 - g$; $H = \hat{V} = 0$;
combined function magnet;

d) $H \neq 0$; $g = H = \hat{V} = 0$; $K_x = K_z = 0$;
solenoid;

e) $\hat{V} \neq 0$; $g = N = H = 0$; $K_x = K_z = 0$;
cavity.

2) Spin

Linearised classical spin motion is described by the equations

$$\frac{d}{ds}\vec{\zeta} = \underline{D}_0 \cdot \vec{\zeta} + \underline{R} \cdot [F \cdot \vec{y} + \vec{c}_2] \qquad (2.2)$$

with

$$\vec{\zeta} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} ; \qquad (2.2a)$$

$$\vec{c}_2 = -\frac{e}{E_0} \cdot \begin{pmatrix} \Delta B_\tau \cdot (1 + a) \\ \Delta B_x \cdot (1 + a\gamma_0) \\ \Delta B_z \cdot (1 + a\gamma_0) \end{pmatrix} ; \qquad (2.2b)$$

$$\underline{D}_0 = \psi' \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \qquad (2.2c)$$

$$\underline{R} = \begin{pmatrix} \ell_\tau & \ell_x & \ell_z \\ -m_\tau & -m_x & -m_z \end{pmatrix} ; \qquad (2.2d)$$

$$\underline{F} = ((F_{\mu\nu})) ; \qquad (2.2e)$$

$F_{12} = -a\gamma_0 \cdot K_z$;

$F_{14} = +a\gamma_0 \cdot K_x$;

$F_{16} = 2H \cdot (1 + a)$;

$F_{21} = -(1 + a\gamma_0) \cdot (N - H')$;

$F_{22} = a\gamma_0 \cdot 2H$;

$F_{23} = (1 + a\gamma_0)(K_z^2 - g) + a\gamma_0 \cdot 2H^2$;

$F_{24} = (a\gamma_0 + 1) \cdot \frac{e\hat{V}}{E_0} \sin\phi \cdot \sum_\mu \delta(s - s_\mu)$;

$F_{26} = -K_z$;

$F_{31} = -(1 + a\gamma_0) \cdot (K_x^2 + g) - a\gamma_0 \cdot 2H^2$;

$F_{32} = -F_{24}$;

$F_{33} = (1 + a\gamma_0) \cdot (N + H')$;

$F_{34} = F_{22}$;

$F_{36} = K_x$;

$F_{\mu\nu} = 0$ otherwise.

Equations (2.1) and (2.2) can be combined into the form

$$\frac{d}{ds}\vec{u} = \hat{\underline{A}}\,\vec{u} + \vec{p} \qquad (2.3)$$

with

$$\vec{u} = \begin{pmatrix} \vec{y} \\ \vec{\zeta} \end{pmatrix} ; \qquad (2.3a)$$

$$\hat{\underline{A}} = \begin{pmatrix} \underline{A} & \underline{0} \\ \underline{G}_0 & \underline{D}_0 \end{pmatrix} ; \qquad (2.3b)$$

$$\vec{p} = \begin{pmatrix} \vec{c}_0 + \vec{c}_1 \\ R\vec{c}_2 \end{pmatrix} ; \qquad (2.3c)$$

$$\underline{G}_0 = \underline{R} \cdot \underline{F} \qquad (2.3d)$$

where the first six components of the eight dimensional vector $\vec{u}(s)$ describe the transverse and longitudinal orbit motion and the last two describe the spin components.

We now require that $\vec{u}$ is periodic

$$\vec{u}(s_0 + L) = \vec{u}(s_0) , \qquad (2.4a)$$

so that

$$\left.\begin{array}{l} \vec{y} \equiv \hat{\vec{y}} \;\text{ with }\; \hat{\vec{y}}(s_0 + L) \doteq \hat{\vec{y}}(s_0) \;\;\text{(closed orbit)}; \\ \vec{\zeta} \equiv \delta\vec{n} \;\text{ with }\; \delta\vec{n}(s_0 + L) = \delta\vec{n}(s_0) \;\;\text{(change of } \vec{n}\text{-axis)} \end{array}\right\} \qquad (2.4b)$$

i.e. the orbit components of $\vec{u}$ now give the six dimensional closed orbit while the spin components give the change in the $\vec{n}$-axis due to the closed orbit shift.

2.2 <u>The transfer matrix</u>

The solution for the inhomogeneous equation (2.3) can be written in the form

$$\begin{pmatrix} \vec{u}(s) \\ 1 \end{pmatrix} = \hat{\underline{M}}(s,s_0) \begin{pmatrix} \vec{u}(s_0) \\ 1 \end{pmatrix} \qquad (2.5)$$

where the 9-dimensional transfer matrix $\hat{\underline{M}}(s,s_0)$ satisfies the relations

$$\frac{d}{ds}\hat{\underline{M}}(s,s_0) = \begin{pmatrix} \hat{\underline{A}} & \vec{p} \\ 0 & 0 \end{pmatrix} \hat{\underline{M}}(s,s_0) ; \qquad (2.5a)$$

$$\hat{\underline{M}}(s,s_0) = \underline{1} . \qquad (2.5b)$$

If we write $\hat{\underline{M}}(s,s_0)$ as

$$\hat{\underline{M}}(s,s_0) = \begin{pmatrix} \underline{M}(s,s_0) & \vec{a}(s,s_0) \\ \underline{0} & 1 \end{pmatrix} \qquad (2.6)$$

with

$$\underline{M}(s,s_0) = \begin{pmatrix} \underline{M}_0(s,s_0) & \underline{0} \\ \underline{G}(s,s_0) & \underline{D}(s,s_0) \end{pmatrix} ; \qquad (2.7)$$

$$\left\{\begin{array}{l} \underline{M}_0(s,s_0) = \text{transfer matrix for the orbit;} \\ \underline{G}(s,s_0) = \text{spin-orbit coupling matrix} \end{array}\right.$$

we then obtain

$$\frac{d}{ds}\begin{pmatrix} \underline{M}(s,s_0) & \vec{a}(s,s_0) \\ \underline{0} & 1 \end{pmatrix} = \begin{pmatrix} \hat{\underline{A}} & \vec{p} \\ \underline{0} & 0 \end{pmatrix} \begin{pmatrix} \underline{M}(s,s_0) & \vec{a}(s,s_0) \\ \underline{0} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\underline{A}} \cdot \underline{M}(s,s_0) & \hat{\underline{A}} \cdot \vec{a}(s,s_0) + \vec{p} \\ \underline{0} & 0 \end{pmatrix} ;$$

$$\begin{pmatrix} \underline{M}(s_0,s_0) & \vec{a}(s_0,s_0) \\ \underline{0} & 1 \end{pmatrix} = \underline{1}$$

or

$$\frac{d}{ds}\,\vec{a}(s,s_0) = \hat{\underline{A}}\cdot\vec{a}(s,s_0) + \vec{p}\ ;\ \vec{a}(s_0,s_0) = \vec{0}\ ; \qquad (2.8a)$$

$$\frac{d}{ds}\,\underline{M}(s,s_0) = \hat{\underline{A}}\cdot\underline{M}(s,s_0) \qquad ;\ \underline{M}(s_0,s_0) = \underline{1}\ ; \qquad (2.8b)$$

Equ. (2.8a) has the solution

$$\vec{a}(s,s_0) = \underline{M}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{M}(s_0,\tilde{s})\cdot\vec{p}(\tilde{s})$$

$$= \int_{s_0}^{s} d\tilde{s}\cdot\underline{M}(s,\tilde{s})\cdot\vec{p}(\tilde{s})\ , \qquad (2.9)$$

where the vector $\vec{p}(s)$ is given by (2.3c).

Furthermore, from equ. (2.8b) together with (2.3b) and (2.7) we have the relations

$$\frac{d}{ds}\begin{pmatrix}\underline{M}_0 & \underline{0}\\ \underline{G} & \underline{D}\end{pmatrix} = \begin{pmatrix}\underline{A} & \underline{0}\\ \underline{G}_0 & \underline{D}_0\end{pmatrix}\begin{pmatrix}\underline{M}_0 & \underline{0}\\ \underline{G} & \underline{D}\end{pmatrix}$$

$$= \begin{pmatrix}\underline{A}\,\underline{M}_0 & \underline{0}\\ \underline{G}_0\,\underline{M}_0 + \underline{D}_0\,\underline{G} & \underline{D}_0\,\underline{D}\end{pmatrix}\ ;$$

$$\begin{pmatrix}\underline{M}_0(s,s_0) & 0\\ \underline{G}(s,s_0) & \underline{D}(s_0,s_0)\end{pmatrix} = \underline{1}$$

or

I) $\frac{d}{ds}\,\underline{M}_0(s,s_0) = \underline{A}(s)\cdot\underline{M}_0(s,s_0)\ ;\ \underline{M}_0(s_0,s_0) = \underline{1}\ ;$

$$\Longrightarrow\ \underline{M}_0(s+\Delta s,\ s) \simeq \underline{1} + \Delta s\cdot\underline{A}(s)\ ; \qquad (2.10)$$

II) $\frac{d}{ds}\,\underline{D}(s,s_0) = \underline{D}_0(s)\cdot\underline{D}(s,s_0)\ ;\ D(s_0,s_0) = 1\ ;$

$$\underline{D}_0(s) = \psi'(s)\cdot\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad \text{following eqn. (2.2c)}\ ;$$

$$\Longrightarrow\ \underline{D}(s,s_0) = \begin{pmatrix}\cos[\psi(s)-\psi(s_0)] & \sin[\psi(s)-\psi(s_0)]\\ -\sin[\psi(s)-\psi(s_0)] & \cos[\psi(s)-\psi(s_0)]\end{pmatrix}\ ; \qquad (2.11)$$

III) $\frac{d}{ds}\,\underline{G}(s,s_0) = \underline{G}_0(s)\cdot\underline{M}_0(s,s_0) + \underline{D}_0(s)\cdot\underline{G}(s,s_0)\ ;\ \underline{G}(s_0,s_0) = \underline{0}\ ;$

$$\Longrightarrow\ \underline{G}(s,s_0) = \underline{D}(s,s_0)\cdot\int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s_0,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\underline{M}_0(\tilde{s},s_0)$$

$$= \int_{s_0}^{s} d\tilde{s}\cdot\underline{D}(s,\tilde{s})\cdot\underline{G}_0(\tilde{s})\cdot\underline{M}_0(\tilde{s},s_0)\ . \qquad (2.12)$$

Remark: The above relations are also useful for describing the orbital transfer matrix $\hat{\underline{M}}_0$:

$$\hat{\underline{M}}_0(s,s_0) = \begin{pmatrix} \underline{M}_0(s,s_0) & \vec{a}_0(s,s_0) \\ 0 & 1 \end{pmatrix} \tag{2.13}$$

with

$$\begin{pmatrix} \vec{y}(s) \\ 1 \end{pmatrix} = \hat{\underline{M}}_0(s,s_0) \begin{pmatrix} \vec{y}(s_0) \\ 1 \end{pmatrix} . \tag{2.14}$$

For this we write (2.1) in the form

$$\frac{d}{ds} \begin{pmatrix} \vec{y}(s) \\ 1 \end{pmatrix} = \begin{pmatrix} \underline{A} & \vec{c} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{y}(s) \\ 1 \end{pmatrix} \tag{2.15}$$

with

$$\vec{c}(s) = \vec{c}_0(s) + \vec{c}_1(s) . \tag{2.16}$$

The matrix $\hat{\underline{M}}_0(s,s_0)$ then obeys the equation

$$\frac{d}{ds} \hat{\underline{M}}_0(s,s_0) = \begin{pmatrix} \underline{A} & \vec{c} \\ 0 & 0 \end{pmatrix} \hat{\underline{M}}_0(s,s_0) ; \tag{2.17a}$$

$$\hat{\underline{M}}_0(s_0,s_0) = \underline{1} \tag{2.17b}$$

so that using (2.13)

$$\frac{d}{ds} \underline{M}_0(s,s_0) = \underline{A}\, \underline{M}_0(s,s_0) ; \; \underline{M}_0(s_0,s_0) = 1 ; \tag{2.18}$$

$$\frac{d}{ds} \vec{a}_0(s,s_0) = \underline{A}\, \vec{a}_0(s,s_0) + \vec{c} ; \; \vec{a}_0(s_0,s_0) = \vec{0} . \tag{2.19}$$

Once $\underline{M}_0(s,s_0)$ is known, (2.19) can be solved in the form:

$$\vec{a}_0(s,s_0) = \underline{M}_0(s,s_0) \cdot \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0(s_0,\tilde{s}) \cdot \vec{c}(\tilde{s})$$

$$= \int_{s_0}^{s} d\tilde{s} \cdot \underline{M}_0(s,\tilde{s}) \cdot \vec{c}(\tilde{s}) . \tag{2.20}$$

This is a convenient form for handling the transfer matrix of a tilted, displaced solenoid (see Appendix I).

2.3 Thin lens approximation

An approximate but very convenient way to solve the equations of spin orbit motion is to divide the magnetic elements into a sufficient number of thin slices of length Δs and develop the solution $\hat{\underline{M}}(s+\Delta s,s)$ in powers of Δs. If only the terms linear in Δs are retained, then from eqns. (2.6) and (2.7) together with (2.9), (2.3c), (2.11) and (2.12) we obtain the thin lens approximation:

$$\hat{\underline{M}}(s+\Delta s,s) = \begin{bmatrix} \underline{M}_0(s+\Delta s,s) & \underline{0} & [\vec{c}_0(s)+\vec{c}_1(s)]\cdot\Delta s \\ \underline{G}_0(s)\cdot\Delta s & \underline{D}(s+\Delta s,s) & \underline{R}(s)\cdot\vec{c}_2(s) \\ \underline{0} & \underline{0} & 1 \end{bmatrix} \tag{2.21}$$

with

$$\underline{D}(s+\Delta s,s) = \begin{pmatrix} \cos[\psi(s+\Delta s)-\psi(s)] & \sin[\psi(s+\Delta s)-\psi(s)] \\ -\sin[\psi(s+\Delta s)-\psi(s)] & \cos[\psi(s+\Delta s)-\psi(s)] \end{pmatrix} . \tag{2.22}$$

The orbit matrix $\underline{M}_0$ must of course remain symplectic during the linearization. This can be achieved by writing it in the form

$$\underline{M}_0(s+\Delta s,s) = \underline{M}_D(s+\Delta s,\, s+\tfrac{\Delta s}{2}) \cdot [\underline{1} + \underline{C}(s)\cdot\Delta s] \cdot \underline{R}(\Delta\theta) \cdot \underline{M}_D(s+\tfrac{\Delta s}{2},s) \tag{2.23}$$

$$\underline{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -(G_1+H^2) & 0 & N & 0 & 0 & K_x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ N & 0 & -(G_2+H^2) & 0 & 0 & K_z \\ -K_x & 0 & -K_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{e\hat{V}}{E_0}\cdot k\cdot\frac{2\pi}{L}\cdot\cos\phi\cdot\sum_{\mu}\delta(s-s_\mu) & 0 \end{pmatrix} ; \tag{2.24}$$

$$\underline{R}(\Delta\theta) = \begin{pmatrix} \cos\Delta\theta & 0 & +\sin\Delta\theta & 0 & 0 & 0 \\ 0 & \cos\Delta\theta & 0 & +\sin\Delta\theta & 0 & 0 \\ -\sin\Delta\theta & 0 & \cos\Delta\theta & 0 & 0 & 0 \\ 0 & -\sin\Delta\theta & 0 & \cos\Delta\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.25a)$$

$$\text{with} \quad \Delta\theta = H(s) \cdot \Delta s \qquad (2.25b)$$

and

$$\underline{M}_D(s+\ell,s) = \begin{pmatrix} 1 & \ell & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.26)$$

(= transfer matrix for a drift of length ℓ) .

In linear order, the r.h.s. of eqn. (2.23) agrees with the r.h.s. of eqn. (2.10). Furthermore, all factor matrices on the r.h.s. of (2.23) and therefore $\underline{M}_0(s+\Delta s,s)$ itself are symplectic.

Remark: The rotation matrix $\underline{R}(\Delta\theta)$ commutes with $\underline{M}_0$ and also with $\underline{C}(s)$ if $N = 0$, $G_1 = G_2 = 0$. Therefore, for a pure solenoid field the factors $\underline{R}(\Delta\theta)$ in (2.23) can be extracted and combined into one rotation $\underline{R}(\theta)$:

$$R(\theta) = R(\Delta\theta_1) \cdot R(\Delta\theta_2) \ \ldots \ R(\Delta\theta_n)$$

with

$$\theta = \sum_{\nu=1}^{n} \Delta\theta_\nu$$

which only needs to be applied once.

2.4 The equation for $\delta\vec{n}$

We now use this thin lens approximation for calculating the periodic solution of the spin orbit vector $\vec{u}(s)$. As in (2.4b) this will lead to the closed orbit $\vec{y}$ and the tilt of the $\vec{n}$-axis $\delta\vec{n}$.

$\vec{y}$ and $\delta\vec{n}$ at position s_0 can be obtained immediately by writing (2.4a) and (2.5) in the form

$$\hat{\underline{M}}(s_0+L,s_0) \cdot \begin{bmatrix} \vec{y}(s_0) \\ \delta\vec{n}(s_0) \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{y}(s_0) \\ \delta\vec{n}(s_0) \\ 1 \end{bmatrix} \qquad (2.27a)$$

and extracting the eigenvector with eigenvalue 1. The normalization of the eigenvector is fixed by requiring that the 9th component be unity. $\vec{u}(s)$ at other positions is obtained by applying (2.5) which we rewrite here as:

$$\begin{bmatrix} \vec{y}(s) \\ \delta\vec{n}(s) \\ 1 \end{bmatrix} = \hat{\underline{M}}(s,s_0) \begin{bmatrix} \vec{y}(s_0) \\ \delta\vec{n}(s_0) \\ 1 \end{bmatrix} . \qquad (2.27b)$$

Using (2.6), eqn. (2.27a) can also be written as

$$\underline{M}(s_0+L,s_0)\ \vec{u}(s_0) + \vec{a}(s_0+L,s_0) = \vec{u}(s_0) \ ,$$

from which we immediately obtain

$$[\underline{1} - \underline{M}(s_0+L,s_0)]\ \vec{u}(s_0) = \vec{a}(s_0+L,s_0) \ ;$$

$$\vec{u}(s_0) = [1 - \underline{M}(s_0+L,s_0)]^{-1} \cdot \vec{a}(s_0+L,s_0) \ . \qquad (2.28)$$

This provides another way to calculate $\hat{\vec{u}}(s_0)$.

In the following we consider that $\hat{\vec{u}}(s)$ and in particular $\delta\vec{n}(s)$ are already known.

3. Harmonic orbit optimization

From eqn. (2.27), with the help of (2.2b), (2.3c), (2.6) and (2.9) it is clear that $\delta\vec{n}(s)$ depends on the field errors and correction fields. The aim is then to find correction fields such that the changes in the $\vec{n}$-axis caused by field errors can be corrected. This problem has already been treated in Ref. 1. In this work we wish to extend the formalism so as to handle the possibility of enhancing the correction to $\delta\vec{n}$ at those positions in the ring where it has the most damaging effect on polarization. This will be achieved with the aid of a a periodic weight function g(s).

We recall that according to eqns. (2.2), (2.2c) and (2.4), $\delta\vec{n}(s)$ satisfies:

$$\frac{d}{ds}\,\delta\vec{n} = \psi' \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \delta\vec{n} + \vec{d} \qquad (3.1a)$$

with the periodicity condition

$$\delta\vec{n}(s+L) = \delta\vec{n}(s) \qquad (3.1b)$$

where

$$\vec{d}(s) \equiv \begin{pmatrix} d_1(s) \\ d_2(s) \end{pmatrix} = \underline{R} \cdot [\underline{F} \cdot \vec{\hat{y}}(s) + \vec{c}_2] \ ; \qquad (3.2a)$$

$$\vec{d}(s+L) = \vec{d}(s) \ . \qquad (3.2b)$$

We now introduce a periodic weight factor

$$g(s) = g(s+L) \qquad (3.3)$$

which will be used to emphasise those parts of the ring where $\delta\vec{n}(s)$ has the most damaging influence on the polarization and we put

$$\delta\vec{\bar{n}}(s) = g(s)\cdot\delta\vec{n}(s) \ . \qquad (3.4)$$

Alternatively, by setting g(s) to zero in some parts of the ring, we can consider that g(s) provides a means of masking out those parts of the ring where $\delta\vec{n}$ has the least damaging effect.

The new function $\delta\vec{\bar{n}}(s)$ obeys a modified form of eqn. (3.1a), namely:

$$\frac{d}{ds}\begin{pmatrix} \delta\bar{n}_1 \\ \delta\bar{n}_2 \end{pmatrix} = \psi' \cdot \begin{pmatrix} \delta\bar{n}_2 \\ -\delta\bar{n}_1 \end{pmatrix} + g'(s) \cdot \begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix} + g(s) \cdot \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

or

$$\frac{d}{ds}\,[\delta\bar{n}_1 - i\cdot\delta\bar{n}_2] = i\psi' \cdot [\delta\bar{n}_1 - i\cdot\delta\bar{n}_2] + f(s) \ ; \qquad (3.5a)$$

$$[\delta\bar{n}_1 - i\cdot\delta\bar{n}_2]_s = [\delta\bar{n}_1 - i\cdot\delta\bar{n}_2]_{s+L} \qquad (3.5b)$$

with

$$f(s) = g(s) \cdot (d_1 - i\cdot d_2) + g'(s) \cdot (\delta n_1 - i\cdot\delta n_2) = f(s+L) \qquad (3.6)$$

whereby $\delta\vec{n}$ and $\vec{d}$ are to be obtained from the solutions to (2.27).

Eqn. (3.2a) can also be written as

$$\frac{d}{ds}\left\{e^{-i\cdot\psi(s)} \cdot (\delta\bar{n}_1 - i\cdot\delta\bar{n}_2)\right\} = e^{-i\cdot\psi(s)} \cdot f(s) \ . \qquad (3.7)$$

By integrating (3.7) from (s-L) to s and applying the periodicity conditions (3.5b)

$$\psi(s) - \psi(s+L) = 2\pi\nu$$

we find

$$[\delta\bar{n}_1 - i\cdot\delta\bar{n}_2] = \frac{i}{2} \cdot \frac{1}{\sin\pi\nu} \cdot e^{i[\psi(s)-\pi\nu]} \cdot \int_{s-L}^{s} d\tilde{s} \cdot e^{-i\cdot\psi(\tilde{s})} \cdot f(\tilde{s}) \ .$$

If $f(\tilde{s})$ (which is periodic) is expanded in a Fourier series:

$$f(\tilde{s}) = \sum_{k=-\infty}^{+\infty} f_k \cdot e^{i\cdot k\cdot 2\pi\cdot\frac{\tilde{s}}{L}} \qquad (3.8a)$$

$$\Longrightarrow \qquad f_k = \frac{1}{L} \cdot \int_{s_0}^{s_0+L} d\tilde{s} \cdot f(\tilde{s}) \cdot e^{-i\cdot k\cdot 2\pi\cdot\frac{\tilde{s}}{L}} \qquad (3.8b)$$

$\delta\vec{n}$ can also be written as

$$[\delta\bar{n}_1(s) - i\cdot\delta\bar{n}_2(s)] = \frac{1}{2}\cdot\frac{1}{\sin\pi\nu}\cdot e^{i[\psi(s)-\pi\nu]}$$

$$\times \sum_k f_k \cdot \int_{s-L}^{s} d\tilde{s} \cdot e^{i[k\cdot 2\pi\cdot\frac{\tilde{s}}{L} - \psi(\tilde{s})]} \quad . \qquad (3.9)$$

If, as in Ref. 1, the phase function has the form

$$\psi(s) = \psi(s_0) + 2\pi\nu\cdot\frac{s-s_0}{L} \qquad (3.10a)$$

$$\Longrightarrow \qquad \psi(s) - \psi(\tilde{s}) = 2\pi\nu\cdot\frac{s-\tilde{s}}{L} \qquad (3.10b)$$

eqn. (3.9) finally becomes

$$[\delta\bar{n}_1(s) - i\cdot\delta\bar{n}_2(s)] = -i\cdot\frac{L}{2\pi}\cdot\sum_{k=-\infty}^{+\infty} f_k\cdot\frac{e^{i\cdot 2\pi k\cdot\frac{s}{L}}}{k-\nu} \quad . \qquad (3.11)$$

This equation describes the connection between the weighted perturbation $\delta\vec{n}$ of the $\vec{n}$-axis and the Fourier coefficients f_k of the function f(s) defined in (3.6). If g'(s) is zero, so that g(s) is constant, these equations just reduce to those of Ref. 1, Section 5.

4. Correction schemes

From eqn. (3.11) it is clear that the largest contributions to $\delta\vec{n}$ come from those harmonics, f_k, for which

$$k \approx \nu \; .$$

It is these Fourier components which we will try to minimize with the aid of suitable correction coils.

To do this we first of all separate f(s) (eqn. (3.6)) into two parts

$$f(s) = \tilde{f}(s) + f^{(o)}(s) \qquad (4.1)$$

where $\tilde{f}(s)$ describes the effect of field errors $\Delta\tilde{B}_x$, $\Delta\tilde{B}_z$, $\Delta\tilde{B}_s$ and of the vector $\vec{c}_0$ (see (2.1b)) and $f^{(o)}(s)$

$$f^{(o)}(s) = g(s)\cdot(d_1^{(o)} - i\cdot d_2^{(o)}) + g'(s)\cdot(\delta n_1{}^{(o)} - i\cdot\delta n_2{}^{(o)})$$

$$= f_R^{(o)}(s) - i\cdot f_I^{(o)}(s) \; ; \qquad (4.2a)$$

$$\left\{ \begin{array}{l} f_R^{(o)}(s) = g(s)\cdot d_1^{(o)} + g'(s)\cdot\delta n_1^{(o)} \; ; \\ f_I^{(o)}(s) = g(s)\cdot d_2^{(o)} + g'(s)\cdot\delta n_2^{(o)} \end{array} \right. \qquad (4.2b)$$

describes the influence of horizontal correction fields $\Delta B_x^{(o)}(s)$. As in Ref. 1 we rely on corrections to the vertical closed orbit only. This division of f(s) into two components is possible due to the linearity of the defining equations (2.3) and (2.4) for $\vec{y}$ and $\delta\vec{n}$. Thus $d_1^{(o)}$ and $d_2^{(o)}$ are given (see eqn. (3.2)) by

$$\vec{d}^{(o)}(s) \equiv \begin{pmatrix} d_1^{(o)}(s) \\ d_2^{(o)}(s) \end{pmatrix} = \underline{R}\cdot[\underline{F}\cdot\vec{y}^{(o)}(s) + \vec{c}_2^{(o)}(s)] \qquad (4.3)$$

with (see eqn. (2.2b))

$$\vec{c}_2^{(o)}(s) = -\frac{e}{E_0}\cdot\begin{bmatrix} 0 \\ \Delta B_x^{(o)}(s) \\ 0 \end{bmatrix} , \qquad (4.4)$$

while according to (2.3) and (2.4), $\vec{y}^{(o)}$ and $\delta\vec{n}^{(o)}$ obey:

$$\frac{d}{ds}\begin{pmatrix}\vec{y}^{(o)}\\ \delta\vec{n}^{(o)}\end{pmatrix} = \hat{\underline{A}}\begin{pmatrix}\vec{y}^{(o)}\\ \delta\vec{n}^{(o)}\end{pmatrix} + \begin{pmatrix}c_1^{(o)}\\ \underline{R}\, c_2^{(o)}\end{pmatrix} ; \tag{4.5a}$$

$$\begin{pmatrix}\vec{y}^{(o)}(s+L)\\ \delta\vec{n}^{(o)}(s+L)\end{pmatrix} = \begin{pmatrix}\vec{y}^{(o)}(s)\\ \delta\vec{n}^{(o)}(s)\end{pmatrix} \tag{4.5b}$$

with (see eqn. (2.1c))

$$(\vec{c}_1^{(o)})^T = (0,\ 0,\ 0,\ \frac{e}{E_0}\,\Delta B_x^{(o)},\ 0,\ 0)\ . \tag{4.6}$$

Eqns. (4.5a) and (4.5b) are of course to be solved using the methods of sections 2.3 and 2.4.

The separation of f(s) into components $\tilde{f}(s)$ and $f^{(o)}(s)$ leads to a corresponding separation of the Fourier coefficients f_k

$$f_k = \tilde{f}_k + f_k^{(o)}, \tag{4.7}$$

where in particular $f_k^{(o)}$ is given by eqns. (3.8b) and (4.2)

$$\begin{aligned} f_k^{(o)} &= \frac{1}{L}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot[f_R^{(o)}(\tilde{s}) - i\cdot f_I^{(o)}(\tilde{s})]\cdot e^{-i\cdot k\cdot 2\pi\cdot\frac{\tilde{s}}{L}} \\ &= a_k - ib_k\ ; \end{aligned} \tag{4.8}$$

$$\begin{cases} a_k = \frac{1}{L}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\{f_R^{(o)}(\tilde{s})\cdot\cos(2\pi k\cdot\frac{\tilde{s}}{L}) - f_I^{(o)}(\tilde{s})\cdot\sin(2\pi k\cdot\frac{\tilde{s}}{L})\}\ ; \\ b_k = \frac{1}{L}\cdot\int_{s_0}^{s_0+L} d\tilde{s}\cdot\{f_R^{(o)}(\tilde{s})\cdot\sin(2\pi k\cdot\frac{\tilde{s}}{L}) + f_I^{(o)}(\tilde{s})\cdot\cos(2\pi k\cdot\frac{\tilde{s}}{L})\}\ . \end{cases} \tag{4.9}$$

By writing the correction field $\Delta B_x^{(o)}$ as

$$\Delta B_x^{(o)}(s) = \sum_\mu \Delta\hat{B}_\mu\cdot\delta(s-s_\mu) \tag{4.10}$$

where the summand

$$\Delta\hat{B}_\mu\cdot\delta(s-s_\mu)$$

describes pointlike correction coils at positions $s = s_\mu$, the real and imaginary parts, a_k and b_k in (4.8), can be put in the form:

$$\begin{aligned} a_k &= \sum_\mu A_{k\mu}\cdot\Delta\hat{B}_\mu\ ; \\ b_k &= \sum_\mu B_{k\mu}\cdot\Delta\hat{B}_\mu \end{aligned} \tag{4.11}$$

where the coefficients $A_{k\mu}$ and $B_{k\mu}$ are determined by eqns. (4.2)-(4.9) and the coil positions s_μ:

$$\Delta B_x^{(o)} = \Delta\hat{B}_\mu\cdot\delta(s-s_\mu)\ .$$

The further development runs in just the same way as in Ref. 1.

We consider the effect of a family of 8 correction coils with fields $\Delta\hat{B}_1$, $\Delta\hat{B}_2$, ... $\Delta\hat{B}_8$.

By writing k = r, r + 1, r + 2, r + 3, eqn. (4.11) takes the form

$$\begin{pmatrix} a_r\\ b_r\\ a_{r+1}\\ b_{r+1}\\ a_{r+2}\\ b_{r+2}\\ a_{r+3}\\ b_{r+3}\end{pmatrix} = \underline{K}\cdot\begin{pmatrix}\Delta\hat{B}_1\\ \Delta\hat{B}_2\\ \Delta\hat{B}_3\\ \Delta\hat{B}_4\\ \Delta\hat{B}_5\\ \Delta\hat{B}_6\\ \Delta\hat{B}_7\\ \Delta\hat{B}_8\end{pmatrix} \tag{4.12}$$

with

$$\underline{K} = \begin{pmatrix} A_{r1} & A_{r2} & A_{r3} & A_{r4} & A_{r5} & A_{r6} & A_{r7} & A_{r8} \\ B_{r1} & B_{r2} & B_{r3} & B_{r4} & B_{r5} & B_{r6} & B_{r7} & B_{r8} \\ A_{r+1,1} & A_{r+1,2} & A_{r+1,3} & A_{r+1,4} & A_{r+1,5} & A_{r+1,6} & A_{r+1,7} & A_{r+1,8} \\ B_{r+1,1} & B_{r+1,2} & B_{r+1,3} & B_{r+1,4} & B_{r+1,5} & B_{r+1,6} & B_{r+1,7} & B_{r+1,8} \\ A_{r+2,1} & A_{r+2,2} & A_{r+2,3} & A_{r+2,4} & A_{r+2,5} & A_{r+2,6} & A_{r+2,7} & A_{r+2,8} \\ B_{r+2,1} & B_{r+2,2} & B_{r+2,3} & B_{r+2,4} & B_{r+2,5} & B_{r+2,6} & B_{r+2,7} & B_{r+2,8} \\ A_{r+3,1} & A_{r+3,2} & A_{r+3,3} & A_{r+3,4} & A_{r+3,5} & A_{r+3,6} & A_{r+3,7} & A_{r+3,8} \\ B_{r+3,1} & B_{r+3,2} & B_{r+3,3} & B_{r+3,4} & B_{r+3,5} & B_{r+3,6} & B_{r+3,7} & B_{r+3,8} \end{pmatrix} . \quad (4.13)$$

On inversion:

$$\begin{pmatrix} \Delta\hat{B}_1 \\ \Delta\hat{B}_2 \\ \Delta\hat{B}_3 \\ \Delta\hat{B}_4 \\ \Delta\hat{B}_5 \\ \Delta\hat{B}_6 \\ \Delta\hat{B}_7 \\ \Delta\hat{B}_8 \end{pmatrix} = \underline{K}^{-1} \cdot \begin{pmatrix} a_r \\ b_r \\ a_{r+1} \\ b_{r+1} \\ a_{r+2} \\ b_{r+2} \\ a_{r+3} \\ b_{r+3} \end{pmatrix} \quad (4.14)$$

we can calculate those fields $\Delta\hat{B}_\mu$ which are required for changing the quantities $a_r = \mathrm{Re}\, f_r^{(o)}$, $b_r = -\mathrm{Im}\, f_r^{(o)}$ independently of each other, so that the dangerous Fourier coefficients

$$f_k = \tilde{f}_k + f_k^{(o)} \quad (k \approx \nu)$$

can be made to vanish systematically.

For example, to correct a_r, we write the system (4.12) as the column vector

$$\begin{pmatrix} a_r \\ b_r \\ a_{r+1} \\ b_{r+1} \\ a_{r+2} \\ b_{r+2} \\ a_{r+3} \\ b_{r+3} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \quad (4.16)$$

The other components b_r, a_{r+1}, b_{r+1}, a_{r+2}, b_{r+2}, a_{r+3}, b_{r+3} can be treated in the same way.

With this correction scheme we are in the position to minimize the weighted tilt $\delta\vec{\bar{n}} = \delta\vec{n}\cdot g$.

5. Conclusion

As explained in Ref. 1, normally we cannot measure the closed orbit to sufficient accuracy to calculate the corrections f_k^{ℓ}. Instead, we would operate empirically, by calculating the correction coil strengths $\Delta\hat{B}_\mu$ up to an overall scale factor and then maximising the measured polarization by varying the scale factor.

In the formalism developed here, we have more freedom than in Ref. 1; here we can experiment with the weighting function to see if, for example, it is safe to ignore tilts $\delta\vec{n}$ in those straight sections where the $\vec{n}$ vector should be longitudinal in comparison with tilts $\delta\vec{n}$ in the arcs. Or it might be, that it is more important to correct $\delta\vec{n}$ in regions where the horizontal dispersion is large than in regions where the horizontal beta function is large. One would then choose g(s) appropriately.

Finally, we point out that the present formalism is only slightly more complicated than in Ref. 1; here, we must calculate not only the closed orbit corresponding to a particular correction field excitation, but also $\delta\vec{n}(s)$ (see eqn. (4.2b)). This requires some modifications to the program FODO /2/; in particular a $\delta\vec{n}(s)$ must be stored for each closed orbit $\vec{y}(s)$.

APPENDIX I

The transfer matrix for a tilted and displaced solenoid

To calculate the transfer matrix of a displaced and tilted solenoid we supplement the usual (s,x,z) coordinate system with a second coordinate system (τ,u,v) fixed to the solenoid. The τ-axis lies along the solenoid axis (see Fig. 1).

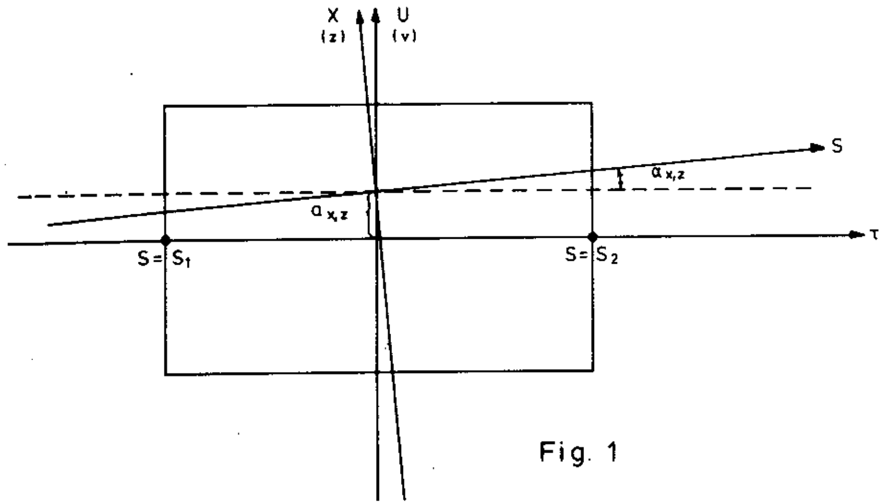

Figure 1

In the (τ,u,v) system the orbital equations (2.1) become

$$u' = p_u + H \cdot p_v \;;$$

$$p_u' = - H^2 \cdot u + H \cdot p_v \;;$$

$$v' = - H \cdot u + p_v \;;$$

$$p_v' = - H \cdot p_u - H^2 \cdot v$$

or, by eliminating p_u, p_v:

$$u'' = 2H \cdot v' + H' \cdot v \;;$$
$$v'' = - 2H \cdot u' - H' \cdot u \,. \qquad (I.1)$$

If $|\alpha_x| \ll 1$, $|\alpha_z| \ll 1$ we have

$$\tau = s \;;$$
$$u = x + a_x + s \cdot \sin\alpha_x \;; \qquad (I.2)$$
$$v = z + a_z + s \cdot \sin\alpha_z$$

and by substituting of (I.2) into (I.1) we obtain

$$x'' = 2H \cdot (z' + \sin\alpha_z) + H' \cdot (z + a_z + s \cdot \sin\alpha_z)$$

$$= 2H \cdot z' + H' \cdot z + 2H \cdot \sin\alpha_z + H' \cdot (a_z + s \cdot \sin\alpha_z) \;;$$

$$z'' = - 2H \cdot (x' + \sin\alpha_x) - H' \cdot (x + a_x + s \cdot \sin\alpha_x)$$

$$= - 2H \cdot x' - H' \cdot x - 2H \cdot \sin\alpha_x - H' \cdot (a_x + s \cdot \sin\alpha_x) \,.$$

By using the variables

$$p_x = x' - H \cdot z ;$$

$$p_z = z' + H \cdot x$$

this may also be written in the form

$$x' = p_x + H \cdot z ;$$

$$p_x' = - H^2 \cdot x + H \cdot p_z + 2H \cdot \sin\alpha_z + H' \cdot (a_z + s \cdot \sin\alpha_z) ;$$

$$z' = p_z - H \cdot x ;$$

$$p_z' = - H \cdot p_x - H^2 \cdot z - 2H \cdot \sin\alpha_x - H' \cdot (a_x + s \cdot \sin\alpha_x) \qquad (I.3)$$

where H and H' are given by eqn. (2.1d) (see Fig. 1):

$$H = \frac{1}{2} \frac{e}{E_0} B_\tau^{(o)} ; \qquad (I.4a)$$

$$H' = H \cdot \delta(s-s_1) - H \cdot \delta(s-s_2). \qquad (I.4b)$$

The variables σ and p_σ which (eqn. (2.1a)) are constants:

$$\begin{cases} \sigma' = 0 \\ p_\sigma' = 0 \end{cases} \Longrightarrow \begin{cases} \sigma(s) = \sigma(s_1) \\ p_\sigma(s) = p_\sigma(s_1) \end{cases} \qquad (I.5)$$

need not be considered in the following.

To integrate eqn. (I.3) in the domain

$$s_1 \leq s \leq s_2$$

we must distinguish three cases:

a) $s_1 - \varepsilon < s < s_1 + \varepsilon$ $(0 < \varepsilon \to 0)$.

In this region (I.3) and (I.4b) become (with $s_1 = -\frac{\ell}{2}$):

$$\frac{d}{ds}\begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix} = H \cdot \delta(s-s_1) \cdot \begin{pmatrix} 0 \\ (a_z + s \cdot \sin\alpha_z) \\ 0 \\ -(a_x + s \cdot \sin\alpha_x) \end{pmatrix}$$

and we have the solution

$$x(s_1 + \varepsilon) = x(s_1 - \varepsilon) ;$$

$$p_x(s_1 + \varepsilon) = p_x(s_1 - \varepsilon) + H \cdot (a_z - \frac{\ell}{2} \cdot \sin\alpha_z) ;$$

$$z(s_1 + \varepsilon) = z(s_1 - \varepsilon) ;$$

$$p_z(s_1 - \varepsilon) = p_z(s_1 + \varepsilon) - H \cdot (a_x - \frac{\ell}{2} \cdot \sin\alpha_x) .$$

This leads (I.5) to the transfer matrix

$$\hat{\underline{M}}_0(s_1+0,s_1-0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & +H\cdot(a_z - \frac{\ell}{2} \cdot \sin\alpha_z) \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -H\cdot(a_x - \frac{\ell}{2} \cdot \sin\alpha_x) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (I.6)$$

b) $s_1 + 0 < s < s_2 - 0$.

The equations of motion are

$$\frac{d}{ds}\begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix} = \begin{pmatrix} 0 & 1 & H & 0 \\ -H^2 & 0 & 0 & H \\ -H & 0 & 0 & 1 \\ 0 & -H & -H^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix} + 2H \cdot \begin{pmatrix} 0 \\ \sin\alpha_z \\ -\sin\alpha_x \\ 0 \end{pmatrix} .$$

In the representation

$$\hat{\underline{M}}_0 = \begin{pmatrix} \underline{M}_0 & \vec{a}_0 \\ 0 & 1 \end{pmatrix} ; \quad \vec{a}_0^T = (a_{01}, a_{02}, a_{03}, a_{04}, a_{05}, a_{06}) \qquad (I.7)$$

for the 7-dimensional transfer $\hat{\underline{M}}_0$ we find for $\underline{M}_0$:

$$\underline{M}_0(s,s_1+0) = \begin{pmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2H}\cdot\sin 2\theta & \frac{1}{2}\sin 2\theta & \frac{1}{2H}(1-\cos 2\theta) & 0 & 0 \\ H\cdot\frac{1}{2}\sin 2\theta & \frac{1}{2}(1+\cos 2\theta) & -H\cdot\frac{1}{2}(1-\cos 2\theta) & \frac{1}{2}\sin 2\theta & 0 & 0 \\ -\frac{1}{2}\sin 2\theta & -\frac{1}{2H}(1-\cos 2\theta) & \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2H}\sin 2\theta & 0 & 0 \\ H\cdot\frac{1}{2}(1+\cos 2\theta) & -\frac{1}{2}\sin 2\theta & -H\cdot\frac{1}{2}\sin 2\theta & \frac{1}{2}(1+\cos 2\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (I.7a)$$

with $\theta = H \cdot (s - s_1)$

and for $\vec{a}_0$ (see (2.20)):

$$\vec{a}_0(s,s_1+0) = \int_{s_1+0}^{s} d\tilde{s} \cdot M_0(s,\tilde{s}) \cdot \vec{c}(\tilde{s})$$

with $\vec{c}^T = 2H \cdot (0, \sin\alpha_z, -\sin\alpha_x, 0, 0, 0)$;

$$\Longrightarrow \quad a_{01}(s,s_1+0) = \sin\alpha_z \cdot \frac{1-\cos 2\theta}{2H} - \sin\alpha_x \cdot [(s-s_1) - \frac{\sin 2\theta}{2H}] ;$$

$$a_{02}(s,s_1+0) = H \cdot \{\sin\alpha_z \cdot [(s-s_1) + \frac{\sin 2\theta}{2H}] - \sin\alpha_x \cdot \frac{1-\cos 2\theta}{2H}\} ;$$

$$a_{03}(s,s_1+0) = -\sin\alpha_z \cdot [(s-s_1) + \frac{\sin 2\theta}{2H}] - \sin\alpha_x \cdot \frac{1-\cos 2\theta}{2H} ;$$

$$a_{04}(s,s_1+0) = -H \cdot \{\sin\alpha_z \cdot \frac{1-\cos 2\theta}{2H} + \sin\alpha_x \cdot [(s-s_1) + \frac{\sin 2\theta}{2H}]\};$$

$$a_{05}(s,s_1+0) = 0 ;$$

$$a_{06}(s,s_1+0) = 0 . \qquad (I.7b)$$

c) $s_2 - \varepsilon < s < s_2 + \varepsilon$.

$$\frac{d}{ds}\begin{pmatrix} x \\ p_x \\ z \\ p_z \end{pmatrix} = -H \cdot \delta(s-s_2) \cdot \begin{pmatrix} 0 \\ (a_z + s \cdot \sin\alpha_z) \\ 0 \\ -(a_x + s \cdot \sin\alpha_x) \end{pmatrix}$$

with $s_2 = +\frac{\ell}{2}$;

$$\hat{M}_0(s_2+0,s_2-0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -H\cdot(a_z + \frac{\ell}{2}\cdot\sin\alpha_z) \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & +H\cdot(a_x + \frac{\ell}{2}\cdot\sin\alpha_x) \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (I.8)$$

We now have the transfer matrices for all three regions

$$s_1 - \varepsilon < s < s_1 + \varepsilon \ ;$$

$$s_1 + \varepsilon < s < s_2 - \varepsilon \ ;$$

$$s_2 - \varepsilon < s < s_2 + \varepsilon \ ;$$

$$(\varepsilon \to 0) \quad .$$

The matrix for the whole region

$$s_1 \leqslant s \leqslant s_2$$

is of course the product of all three:

$$\hat{\underline{M}}_0(s_2+\varepsilon, s_1-\varepsilon) = \hat{\underline{M}}_0(s_2+\varepsilon, s_2-\varepsilon) \cdot \hat{\underline{M}}_0(s_2-\varepsilon, s_1+\varepsilon) \cdot \hat{\underline{M}}_0(s_1+\varepsilon, s_1-\varepsilon). \qquad (I.9)$$

The corresponding matrices $\underline{G}$ and $\underline{D}$ for the spin motion in each region can now also be calculated using these equations for $\hat{\underline{M}}_0$ together with eqns. (2.11) and (2.12).

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