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A SOLENOID SPIN ROTATOR FOR LARGE ELECTRON STORAGE RINGS

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We present details of a design of a solenoid spin rotator that would enable longitudinal polarization to be achieved at electron storage rings.[1] The advantages of the system are the ease of helicity reversal (since no orbit displacements are needed), the high degree of polarization attainable (~90%) and the comparative insensitivity of the polarization to energy changes.

1. INTRODUCTION

In electron storage rings, the electron spins become polarized antiparallel to the magnetic bending field as a result of synchrotron-radiation emission (the Sokolov-Ternov effect).[2] The maximum polarization obtainable from this effect is 92.4%

However, most of the interest among high-energy physicists for use of polarization lies not with this naturally occurring transverse polarization, but with longitudinal polarization, which would enable spin dependent effects in weak-interaction physics to be investigated.[3] Since the Sokolov-Ternov mechanism only works effectively if the equilibrium spin direction is close to vertical in the arc bending magnets, it is then necessary to devise optical systems that rotate the spins from vertical to longitudinal just before the particles reach the interaction point and to rotate them back to the vertical before they reach the arc again. Such devices are called 'spin rotators' and, as a glance at the literature shows,[4] they are neither trivial nor inexpensive. Several kinds of scheme have been proposed[1,4,5,6] for use in future machines such as TRISTAN, HERA and VEPP-4, but so far none has been installed in an existing ring.

The basic principles of spin rotators can be understood by reference to the Bargmann–Michel–Telegdi (BMT) equation,[7,8] which describes electron spin motion in electromagnetic fields. In the following, two consequences of this equation are of importance:

(a) Fields transverse to the particle momentum precess the spin around an axis parallel to the field direction (Fig. 1a). A rotation of the spin by 90° with respect to the outgoing particle direction requires an integrated field

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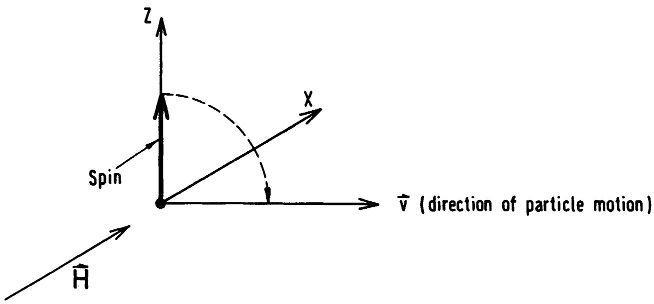


FIGURE 1a Precession of a spin vector in a magnetic field, *H*, transverse to the particle momentum.

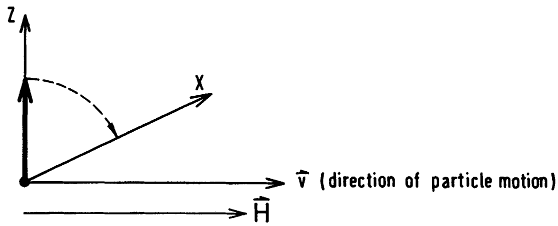


FIGURE 1b Precession of a spin vector in a field parallel to the particle momentum.

strength of about 23 kG-m independently of the particle energy. In general, the spin rotates with respect to the outgoing momentum vector by an angle

$$\Theta_s = \alpha\gamma\Theta_B, \tag{I.1}$$

where Θ_B is the angle of particle deflection, $a = g - 2/2$, where g is the electron g-factor, γ is the Lorentz factor and $a\gamma$ is called the spin tune.

(b) Fields parallel to the particle momentum again precess the spin around the field direction (Fig. 1b). In this case a field integral of about $a\gamma \times 23$ kG-m is required to attain a 90° spin rotation. At a beam energy of 27.5 Gev, $a\gamma$ is 62.4.

Thus it seems clear that spin rotators employing transverse fields would be preferred and indeed the first rotator to be designed used the vertical *S*-bend geometry.[9] This device has an important disadvantage:

- To switch between positive and negative helicity, the vertical beam displacement must be reversed so that the particle trajectory at the interaction point changes.
- Vertical bending also introduces vertical dispersion and thus excites vertical betatron oscillations. This latter can lead to a reduction of the degree of polarization.

Since the early days, more sophisticated schemes have been developed consisting of a combination of horizontal and vertical bends such as in the 'Mini Rotator' of Buon and Steffen for HERA.[10] The rotators are installed outside the interaction region and form part of the arc. A helicity change again requires polarity inversion in the vertical bend magnets and the resulting vertical trajectory displacements must be accommodated either by use of large-aperture rotator magnets or by arranging that the rotator magnets can be moved vertically. There is, however, no displacement of the trajectory in the interaction region (I.R.) but special constraints on the optics ('spin matching conditions') are needed to reduce the depolarization by vertical betatron oscillations.[11,12,13,14,15,16] There are two further consequences of fitting these rotators into the HERA arc:

(a) In practice the sections of rotator in which the equilibrium spin direction is not parallel to the bending field result in significant lowering of the maximum polarization attainable by the Sokolov-Ternov effect.
(b) Since the arrangement of the horizontal bends in the HERA rotator is symmetric with respect to the interaction point, at energies away from the design energy the equilibrium spin direction is not exactly vertical in the arcs and strong depolarization effects resulting from synchrotron and horizontal betatron oscillations can occur.[10]

These considerations were the starting point of our quest to design a spin rotator with no moving elements, no vertical beam deflection and a degree of polarization approaching 90%. A further major impetus came from the realisation that reasonably priced superconducting solenoids of the required field strength are now available from manufacturers.

2. THE ELEMENTS OF THE SOLENOID SPIN ROTATOR

The essential features of the proposed solenoid system are shown in Fig. 2a. A solenoid with an integrated field strength of $a\gamma/(1+a)\times 23.1$ kG-m placed at the

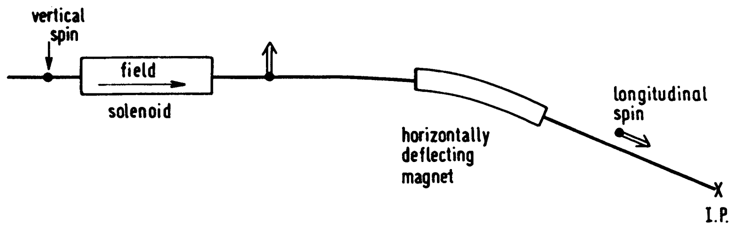


FIGURE 2a Essential features of a rotation system showing how solenoids and dipoles are combined. The spin motion is also indicated.

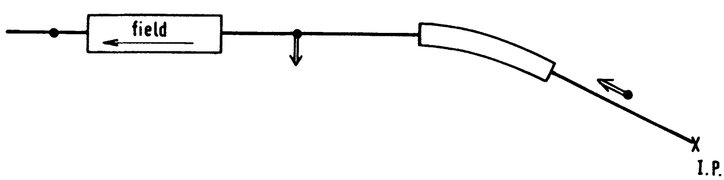


FIGURE 2b As in (a) but with reversed helicity.

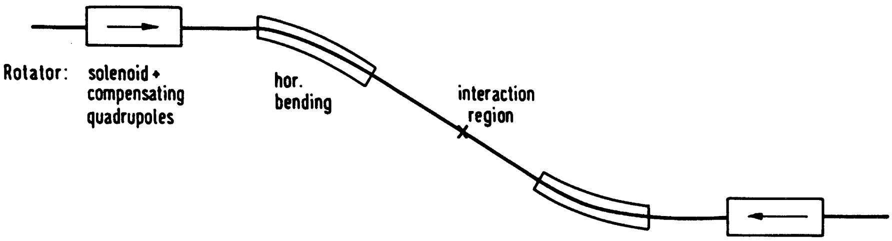


FIGURE 3a A rotation system with antisymmetric dipole arrangement.

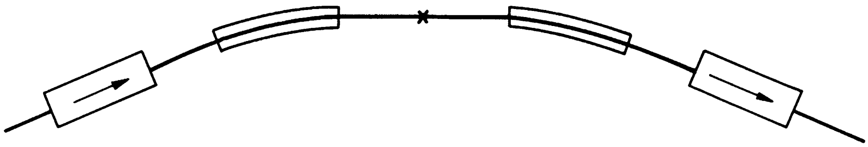


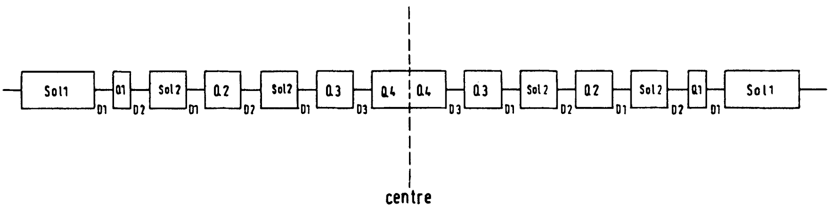
FIGURE 3b A rotation system with symmetric dipole arrangement.

entrance to the interaction region rotates vertical spins leaving the arc into the radial direction and a dipole placed just before the interaction point (I.P.) rotates the spin by 90° into the longitudinal direction. At the same time the dipole causes a horizontal beam deflection of $90°/a\gamma$ i.e. 25 mrad at 27.5 GeV. The helicity at the I.P. can be changed by simply reversing the solenoid polarity (Fig. 2b) and the central particle trajectory is the same for both helicities. Following the interaction point, the spin must be returned to the vertical direction by a second rotator system.

There are then two possibilities, as illustrated in Fig. 3. In the anti-symmetric scheme of Fig. 3a, the solenoids and dipoles on one side of the interaction point have polarities opposite to those on the other side. Therefore, even if the system is not run at design energy, so that the equilibrium spin direction is not parallel to the beam at the I.P., the spins return exactly to the vertical in the arc. Thus, away from design energy, depolarization effects due to horizontal betatron motion and synchrotron motion do not occur.[17,18]

In the symmetric scheme of Fig. 3b, the solenoids and dipoles have the same polarity and it is evident that this would be a mono-energetic device in the same way as the symmetric Mini-Rotator. In addition, it is not trivial to arrange that dispersion vanishes both in the rf accelerating system and at the I.P. with such a scheme. Therefore, although it is easier to fit the symmetric scheme into the HERA tunnel, in the following, the antisymmetric scheme is preferred.

So far, we have ignored the fact that a solenoid which rotates the spin by 90° will, owing to the combined action of the longitudinal central field and the radial end fields, rotate the plane of the betatron oscillations by $90°/2(1+a)$. Thus the beam becomes 'twisted' [19,20] and the vertical and horizontal betatron oscillations become coupled at the I.P. In the antisymmetric scheme, the second solenoid



Sol 1 = Solenoid 5,0 m long 7.2 T } at 27.5 GeV
Sol 2 = Solenoid 2,5 m long 7.2 T }

Quadrupoles: Q1 = 1m long K = 0,007 m^{-2}
Q2 = 2m long K = -0,103 m^{-2}
Q3 = 2m long K = 0,124 m^{-2}
Q4 = 2m long K = -0,116 m^{-2}

Drift spaces: D1 = 40 cm
D2 = 70 cm
D3 = 120 cm

FIGURE 4 Internal layout of the proposed rotator.

would 'untwist' the beam before it reaches the arc. However, optical complications arise from the intervening dipoles, which generate dispersion and additional betatron excitations (see section IV).

In order to maintain a conventional uncoupled optics at the I.P. and to avoid generation of vertical emittance, it is then necessary to modify the simple solenoid rotator concept by introducing extra quadrupoles into the rotator. For example, a set of 2 normal and 2 skew quadrupoles can be placed at the end of the solenoid or, as proposed for VEPP-4,[5,21] a set of 6 normal quadrupoles can be inserted between two solenoids of half length. Since there is no field on the axis of the quadrupoles, they have no effect on the equilibrium spin direction.

Inspired by the latter approach, we finally split the solenoid into 6 slices with normal quadrupoles interleaved. By this means, it is possible not only to untwist the beam but to ensure a "spin-transparent" solenoid-quadrupole arrangement (see below).

The arrangement currently favoured is shown in Fig. 4, which lists all parameters. The total length of the rotator is 41.6 m and the solenoids have central fields of about 72 kG. The whole arrangement transforms a flat beam into a flat beam and rotates the spin by 90°. We should also point out that in principle a whole range of similar rotators may be designed (Appendix II). Furthermore, if the solenoids are turned off for running with vertical polarization, the quadrupoles can be retuned to create a normal uncoupled optics.

3. THE PRACTICAL ENGINEERING LAYOUT

A schematic plan view of a proposed layout of the system is shown in Fig. 5 where the basic elements of Fig. 3a and Fig. 4 are combined. The labels that indicate the major components will be used in later discussions.

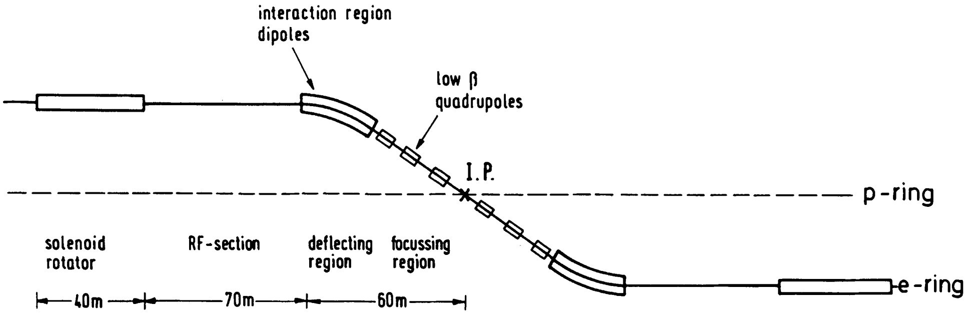


FIGURE 5 Schematic plan view of the whole rotation system. The main features are labelled.

Since in HERA the electron ring must coexist with the proton ring and fit into the proposed tunnel, the arrangement of the rotator system is subject to a number of constraints:

(a) Near the I.P., the low-beta electron quadrupoles must be arranged to leave space for the low-beta proton quadrupoles.
(b) The dispersion must be zero at the I.P. and in the rf section and the beta functions in the rf section must be small.
(c) The interaction-region dipole section which at 27.5 GeV must deflect the beam by 25 mrad should for reasons of optics (see next section) be long. However, the lateral separation of the rf sections must not be too large because the usable tunnel cross section is limited.
(d) The limited available length of the straight sections also restricts the allowed length of the rotator.

Fig. 6 shows[22] a possible solution to these problems. The total straight section

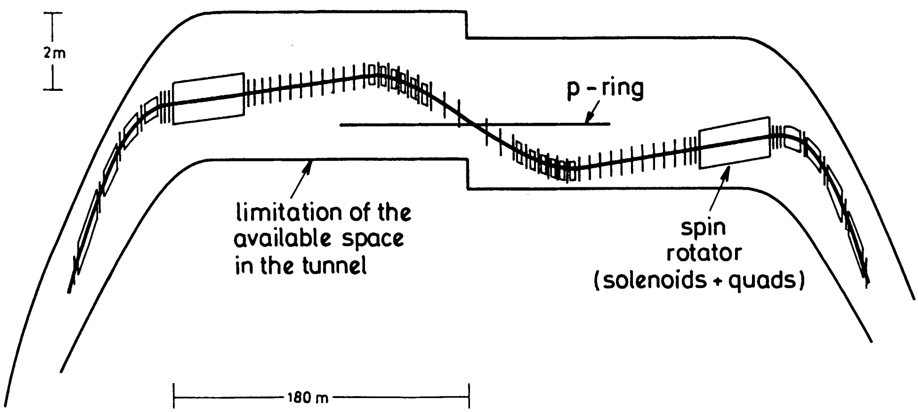


FIGURE 6 Sketch indicating how the proposed system would fit into the available space in the antisymmetric version of the HERA tunnel together with the dimensions. The vertical bars represent quadrupoles and the parallelograms represent dipoles. The rf cavities are not shown.

space required is 2×180 m and as can be seen, by tilting the whole assembly it can be made compatible with the layout of the existing tunnel.

4. DEPOLARIZATION EFFECTS AND 'SPIN MATCHING'

Preamble: In the preceding description of spin manipulations with solenoids, we have only covered the behavior of spins of electrons travelling exactly on the (central) design orbit which passes down the centre of the solenoid and quadrupoles. As we have seen, with sufficiently strong solenoids, spin manipulation for such electrons is straightforward. In reality, however, the electrons execute horizontal and vertical betatron oscillations around the central orbit. As a result, they are subjected to additional fields as they travel off axis through quadrupoles and solenoids and in general their spin precession acquires a small additional orbit-dependent component.

In addition, an ensemble of electrons with energies distributed around the design energy will emerge from a solenoid with a small spread of angles of precession around the beam direction and will emerge from a dipole with a small spread of angles of precession around the dipole field direction. The generation of this spread of spin directions in the magnets can create serious depolarization effects unless the optics is organised so that the small precessions cancel.

An essential part of the present proposal is that it has been possible to design the rotator (Fig. 5) system so that these depolarizing effects indeed largely cancel: a spin entering the first rotator and pointing exactly vertically emerges from the second rotator again pointing almost exactly vertically independently of its horizontal betatron coordinates, x, x' and energy on entering the first rotator.

Detailed Discussion: As explained in detail in Appendix I, the precession errors due to orbital motion can be described in terms of angles α, β which specify a spin orientation with respect to the equilibrium spin direction, $\mathbf{n}$. Changes in α, β caused by passage through a section of the ring can be related to the particle coordinates at the entrance to the section with the help of a 2×6-dimensional transfer matrix G

$$\begin{pmatrix}\Delta\alpha\\ \Delta\beta\end{pmatrix} = G\mathbf{y}, \tag{4.1}$$

where $\mathbf{y}=(x, x', z, z', l, \delta)$ and x, x' describe horizontal motion, z, z' describe vertical motion and l, δ describe longitudinal displacement and energy deviation. In addition G may be subdivided into three 2×2-dimensional matrices g_x, g_z, g_s

$$G=(g_x, g_z, g_s), \tag{4.2}$$

where the g's describe perturbations due to motion along the three axes separately. If g_x (or g_z, g_s) for a section of the ring is zero we describe the section as being "horizontally (or vertically, longitudinally) spin matched".

The basic concepts presented in the Preamble above can now be restated by saying that it is in systems where g_x, g_z, g_s are non-zero that betatron and synchrotron oscillations can cause small changes in α, β and thus cause depolarization. These oscillations are excited by synchrotron photon emission in dipole magnets and it can be shown in the linear theory[11,12,23,24,25] that these notions can be combined in the following statement:

(a) If the matrices g_x, g_z, g_s are non-zero for one circuit around the ring beginning and ending at a slice of dipole which is exciting beam oscillations, then depolarization can occur and there are depolarizing resonances centered at energies where $a\gamma = k \pm Q_i$ (k = integer, Q_i = orbital tune).
(b) The total depolarizing effect is obtained by summing over all dipoles. The contribution of a particular dipole is proportional to the degree of beam excitation in that dipole.

In a perfectly aligned ring consisting only of horizontally bending dipoles and quadrupoles, the $\mathbf{n}$-axis is vertical and g_x and g_s for these elements are zero (see Appendix I). Thus in such a ring, the one-circuit g_x and g_s matrices at any dipole are zero and cause no depolarization effects. Although g_z is non-zero, the beam has zero thickness in the perfect machine and there is no depolarization contribution.

If, however, a spin rotator section is introduced so that $\mathbf{n}$ is locally radial or longitudinal, the local g_x and g_s become non-zero (Eq. (AI-9), (AI-10)). Thus the one-circuit g_x and g_s can also be non-zero for any dipole inside or outside the rotator region. The non-zero one-circuit elements arise solely from the rotator region as $\mathbf{n}$ is vertical in the arcs.

Since in solenoid schemes, it is the main ring dipoles that are responsible for most of the beam excitation around the machine, we see that it is essential to arrange for a horizontal and longitudinal spin match for the section between the entrance to the first rotator and the exit from the second rotator.

In the present scheme, this spin match is achieved by ensuring that g_x is zero for each rotator (Appendix II) and by ensuring that g_x and g_s are zero across the quadrupole and dipole section between the rotators (the "straight section"). The contribution to g_s originating in the solenoids and representing small additional precessions around the beam direction is zero because the solenoids have opposite polarities.

In a straight section containing only drifts and quadrupoles, g_x is generally non-zero but g_s is automatically zero (Eq. (AI-5)). Usually however, g_s becomes non-zero as soon as horizontal dipoles are introduced and if the $\mathbf{n}$-axis has a horizontal component. That this is so is immediately clear from Eq. (AI-9) but in physical terms can be traced to the fact that the dipoles create additional horizontal dispersion. Since the main contribution to the spin perturbation comes from the change of x' due to horizontal focusing, (see e.g. Eq. (AI-10) and accompanying comments), the spin perturbation acquires an energy-deviation dependence resulting from the change of track direction caused by the focusing of the additional dispersion and g_s is correspondingly non-zero. The spin perturbation can be made small if the total change in x' due to focusing is made to vanish both for the pure betatron part and for the horizontal-dispersion part[11,12,25] of the orbit. The betatron condition forces g_x to vanish and by evaluation of the straight-section matrix one sees that the simultaneous application of the dispersion condition causes g_s to become small; g_s then only contains small terms arising from additional precession of off-energy electrons in the dipoles. If only one of the conditions is applied, the corresponding spin perturbation will be zero, but g_x and g_s need not both be zero or small.

In this particular system with its symmetric quadrupole arrangement and antisymmetric dipoles, the optical requirements are actually much simpler since it may be shown that as soon as g_x is made zero by enforcing the betatron condition, g_s is automatically also zero.[26] This is facilitated by the cancellation of the antisymmetric extra spin precessions of off-energy electrons in the dipoles. Because the focusing effects are symmetric around the I.P., the betatron condition is satisfied by using an optics for which

$$\frac{\beta'_x}{2} = \tan \phi_x, \tag{4.3}$$

where ϕ_x is the horizontal betatron phase advance between the I.P. and the entrance to the outgoing rotator and β'_x is the slope of the horizontal beta function† at the same point. Since $\mathbf{n}$ is radial between the rotator and the I.R. dipoles, g_z in the quadrupoles is zero in this region. Between the incoming and outgoing I.R. dipoles, where $\mathbf{n}$ is not radial, g_z is non-zero but again, as the beam has zero height, this is not dangerous.

Once g_s and g_x are zero, we could in principle allow a quite arbitrary horizontal dispersion profile in the straight section. However, since the dipole polarities are antisymmetric, it has been convenient to design the optics so that the dispersion is antisymmetric and vanishes at the outer ends of the I.R. dipoles (Fig. 7) and in the rf section.

Having dealt with the spin matching of the whole I.R. for excitations generated in the arcs, we return to consideration of the effect of beam excitation in the I.R. dipoles. Ideally, if there were sufficient space in the machine tunnel, one would at least arrange for horizontal spin matching between the exit of the first rotator and the entrance to the first I.R. dipole, between the I.R. dipoles and between the exit of the second I.R. dipole and the entrance to the second rotator.

In practice, this requires extra space for the insertion of strings of matchmaking quadrupoles that is not available. Thus in this scheme, no attempt has been made to impose these extra matching conditions. Instead, the I.R. dipoles, which must deflect the beam by 25 mrad at 27.5 GeV, have been made as long as possible so that the synchrotron-radiation power is low. In addition, the dipoles have been divided into sections so that those nearest the I.P. are especially weak and the high-energy physics experiment at the I.P. is not subjected to excessive synchrotron-radiation background. By placing quadrupoles between the dipoles, it is then possible to have small beta functions at the I.P., no dispersion (η_x) at the I.P. (but η'_x not 0), and vanishing dispersion at the outer ends of the I.R. dipoles. It is also possible to arrange for small horizontal beta functions over the region of

† Should not be confused with the previously used β.

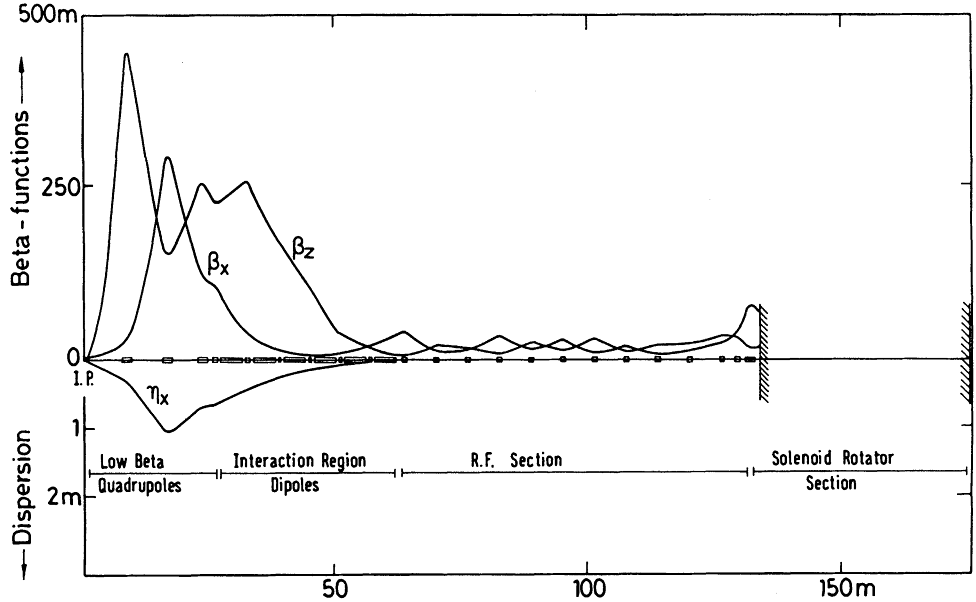


FIGURE 7 Beta functions and horizontal dispersion near the interaction point, in the interaction region dipoles and in the rf section.

the dipoles where the dispersion is large and a small dispersion where the beta function is large (Fig. 7).

By these means, the horizontal betatron excitation is made small and the extra spin matching is not needed. Because the I.R. dipoles are not too strong, the upper limit to achievable Sokolov-Ternov polarisation is not significantly suppressed. A version of an optics satisfying these conditions and which according to SLIM[8] leads to polarizations of 89% for a perfect machine with four interaction regions equipped is shown in Fig. 7.

Finally we note that radiation in the I.R. dipoles causes a small change in average beam energy between the rotators so that the **n**-axis rotation is not exactly cancelled by using solenoids of opposite polarity. Calculations indicate that this is not an important problem when the energy step is limited by use of weak I.R. dipoles. If it were, a small readjustment in solenoid strengths would be enough to ensure that **n** is again vertical in the arcs. A similar comment applies to energy changes caused by the rf system if it is placed between the rotators.

Horizontal shifts in the closed orbit resulting from energy gain and loss effects in the arc dipoles and rf system should again cause no shift in the equilibrium spin direction (**n**-axis) if the I.R. is horizontally spin matched as described.

This completes the description of the currently favoured spin-matching scheme. In Appendix III we describe an alternative.

5. SUMMARY

In this article we have shown how solenoid spin rotators can be conceived which have important advantages compared with previously proposed schemes:

– the basic polarization for the perfect machine, as calculated by SLIM, can reach almost 90%.
– the degree of polarization is essentially independent of the energy.
– helicity reversal is simple since there are no moving parts.
– the natural beam height is zero.
– no spin matching of the arcs is necessary.

We have also shown how such a scheme could be fitted into a version of the layout of the HERA interaction region. In future studies, we intend to investigate methods for correcting the effects of orbit errors and magnet misalignments.[17,29]

6. ACKNOWLEDGEMENTS

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APPENDICES

In these appendices we give more detailed explanation of topics alluded to in the main text. In particular we go into more detail on the subject of spin matching and show how other types of spin rotator systems might be conceived.

APPENDIX I

The Linear Matrix Theory

Calculations of the polarization in storage rings are most conveniently carried out using the linearised transport-matrix formalism of the computer program SLIM of A. Chao,[8] which we briefly review here.[23,24]

In electron storage rings, the polarization vector points along the **n**-axis, a periodic unit spin vector defined uniquely for each point on the closed orbit which transforms into itself when transported once around the ring on the closed orbit. We associate with **n** two other unit vectors **m** and **l** so that the set **n**, **m**, **l** forms a right-handed unit-vector basis. **m** and **l** also precess like spin vectors. In a flat machine, they would precess by an angle $2\pi\gamma a$ around **n** for one circuit of the ring.

In the presence of depolarizing effects, when a spin may not be exactly aligned along the **n**-axis, the spin vector is written:

$$\mathbf{S}=\mathbf{n}+\alpha\mathbf{m}+\beta\mathbf{l}, \tag{AI-1}$$

where $\alpha^2+\beta^2 \ll 1$.

Thus spin perturbations may be represented in terms of two angles α, β and the complete state of an electron may be represented in terms of an eight component vector:

$$(x, x', z, z', l, \delta, \alpha, \beta), \tag{AI-2}$$

where x, x' describe horizontal motion, z, z' describe vertical motion and l, δ describe longitudinal displacement and energy deviation.

The combined linearized transverse, longitudinal and spin-perturbation motion may then be handled using 8×8 transport matrices with the structure:

$$\begin{pmatrix} M_{6\times6} & O_{6\times2} \\ G_{2\times6} & I_{2\times2} \end{pmatrix}, \tag{AI-3}$$

where $M_{6\times6}$ is a 6×6 matrix describing the (coupled) transverse and longitudinal motion[23,24] and $G_{2\times6}$ is a 2×6 matrix describing the dependence of changes in α and β on the betatron and synchrotron motion. G depends on $a\gamma$ and on the x, z, s components of **m** and **l** and on the optical properties of the particular magnet. $O_{6\times2}$ is a 6×2 null matrix resulting from the independence of orbital motion from the spin motion. $I_{2\times2}$ is a unit matrix.

Clearly, columns 1 and 2 of G (which we call g_x) describe changes of α and β due to horizontal motion, columns 3 and 4 (g_z) describe perturbations due to vertical motion and columns 5 and 6 (g_s) describe perturbations due to energy oscillations.

The spin-perturbation properties of short sections of ring may be studied by constructing the 8×8 matrix for the section and extracting the G matrix.

For drift spaces, quadrupoles, skew quadrupoles, dipoles and solenoids and in the absence of closed-orbit errors, the 8×8 matrices take the following forms as may be shown by executing the integrals of Eq. (8-22) in Ref. 24. (See also Ref. 8).

Drift space of length L:

$$M_{drift}=\begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{AI-4}$$

$$g_x=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad g_z=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad g_s=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Quadrupole (the cosine, sine cosh and sinh terms are represented symbolically by letters):

$$M_{quad}=\begin{pmatrix} C_x & S_x & 0 & 0 & 0 & 0\\ \hat{C}_x & \hat{S}_x & 0 & 0 & 0 & 0\\ 0 & 0 & C_z & S_z & 0 & 0\\ 0 & 0 & \hat{C}_z & \hat{S}_z & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_x=\begin{pmatrix} (1+a\gamma)\hat{C}_x\mathbf{l}_z & (1+a\gamma)(\hat{S}_x-1)\mathbf{l}_z\\ -(1+a\gamma)\hat{C}_x\mathbf{m}_z & -(1+a\gamma)(\hat{S}_x-1)\mathbf{m}_z \end{pmatrix} \tag{AI-5}$$

$$g_z=\begin{pmatrix} -(1+a\gamma)\hat{C}_z\mathbf{l}_x & -(1+a\gamma)(\hat{S}_z-1)\mathbf{l}_x\\ (1+a\gamma)\hat{C}_z\mathbf{m}_x & (1+a\gamma)(\hat{S}_z-1)\mathbf{m}_x \end{pmatrix}$$

$$g_s=\begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}.$$

Skew quadrupole:

$$M_{skew}=\begin{pmatrix} C & S & R & T & 0 & 0\\ \hat{C} & \hat{S} & \hat{R} & \hat{T} & 0 & 0\\ R & T & C & S & 0 & 0\\ \hat{R} & \hat{T} & \hat{C} & \hat{S} & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_x=\begin{pmatrix} (1+a\gamma)(\hat{C}\mathbf{l}_z-\hat{R}\mathbf{l}_x) & (1+a\gamma)((\hat{S}-1)\mathbf{l}_z-\hat{T}\mathbf{l}_x)\\ -(1+a\gamma)(\hat{C}\mathbf{m}_z-\hat{R}\mathbf{m}_x) & -(1+a\gamma)((\hat{S}-1)\mathbf{m}_z-\hat{T}\mathbf{m}_x) \end{pmatrix}$$

$$g_z=\begin{pmatrix} (1+a\gamma)(\hat{R}\mathbf{l}_z-\hat{C}\mathbf{l}_x) & (1+a\gamma)(\hat{T}\mathbf{l}_z-(\hat{S}-1)\mathbf{l}_x)\\ -(1+a\gamma)(\hat{R}\mathbf{m}_z-\hat{C}\mathbf{m}_x) & -(1+a\gamma)(\hat{T}\mathbf{m}_z-(\hat{S}-1)\mathbf{m}_x) \end{pmatrix} \tag{AI-6}$$

$$g_s=\begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix}$$

Solenoid consisting of an idealised thin radial entrance end field, a longitudinal central field of length L and a radial exit field, with

$$M_{sol}=M_{exit}\cdot M_{central}\cdot M_{entrance}$$

$$M_{\substack{entrance\\ exit}}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & \pm\frac{R}{2} & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ \mp\frac{R}{2} & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_x = \begin{pmatrix} \pm(1+a\gamma)R\mathbf{l}_x/2 & 0 \\ \mp(1+a\gamma)R\mathbf{m}_x/2 & 0 \end{pmatrix} \tag{AI-7}$$

$$g_z = \begin{pmatrix} \pm(1+a\gamma)R\mathbf{l}_z/2 & 0 \\ \mp(1+a\gamma)R\mathbf{m}_z/2 & 0 \end{pmatrix}$$

$$g_s = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$M_{central} = \begin{pmatrix} 1 & \frac{S}{R} & 0 & \frac{1-C}{R} & 0 & 0 \\ 0 & C & 0 & S & 0 & 0 \\ 0 & -\frac{1-C}{R} & 1 & \frac{S}{R} & 0 & 0 \\ 0 & -S & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_x = \begin{pmatrix} 0 & \psi\mathbf{l}_{0x} \\ 0 & -\psi\mathbf{m}_{0x} \end{pmatrix} \tag{AI-8}$$

$$g_z = \begin{pmatrix} 0 & \psi\mathbf{l}_{0z} \\ 0 & -\psi\mathbf{m}_{0z} \end{pmatrix}$$

$$g_s = \begin{pmatrix} 0 & RL(1+a)\mathbf{l}_{0s} \\ 0 & -RL(1+a)\mathbf{m}_{0s} \end{pmatrix}$$

where

$$R = \frac{2\times\text{orbital twist angle}}{\text{length of central field}} = \frac{e}{pc}B_s$$

and $\psi = a\gamma RL$, $S = \sin(RL)$, $C = \cos(RL)$ and $\mathbf{l}_0$ and $\mathbf{m}_0$ are spin basis components at the entrance to the central field. For presentational clarity here, in the spin-rotation motion around the beam axis used in the spin integrals, the term $1+a$ has been replaced by 1.

Horizontally bending dipole:

$$M_{dipol} = \begin{pmatrix} C & \frac{S}{K} & 0 & 0 & 0 & \frac{1-C}{K} \\ -KS & C & 0 & 0 & 0 & S \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -S & \frac{C-1}{K} & 0 & 0 & 1 & \frac{S}{K}-L \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g_x = \begin{pmatrix} -(1+a\gamma)KS\mathbf{l}_{0z} & (1+a\gamma)(C-1)\mathbf{l}_{0z} \\ (1+a\gamma)KS\mathbf{m}_{0z} & -(1+a\gamma)(C-1)\mathbf{m}_{0z} \end{pmatrix} \tag{AI-9}$$

$$g_z = \begin{pmatrix} 0 & -\hat{S}\mathbf{l}_{0s} - (\hat{C}-1)\mathbf{l}_{0x} \\ 0 & \hat{S}\mathbf{m}_{0s} + (\hat{C}-1)\mathbf{m}_{0x} \end{pmatrix}$$

$$g_s = \begin{pmatrix} 0 & -(1+a\gamma)K\left(L-\dfrac{S}{K}\right)\mathbf{l}_{0z} + KL\mathbf{l}_{0z} \\ 0 & (1+a\gamma)K\left(L-\dfrac{S}{K}\right)\mathbf{m}_{0z} - KL\mathbf{m}_{0z} \end{pmatrix}$$

where K = horizontal curvature, L = length, $S = \sin(KL)$, $C = \cos(KL)$, $\hat{S} = \sin(-a\gamma KL)$, $\hat{C} = \cos(-a\gamma KL)$ and $\mathbf{l}_0, \mathbf{m}_0$ are the spin basis vectors at the entrance of the dipole.

We now note that the G matrix terms contained in these equations have simple physical interpretations.

For example in quadrupoles, if the $\mathbf{n}$-axis is radial ($\mathbf{n}_x = 1, \mathbf{m}_z = 1$), then from Eq. (AI-5):

$$\begin{aligned} \Delta\alpha &= 0 \\ \Delta\beta &= -(1+a\gamma)(\hat{C}_x x + (\hat{S}_x - 1)x') = -(1+a\gamma)\Delta x' \end{aligned} \tag{AI-10}$$

This represents a perturbation of the spin by a small rotation around the vertical axis proportional to the change in the horizontal trajectory direction and is just a restatement of the linearised BMT equation. Note that when $\mathbf{n}$ has a horizontal component, the vanishing of g_x implies that $\Delta x'$ vanishes and vice versa. That this is also the case for an arbitrary string of drifts and normal quadrupoles can be seen by inspecting the structure of g_x in Eq. (AI-13) below.

If $\mathbf{n}$ is vertical in a quadrupole, g_x is zero since there is no precession of $\mathbf{n}$ around a vertical field. Likewise in a dipole g_s is zero if $\mathbf{n}$ is vertical.

Similar analysis may be applied to solenoids and it is found for example (Eq. (AI-7), (AI-8)) that when a solenoid rotates an initially vertical $\mathbf{n}$-axis by $\pi/2$ into the horizontal, the horizontal betatron motion causes perturbations to the emerging spins of the form ($\mathbf{m}_{x0} = 1, \mathbf{n}_{z0} = -1$)

$$\begin{aligned} \Delta\alpha &= 0 \\ \Delta\beta &= -(a\gamma+1)\frac{\pi}{8L}x + \frac{1}{2}\left[-2a\gamma\frac{\pi}{2} + (a\gamma+1)\right]x'. \end{aligned} \tag{AI-11}$$

This represents a perturbation of the spin at the output by a small rotation around the vertical axis and is related to the forwards/backwards tilting that occurs in the weak-solenoid case.[20]

We now see that by combining a solenoid with a $\pi/2$ spin rotation with a set of quadrupoles, it might be possible to arrange that the spin precession errors cancel since the effects in the solenoid and the quadrupoles are of the same order, viz. $a\gamma x'$. If the quadrupoles follow the solenoid, the precession errors are also around the same axis.

It is this kind of behaviour that is at the basis of schemes which combine solenoids and quadrupoles to achieve spin matching. A way to achieve spin matching in the presence of weak solenoids has already been discussed in Ref. 20. The g_s terms for the solenoid describe the small perturbations in the angle of spin precession around the solenoid axis as a result of fractional energy fluctuations, δ.

We recall that the G matrices of quadrupoles and skew-quadrupoles describe the fact that spin perturbations are proportional to the change in track direction (see, e.g., Eq. (AI-10)) and we again expect on the basis of the BMT equation that a similar G matrix behaviour should result for an arbitrary system of drifts, quadrupoles and skew quadrupoles in the absence of dipoles. This is indeed the case; if the 4×4 (transverse motion) matrix has the form

$$M_{general} = \begin{pmatrix} C_x & S_x & R_x & T_x \\ \hat{C}_x & \hat{S}_x & \hat{R}_x & \hat{T}_x \\ R_z & T_z & C_z & S_z \\ \hat{R}_z & \hat{T}_z & \hat{C}_z & \hat{S}_z \end{pmatrix} \tag{AI-12}$$

then the g-matrices have the forms

$$\begin{aligned} g_x &= \begin{pmatrix} (1+a\gamma)(\hat{C}_x\mathbf{l}_z - \hat{R}_z\mathbf{l}_x) & (1+a\gamma)((\hat{S}_x-1)\mathbf{l}_z - \hat{T}_z\mathbf{l}_x) \\ -(1+a\gamma)(\hat{C}_x\mathbf{m}_z - \hat{R}_z\mathbf{m}_x) & -(1+a\gamma)((\hat{S}_x-1)\mathbf{m}_z - \hat{T}_z\mathbf{m}_x) \end{pmatrix} \\ g_z &= \begin{pmatrix} (1+a\gamma)(\hat{R}_x\mathbf{l}_z - \hat{C}_z\mathbf{l}_x) & (1+a\gamma)(\hat{T}_x\mathbf{l}_z - (\hat{S}_z-1)\mathbf{l}_x) \\ -(1+a\gamma)(\hat{R}_x\mathbf{m}_z - \hat{C}_z\mathbf{m}_x) & -(1+a\gamma)(\hat{T}_x\mathbf{m}_z - (\hat{S}_z-1)\mathbf{m}_x) \end{pmatrix} \\ g_s &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{AI-13}$$

Thus, for example, the spin-perturbation properties of a whole string of normal quadrupoles can be simply represented in terms of the six independent parameters of the 4×4 matrix (AI-12). We return to this topic in Appendix III.

Finally, we would also like to point out that some of our work has been carried out using the algebraic-manipulation program REDUCE,[27] which has proved useful for investigating the algebraic properties of products of 8×8 transport matrices.

APPENDIX II

Design of the Spin Rotator

From the earlier discussions, it is clear that the primary design requirement for a spin rotator is that in its 8×8 matrix, the off-diagonal blocks of the 4×4 optical matrix should be zero, so that no optical coupling is introduced, and that g_x should be zero. By choosing $l_s = 1$, this reduces to the requirement that $G(2,1)$ and $G(2,2)$ be zero.

Thus, since the 4×4 optical matrix is symplectic and since for a system of quadrupoles and solenoids, rows and columns five and six of the 6×6 matrix contain only unit diagonal elements, a mimimum of six parameters is needed to enable adjustment of the off-diagonal blocks and the two spin elements to zero. It is clear that the final optical-transfer matrix should also be "reasonable" and allow a comfortable optics to be established in and around the rotator, that the rotator should not be too long, and that the quadrupoles should not be too strong.

In the proposal discussed here, these requirements have been met by dividing the solenoids into sections and interspersing them with an arrangement of normal quadrupoles that is symmetric with respect to the mid-plane (Fig. 4). The six matrix elements have been calculated using the matrices given in Appendix I and by taking into account the spin-basis rotation in the solenoids. The matrix elements have then been adjusted to zero by varying the four independent quadrupole strengths, their lengths, and the spacings between the magnets. Using this approach, a whole family of broadly similar rotators may be designed.

Once horizontal spin matching is achieved for a vertical **n**-axis entering and radial **n**-axis leaving the rotator, it is found that, as expected, the reverse rotator is also matched for a radial **n**-axis entering and a vertical **n**-axis leaving. The g_x elements also remain zero when the rotator solenoid polarities are reversed. We also note that since the G matrix terms for quadrupoles and solenoid end fields depend on $a\gamma+1$ and the G matrix terms for the solenoid central field depend on $a\gamma$, the quadrupole strengths for the decoupled spin matched rotator are slightly energy dependent. However, since $a\gamma\sim 63$, this effect is not large.

APPENDIX III

An Alternative Scheme

In the text, we described a spin-matching scheme in which the rotators were each internally horizontally spin matched and the region between the rotators was separately matched. Here, for completeness we describe an alternative approach in which the whole I.R. including the rotators is matched in one piece and which would be useful if the rotator itself could not be internally matched.

Such a rotator is sketched in Fig. 8a and consists of a 20-m solenoid which rotates **n** by 90° and the beam by about 45°. This is followed by a set of "tuning" skew quadrupoles, and two pairs of quadrupoles and skew quadrupoles with equal separations. These latter four elements serve to rotate the beam back by 45° to remove the coupling introduced by the solenoid. The tuning skew quadrupoles serve to adjust the optics of the whole rotator without introducing extra coupling.

This kind of rotator is not automatically internally matched, but we introduce between the rotators two arbitrary mirror-symmetric systems of normal quadrupoles and drifts (called "ersatz sections") represented by the 4×4 transfer matrices E_1 and E_2 respectively (Fig. 8b). The symmetry ensures symmetry of the beta functions around the I.P. and the elements of E_2 are uniquely determined by the elements of E_1 and the symmetry. As we have seen in Eqs. (AI-12) and (AI-13), the orbital and spin motion properties of E_1 relating to horizontal spin motion are determined by three parameters.

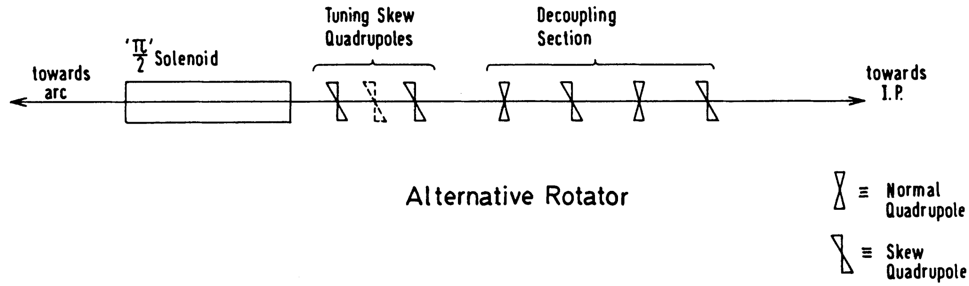

FIGURE 8a An alternative rotator using a solenoid, quadrupoles and skew quadrupoles.

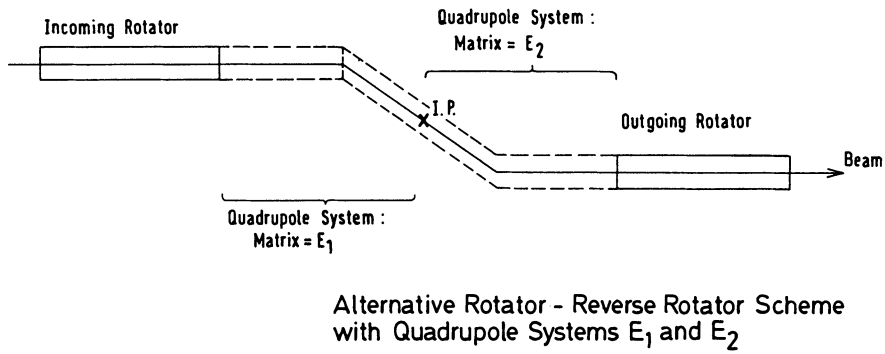

FIGURE 8b Two such rotators combined to make a rotation system. In both rotators the solenoids face the arcs.

Horizontal spin matching of the whole I.R. and rotator sections now consists of choosing a suitable spin basis (Appendix I) and adjusting the three independent (horizontal) matrix elements of E_1 so that the total g_x vanishes. With the kind of rotator shown in Fig. 8a, it is found numerically that in the approximation that $1+a \to 1$ (Eqs. (AI-7), (AI-8)), this is achieved when the horizontal part of the transfer matrix for the whole region is a unit matrix. This may be arranged, for example, by setting

$$E_{1,\text{hor}} = \pm \rho_{\text{hor}}^{-1} \tag{AIII-1}$$

independently of beam energy, where ρ is the optical transfer matrix for the incoming rotator. This implies that the horizontal matrices for each half are either +I or −I. The horizontal beta function at the entrance to the solenoids coming from the arcs is therefore the same as at the interaction point.

The matching procedure then has two steps. The first step involves using the approximation $1+a \to 1$ and tuning the rotator so that its horizontal and vertical transfer properties are reasonable and then creating an optics between the rotators for which Eq. (AIII-1) is satisfied. Inspection of the g_x matrices for each half of the region shows that the spin match is achieved as a result of an exact energy-independent cancellation of the g_x terms of the two halves.

The solution proposed in Eq. (AIII-1) is not unique: any mirror-symmetric quadrupole system with a unit horizontal transfer matrix (which is therefore horizontally spin matched) may be inserted in addition between E_1 and E_2 and the optical flexibility for designing the optics thereby improved.

In the second step, a is given its correct value of 0.0011596, the solenoid strength is proportionally lowered and g_x is again brought to zero by slightly retuning the previously established E_1.

Such schemes were investigated for HERA and again SLIM predicted high polarizations and it is clear that the use of the "ersatz" matrix parametrization is a useful way to investigate spin-matching problems, and to create rotators, especially since there is a wide variety of ways to combine quadrupoles, skew quadrupoles and solenoids to make optically decoupled systems.[5,21]