NEW AND UNIFYING FORMALISM FOR STUDY OF PARTICLE-SPIN DYNAMICS USING TOOLS DISTILLED FROM THEORY OF BUNDLES *

K. Heinemann, J.A. Ellison, University of New Mexico, Albuquerque, New Mexico, USA
D.P. Barber, M. Vogt, DESY, Hamburg, Germany

INTRODUCTION

We return to our study [1] of spin dynamics in storage rings and substantially extend our toolset. To accomplish that, we employ a method developed in the 1980s by R. Zimmer and others for Dynamical-Systems theory [2, 3]. This allows us to generalize the notions of particle-spin motion and field motion. In contrast to [1], we now employ a discrete-time formalism (but a continuous-time treatment would do as well). Four major theorems are presented, the Decomposition Theorem, which allows one to compare different invariant fields, the Invariant Reduction Theorem, which gives new insights into the existence and uniqueness problems of invariant fields (and in particular invariant spin fields), the Cross Section Theorem which supplements the Invariant Reduction Theorem, and the Normal Form Theorem which ties invariant fields with the notion of normal form. It thus turns out that the well established notions of invariant frame field, spin tune, and spin-orbit resonance are generalized by the normal form concept whereas the well established notions of invariant polarization field and invariant spin field are generalized to invariant (E,l)-fields. Here the notation (E,l) will mean that E is a topological space and that the function $l : SO(3) \times E \to E$ is a continuous $SO(3)$-action, i.e., $l(I;x) = x$ and $l(r_1 r_2;x) = l(r_1;l(r_2;x))$. With the flexibility in the choice of (E,l) we also have a unified way to study the dynamics of spin-1/2 and spin-1 particles. Accordingly the special cases $(E,l) = (\mathbb{R}^3, l_{1/2})$ and $(E,l) = (E_1, l_1)$ are discussed in some detail. The origins of our formalism, lying in bundle theory, are pointed out and we briefly mention the relation to Yang-Mills Theory as well.

THE FORMALISM

Particle-spin motion

For given (E,l) each particle carries, in addition to its position z on the torus $\mathbb{T}^d$, an E-valued quantity x we call spin. The one-turn particle-spin map is the function $\mathcal{P}[j,A] : \mathbb{T}^d \times E \to \mathbb{T}^d \times E$ defined by

$$\mathcal{P}[j,A](z,x) = (j(z), l(A(z);x)) \,, \tag{1}$$

where $j \in \text{Homeo}(\mathbb{T}^d)$ is the one-turn particle map (e.g., linear translation on the torus) and $A \in C(\mathbb{T}^d, SO(3))$ is the one-turn spin transfer matrix. Here Homeo($\mathbb{T}^d$) denotes the set of homeomorphisms on $\mathbb{T}^d$, $C(X,Y)$ denotes the set of continuous functions from X to Y and $SO(3)$ is the group of real orthogonal 3×3-matrices of determinant 1 (for the spinor formalism our formalism is obtained by simply replacing $SO(3)$ by $SU(2)$). In our formalism, (1) is the most general description of particle-spin dynamics and the choice of (E,l) depends on the situation, e.g., $(E,l) = (\mathbb{R}^3, l_{1/2})$ for spin-1/2 particles - see below. We work in the framework of topological dynamical systems and therefore A, j, l are continuous functions. This condition could be strengthened to A, j, l being smooth functions or weakened to being Borel measurable functions.

* Work supported by DOE under DE-FG-99ER41104 and by DESY

Field motion and invariant fields

We are primarily interested in the field dynamics induced by the particle-spin dynamics. Let $f : \mathbb{T}^d \to E$ be an E-valued field on $\mathbb{T}^d$ and set $x = f(z)$ in (1). Then after one turn z becomes $j(z)$ and the field value at $j(z)$ becomes $l(A(z); f(z))$. Thus after one turn the field f becomes the field $f' : \mathbb{T}^d \to E$ where $f'(z) := l(A(j^{-1}(z)); f(j^{-1}(z)))$. Thus we have the field map

$$f \mapsto f' = l(A \circ j^{-1}; f \circ j^{-1}) \,, \tag{2}$$

where $\circ$ denotes the composition of functions. We call $f \in C(\mathbb{T}^d, E)$ an "invariant (E,l)-field of (j,A)" if it is mapped by (2) into itself, i.e., if

$$f \circ j = l(A; f) \,. \tag{3}$$

We call (3) the "(E,l)-stationarity equation of (j,A)". Our main focus is on the existence of solutions of (3) as this is what describes the spin equilibrium of a bunch. In the important case where $(E,l) = (\mathbb{R}^3, l_{1/2})$, an invariant (E,l)-field f such that $|f| = 1$ is called an "invariant spin field (ISF)". This completes our introduction to the formalism.

The set $\Sigma_x[f]$ and its invariance

Let $E_x := \{l(r;x) : r \in SO(3)\}$. Then the E_x partition E and each set $\mathbb{T}^d \times E_x$ is invariant under the particle-spin motion of (1) and so we have "decomposed" $\mathbb{T}^d \times E$. Let

$$\Sigma_x[f] := \{z \in \mathbb{T}^d : f(z) \in E_x\} \,. \tag{4}$$

The nonempty sets among the $\Sigma_x[f]$ form a partition of $\mathbb{T}^d$ and tell us how the values of f are distributed, i.e., $z \in \Sigma_x[f]$ iff (=if and only if) $f(z) \in E_x$. It follows from the definition of $\Sigma_x[f]$ and (2) that $\Sigma_x[f'] = j(\Sigma_x[f])$. Thus if f is invariant then every $\Sigma_x[f]$ is invariant under j and $\mathbb{T}^d$ is partitioned into f-dependent invariant sets for the particle dynamics, an interesting fact in its own right.

We can now state three facts related to the existence of invariant fields. Firstly, if there exists an x such that $\Sigma_x[f]$ is not invariant then f is not an invariant field. Secondly, if $\Sigma_x[f]$ is nonempty, let $f_x \in C(\Sigma_x[f], E_x)$ where $f_x(z) =$

$f(z)$. Then f is invariant iff $f_x(j(z)) = l(A(z); f_x(z))$ for every nonempty $\Sigma_x[f]$.

Finally, suppose that j is topologically transitive (e.g., off orbital resonance). This means that a $z_0 \in \mathbb{T}^d$ exists such that $B := \{j^n(z_0) : n = 0, \pm 1, \pm 2, \dots\}$ is dense in $\mathbb{T}^d$, i.e., that the closure $\overline{B}$ of B equals $\mathbb{T}^d$. Let f be invariant and pick x such that $z_0 \in \Sigma_x[f]$ then $B \subset \Sigma_x[f]$. Assume E is Hausdorff (e.g., the topology is from a metric). Then it follows that $\Sigma_x[f]$ is closed and, since $\overline{B} = \mathbb{T}^d$, we have $\Sigma_x[f] = \mathbb{T}^d$. Thus topological transitivity and the Hausdorff property imply an invariant f takes values only in one E_x. The so-called "ISF-conjecture" claims, for $(E, l) = (\mathbb{R}^3, l_{1/2})$, that if j is topologically transitive, then an ISF exists.

FOUR BASIC RESULTS

To show our formalism at work we now present four theorems, the Decomposition Theorem (DT), the Invariant Reduction Theorem (IRT), the Cross Section Theorem (CST), and the Normal Form Theorem (NFT).

The Decomposition Theorem (DT)

Let E be Hausdorff. It is natural to ask about the relation between the dynamics on two distinct invariant sets $\mathbb{T}^d \times E_x$, $\mathbb{T}^d \times E_y$. Consider the particle-spin trajectories defined by $(z(n+1), x(n+1)) = \mathcal{P}[j, A](z(n), x(n))$ where $(z(0), x(0)) = (z_0, x_0)$ is given with $x_0 \in E_x$. Suppose there exists $\beta \in C(E_x, E_y)$ such that for every particle-spin trajectory $(z(n), x(n)) \in \mathbb{T}^d \times E_x$ the function $(z(n), \beta(x(n))) \in \mathbb{T}^d \times E_y$ is a particle-spin trajectory. A necessary and sufficient condition for β to have this property is that $\beta(l(r; \xi)) = l(r; \beta(\xi))$ for all $r \in SO(3), \xi \in E_x$ and this is true iff $r_0 \in SO(3)$ exists such that $\{r_0 r r_0^t : r \in H_x\} \subset H_y$. Here, the subgroup H_η of $SO(3)$ is defined by $H_\eta := \{r \in SO(3) : l(r; \eta) = \eta\}$ for every $\eta \in E$. The proof of this is constructive showing that β can be defined by $\beta(l(r; x)) := l(r r_0^t; y)$. Furthermore, it can be shown that if f is an invariant (E, l)-field of (j, A) which takes values only in E_x then $g \in C(\mathbb{T}^d, E)$, defined by $g(z) := \beta(f(z))$, is an invariant field taking values only in E_y.

The above can be generalized as follows, leading to new ways to treat the spin-1 case and others. So choose (E', l') in addition to (E, l) (possibly $(E', l') = (E, l)$). Let E, E' be Hausdorff and let $x \in E, x' \in E'$ such that $r_0 \in SO(3)$ exists which satisfies $\{r_0 r r_0^t : r \in H_x\} \subset H'_{x'}$ where $H'_{x'} := \{r \in SO(3) : l'(r; x') = x'\}$. This is equivalent to the condition that a $\beta \in C(E_x, E'_{x'})$ exists such that $\beta(l(r; x)) = l'(r; \beta(x))$. The proof of this is constructive showing that β can be defined by $\beta(l(r; x)) := l'(r r_0^t; x')$. Furthermore, if f is an invariant (E, l)-field of (j, A) which takes values only in E_x then the DT tells us that $g \in C(\mathbb{T}^d, E')$, defined by $g(z) := \beta(f(z))$, is an invariant (E', l')-field of (j, A) taking values only in $E'_{x'}$. If β is a homeomorphism, which entails that E_x and $E'_{x'}$ are homeomorphic, then g is invariant iff f is. In summary, the DT classifies invariant fields in terms of the functions β, i.e., in terms of the subgroups H_x of $SO(3)$.

The Invariant Reduction Theorem (IRT)

Let $f \in C(\mathbb{T}^d, E)$, $x \in E$, $\check{\Sigma}_x[f] := \{(z, r) \in (\mathbb{T}^d \times SO(3)) : l(r; x) = f(z)\}$ and $\check{\mathcal{P}}[j, A] \in \text{Homeo}(\mathbb{T}^d \times SO(3))$, with $\check{\mathcal{P}}[j, A](z, r) := (j(z), A(z)r)$. Then the IRT [2, 3] states that f satisfies (3) iff $\check{\Sigma}_x[f]$ is invariant under $\check{\mathcal{P}}[j, A]$ for every $x \in E$. For the definition of invariant reductions, see [4] and our comments on bundle theory below. The IRT renders the existence and uniqueness problems of invariant fields into a problem of Topological Dynamics invoking techniques from Ergodic Theory and Homotopy Theory. In particular the IRT gives a new view of the ISF.

The Cross Section Theorem (CST)

An important aspect of $\check{\Sigma}_x[f]$ is the CST which gives valuable information to be used when applying the IRT to (3). Let $f \in C(\mathbb{T}^d, E)$ take values only in E_x. Thus for every $z \in \mathbb{T}^d$, $f(z) \in E_x$ and so there exists a function $T : \mathbb{T}^d \to SO(3)$ such that $f(z) = l(T(z), x)$ but in general there may not be a continuous T. The CST asserts that a continuous T exists iff the continuous function $p_x[f] : \check{\Sigma}_x[f] \to \mathbb{T}^d$, defined by $p_x[f](z, r) := z$, has a cross section σ, i.e., iff a continuous $\sigma : \mathbb{T}^d \to \check{\Sigma}_x[f]$ exists such that $p_x[f](\sigma(z)) = z$. Most importantly, if $p_x[f]$ has a cross section then the CST gives a natural homeomorphism on $\mathbb{T}^d \times SO(3)$ which maps $\check{\Sigma}_x[f]$ onto $\mathbb{T}^d \times H_x$. The CST will be illustrated in the spin-1/2 case.

The Normal Form Theorem (NFT)

To address the case when a cross section exists we use the NFT, which is closely related to the CST. Let $T \in C(\mathbb{T}^d, SO(3))$, $x \in E$ and $f(z) := l(T(z), x)$. Then f is an invariant field iff $l(T^t(j(z))A(z)T(z); x) = x$ for all z. This motivates the definition of H_x above and we state the NFT as: $T^t(j(z))A(z)T(z) \in H_x$ for all $z \in \mathbb{T}^d$ iff $f(j(z)) = l(A(z); f(z))$ for all $z \in \mathbb{T}^d$. The moral of the NFT is that f and T are effectively the same, i.e., that one can view invariant fields from two different perspectives. The special case where (E, l) and x are such that

$$H_x = G_\nu := \{ \begin{pmatrix} \cos(2\pi n\nu) & -\sin(2\pi n\nu) & 0 \\ \sin(2\pi n\nu) & \cos(2\pi n\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} : n = 0, \pm 1, \pm 2, \dots \}, \tag{5}$$

is the case where spin tunes, ν, exist. The subcase where $\nu = 0$ describes spin-orbit resonances. The terminology NFT is justified as follows: if $T \in C(\mathbb{T}^d, SO(3))$ and if $A'(z) := T^t(j(z))A(z)T(z)$ belongs to a subgroup H of $SO(3)$ then one calls (j, A') an H-normal form of (j, A). The notion of normal form gives a new view on spin tunes and spin-orbit resonances.

SPECIAL CASES

Spin-1/2 particles and $(E, l) = (\mathbb{R}^3, l_{1/2})$

For spin-1/2 particles the most important (E, l) is given by $E = \mathbb{R}^3$ and where $l_{1/2}(r; S) := rS$. Clearly E is Hausdorff and the E_x are concentric spheres centered at $(0,0,0)$ and the field map (2) gives $f'(z) = A(j^{-1}(z))f(j^{-1}(z))$. The invariant $(\mathbb{R}^3, l_{1/2})$-fields f are just the invariant polarization fields describing the spin equilibrium of a bunch and for $|f| = 1$ they are the invariant spin fields. Coming to the NFT with $x = (0,0,1)$ and using

$$H_x := \{r \in SO(3) : l_{1/2}(r;x) = x\} = \{\begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} : \nu \in \mathbb{R}\} =: SO(2) \,,$$

we find that a $T \in C(\mathbb{T}^d, SO(3))$ satisfies $T^t(j(z))A(z)T(z) \in H_x$ for all $z \in \mathbb{T}^d$ iff the third column of T is an invariant spin field. Then T is called an invariant frame field as in [1]. Thus the notion of normal form gives a new view on the notion of invariant frame field and generalizes it from the group $SO(2)$ to an arbitrary subgroup of $SO(3)$. We emphasize that the notions of invariant frame field and invariant spin field are tied to the group $SO(2)$. If $x = (0,0,1)$ and $f \in C(\mathbb{T}^d, \mathbb{R}^3)$ with $|f| = 1$ one observes, by the CST, that $p_x[f]$ has a cross section iff a $T \in C(\mathbb{T}^d, SO(3))$ exists whose third column is f. Thus the CST gives a new view on invariant frame fields. Using simple arguments from Homotopy Theory one can also show [2] that, if $d \geq 2$, $p_x[f]$ does not always have a cross section. We now apply the DT for the case where $(E,l) = (E',l') = (\mathbb{R}^3, l_{1/2})$ and $x = (0,0,1), y = (0,0,y_3)$ and define $\beta \in C(E_x, E_y)$ by $\beta(l_{1/2}(r;x)) := l_{1/2}(r;y)$ where $r \in SO(3)$, i.e., $\beta(S) := y_3 S$ where $S \in \mathbb{R}^3$ with $|S| = 1$. Then if f is an invariant spin field of (j,A) we see that $g \in C(\mathbb{T}^d, \mathbb{R}^3)$, defined by $g(z) := \beta(f(z)) = y_3 f(z)$ is an invariant polarization field of (j,A).

Spin-1 *particles and* $(E,l) = (E_1, l_1)$

For spin-1 particles the most important (E,l) are $(\mathbb{R}^3, l_{1/2})$ and (E_1, l_1) where $E_1 := \{M \in \mathbb{R}^{3\times 3} : M^t = M, Tr(M) = 0\}$ and where the function $l_1 : SO(3) \times E_1 \to E_1$ is defined by $l_1(r;M) := rMr^t$. For brevity we only address the DT in this section and we do that for the case where $(E,l) = (\mathbb{R}^3, l_{1/2})$ and $(E',l') = (E_1, l_1)$. Clearly E' is Hausdorff and we pick $x = (0,0,1)$ and the diagonal matrix $x' = diag(y,y,-2y)$ where y is a real constant and we find $H_x = SO(2), H'_{x'} = SO(2) \bowtie \mathbb{Z}_2$ where

$$SO(2) \bowtie \mathbb{Z}_2 := \{rr' : r \in \mathbb{Z}_2, r' \in SO(2)\} \,, \qquad (6)$$

with $\mathbb{Z}_2 := \{I_{3\times 3}, diag(1,-1,-1)\}$ where $I_{3\times 3}$ is the 3×3 unit matrix. To apply the DT we define $\beta \in C(E_x, E'_{x'})$ by $\beta(l_{1/2}(r;x)) := l'_1(r;x')$ where $r \in SO(3)$, i.e., $\beta(S) := yI_{3\times 3} - 3ySS^t$ where $S \in \mathbb{R}^3$ with $|S| = 1$. Thus if f is an invariant spin field of (j,A) then $g \in C(\mathbb{T}^d, E_1)$, defined by

$$g(z) := \beta(f(z)) = yI_{3\times 3} - 3yf(z)f^t(z) \,, \qquad (7)$$

is an invariant (E_1, l_1)-field of (j,A). This confirms the observation [5] obtained by a different method.

UNDERLYING BUNDLE THEORY

While bundle aspects were not needed in the above outline of our results it is worthwhile to mention them since they are the origin of our formalism (see [2,4]) and therefore supply a steady flow of ideas, many of which not even mentioned here (e.g., algebraic hull, characteristic class, rigidity). More details can be found in [3,6]. The "unreduced" principal bundle underlying our formalism is a product principal bundle $(\mathbb{T}^d \times SO(3), p_d, \mathbb{T}^d, R_d)$ with bundle space $\mathbb{T}^d \times SO(3)$, base space $\mathbb{T}^d$, bundle projection $p_d(z,r) := z$, and structure group $SO(3)$. So $R_d : SO(3) \times \mathbb{T}^d \times SO(3) \to \mathbb{T}^d \times SO(3)$ is an $SO(3)$-action defined by $R_d(r;z',r') := (z', r'r^t)$. The reductions are just the principal subbundles of the unreduced bundle. So they are uniquely determined by their bundle space X which of course is a subspace of $\mathbb{T}^d \times SO(3)$. Perhaps surprisingly, if one restricts oneself to reductions whose structure group is closed and for which X is closed then the well known Reduction Theorem (RT) tells us [2,7] that the reductions are exactly those subbundles of the unreduced principal bundle for which $E' = \check{\Sigma}_x[f]$ where $f \in C(\mathbb{T}^d, E)$ takes only values in E_x and where E is Hausdorff (the structure group of the reduction is H_x). Thus, as its name suggests, the IRT identifies the dynamical invariance of the reductions with the invariance of the labeling fields f. With regard to the bundle aspect, the crucial point of the CST is that the existence of a cross section of the reduction is equivalent to the triviality of the reduction, meaning that the reduction is isomorphic to a product principal bundle. Note also that the $p_x[f]$ are the projections of the reductions. Every (E,l) in the formalism corresponds to a so-called associated bundle (relative to the unreduced bundle). The bundle aspects show the close relationship between our formalism and the use of bundles in Yang-Mills Theory where each (E,l) corresponds to a certain set of fundamental matter fields (e.g., one (E,l) corresponds to quarks another one to leptons etc.).

REFERENCES

[1] D.P. Barber, J.A. Ellison and K.Heinemann, Phys. Rev. ST Accel. Beams **7** (2004) 124002.

[2] K. Heinemann, Doctoral Thesis, University of New Mexico, May 2010.

[3] K. Heinemann, D.P. Barber, J.A. Ellison and M.Vogt. To be submitted.

[4] R. Feres, *Dynamical systems and semisimple groups: an introduction*, Cambridge University Press, Cambridge, 1998. *Handbook of dynamical systems Vol. 1A*. Edited by B. Hasselblatt and A. Katok, North-Holland, Amsterdam, 2002.

[5] D.P. Barber, M. Vogt, Proc.18th Int. Spin Physics Symposium, Charlotteville (Virginia), USA, October 2008, AIP proceedings 1149, (2008).

[6] K. Heinemann, D.P. Barber, J.A. Ellison and M.Vogt. Under preparation for arXiv.

[7] D. Husemoller, *Fibre Bundles*, Third edition, Springer, New York, 1994.