Proton and electron polarisation in storage rings: some basic concepts [1]

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Abstract. An introduction to some basic concepts central to a modern understanding of spin motion in storage rings is given.

INTRODUCTION

A proper understanding of spin–orbit resonance structure at high energy in storage rings can only be obtained with a correct definition of the "spin tune". This in turn requires establishing a proper coordinate system for "measuring" spin precession and that, in turn, requires the notion of the "invariant spin field". This paper shows how to embark on that approach. More comprehensive treatments can be found in [1–6].

THE INVARIANT SPIN FIELD

Particle dynamics in storage rings is described in terms of three pairs of canonical coordinates $\vec{u} = (q_1, p_1, q_2, p_2, q_3, p_3)$ which could, for example, be $(x, p_x, y, p_y, \Delta t, \Delta E)$ where $x,\ p_x,\ y,\ p_y$ describe transverse motion with respect to the curved periodic orbit and $\Delta t, \Delta E$ are the time delay relative to a synchronous particle and the deviation from the "design" energy. The independent variable is the distance along the ring s, ("the azimuth"). There is a corresponding classical Hamiltonian $h_{\mathrm{orb}}(\vec{u}; s)$. In distorted rings $\vec{u}$ describes motion w.r.t. the resulting closed orbit.

We now make the idealisation that the beam can be described in terms of a smooth continuous density, $w(\vec{u}; s)$, which is a scalar function of $\vec{u}$ and the azimuth s. It is normalised to unity. In the absence of dissipation and noise (e.g. due to synchrotron radiation) and ignoring the effect of the tiny Stern–Gerlach forces

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on the orbital motion, w is constant along a phase space trajectory and obeys a relation of the Liouville type: $\frac{\partial w}{\partial s} + \sum_{k=1}^{3} \frac{dq_k}{ds}\frac{\partial w}{\partial q_k} + \frac{dp_k}{ds}\frac{\partial w}{\partial p_k} = 0$ which we write in terms of a Poisson bracket as $\partial w/\partial s = \{h_{\rm orb}, w\}$. If the beam is stable, i.e. if w is the same from turn to turn, then it is 1–turn periodic in s and we write it as $w_{\rm st}$ so that $w_{\rm st}(\vec{u}; s + C) = w_{\rm st}(\vec{u}; s)$, where C is the ring circumference: $w_{\rm st}(\vec{u}; s)$ is a 1–turn periodic *scalar field* on $(\vec{u}, s)$.

In the absence of spin flip, spin motion for electrons and protons moving in electric and magnetic fields is described by the T–BMT equation [1] $d\vec{S}/ds = \vec{\Omega} \times \vec{S}$ where $\vec{S}$ is the rest frame spin expectation value of the particle ("the spin") and $\vec{\Omega}$ depends on the electric and magnetic fields, the velocity and the energy so that it depends on $\vec{u}$ and s. Having assigned a phase space density to each point in phase space we now assign a polarisation $\vec{P}(\vec{u}; s)$ to each point. $\vec{P}$ is the average over particles in an infinitisimal packet of phase space at $\vec{u}$ of the normalised spin expectation values $2\vec{S}/\hbar$. Since the T–BMT equation is linear in the spin and since the spins at $(\vec{u}, s)$ all see the same $\vec{\Omega}(\vec{u}; s)$, $\vec{P}(\vec{u}; s)$ obeys the T–BMT equation $d\vec{P}/ds = \vec{\Omega}(\vec{u}(s); s) \times \vec{P}$ also. This can be rewritten as $\partial\vec{P}/\partial s = \{h_{\rm orb}, \vec{P}\} + \vec{\Omega}(\vec{u}; s) \times \vec{P}$ in analogy with the Liouville equation for $w(\vec{u}; s)$. It is assumed that $\vec{P}(\vec{u}; s)$ is differentiable in all directions in phase space. The polarisation of the whole beam at azimuth s is $\vec{P}_{\rm av}(s) = \int d^6u \; w(\vec{u}; s)\vec{P}(\vec{u}; s)$. If the spin distribution is stable, i.e. if $\vec{P}(\vec{u}; s)$ is the same from turn to turn, then $\vec{P}(\vec{u}; s)$ not only obeys the T–BMT equation, but it is also 1–turn periodic in s and we write it as $\vec{P}_{\rm st}$ so that $\vec{P}_{\rm st}(\vec{u}; s+C) = \vec{P}_{\rm st}(\vec{u}; s)$. We denote the unit vector along $\vec{P}_{\rm st}(\vec{u}; s)$ by $\hat{n}(\vec{u}; s)$. Thus $\hat{n}(\vec{u}; s)$ is a 1–turn periodic *vector field* on $(\vec{u}, s)$ obeying the T–BMT equation. Since the field is invariant from turn to turn we call $\hat{n}(\vec{u}; s)$ the *invariant spin field* [2]. It can be visualised as a field of unit vectors in real space attached to every $\vec{u}$ at every s such that each point $(\vec{u}, s)$ has its own unique $\hat{n}$ with the property that along particle orbits it obeys the T–BMT equation. For one turn $\hat{n}(\vec{M}(\vec{u}; s); s + C) = \hat{n}(\vec{M}(\vec{u}; s); s) = R_{3\times3}(\vec{u}; s)\hat{n}(\vec{u}; s)$ where $\vec{M}(\vec{u}; s)$ is the new phase space vector after one turn starting at $\vec{u}$ and $R_{3\times3}(\vec{u}; s)$ is the corresponding spin transfer matrix. On the closed orbit $\hat{n}(\vec{u}; s)$ becomes $\hat{n}(\vec{0}; s)$ which we denote by $\hat{n}_0(s)$. Many authors make no clear distinction between $\hat{n}$ and $\hat{n}_0$ and many use the symbol $\hat{n}$ for $\hat{n}_0$. This causes confusion. Obviously $\hat{n}_0(s + C) = \hat{n}_0(s)$, i.e. $\hat{n}_0(s)$ is the 1–turn periodic solution of the T–BMT equation on the closed orbit. It is given by the real eigenvector of the 1–turn 3×3 spin transport matrix on the closed orbit.

Examples of the invariant spin field at 800 GeV for a HERA proton optic with a suitable arrangement of Siberian Snakes are shown in figure 1. In these particular simulations the protons only execute stable linear vertical betatron motion of fixed amplitude. The particle coordinates are not 1–turn periodic but at fixed azimuth they lie on a closed elliptical curve at positions depending on their vertical betatron phases. Likewise a spin at some $\vec{u}$ set parallel to $\hat{n}$ and tracked, is not 1–turn periodic but on tracking it turn to turn, it lies on the closed curve, parametrised

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[2] Likewise we could call $w_{\rm st}(\vec{u}; s)$ the "invariant density field".

by the orbital phase, of the field $\hat{n}$. Each picture shows the locus, on the surface of a sphere, of the tip of the $\hat{n}$ vector as the betatron phase varies at a point on the ring where $\hat{n}_0$ is vertical. The parameters are shown in the captions. An invariant emittance of 4π mm mrad corresponds to "1-σ". Both curves of figure 1 are invariant when tracked from turn to turn. Clearly, as the amplitude is increased, the invariant spin field becomes complicated. Near the spin–orbit resonances to be discussed below, the curves become very convoluted. For motion in one orbital plane, the loci on the sphere are closed as, for example, in figure 1. For motion in all three planes, the loci do not close in general although the field $\hat{n}$ is still an invariant of the 1–turn spin–orbit map. If the spins for an ensemble of particles distributed uniformly around the phase space ellipses for figure 1, are all set initially parallel to $\hat{n}_0$ and then tracked, the beam polarisation at that azimuth oscillates. If they are set parallel to $\hat{n}$, the beam polarisation at that azimuth is stationary. In general the maximum *stationary* beam polarisation that can be reached is $P_{\rm lim}(s) = ||\int d^6u \; w_{\rm st}(\vec{u};s)\hat{n}(\vec{u};s)||$. This can be calculated before carrying out simulations of particle acceleration and can give an impression of whether such a simulation would be worthwhile. For motion on a vertical betatron ellipse $P_{\rm lim}$ is just given by the average of $\hat{n}$ over the betatron phase [3]. On the 64π mm mrad ellipse $P_{\rm lim}$ is much smaller than for the 4π mm mrad ellipse — it pays to devise ways to keep the spread of $\hat{n}$ small. Note that for $\vec{u} \neq \vec{0}$, the constraint $\hat{n}(\vec{u};s+C) = \hat{n}(\vec{u};s)$

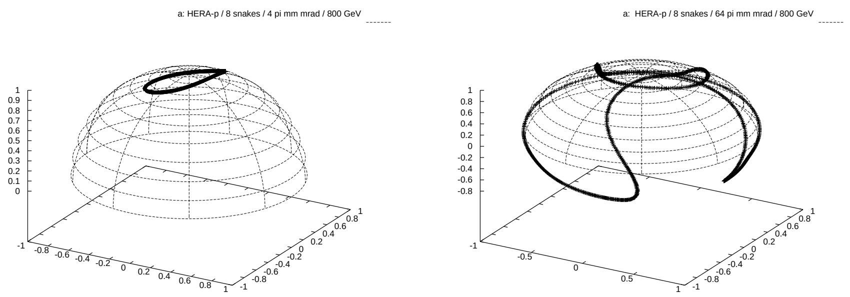

FIGURE 1. The field $\hat{n}$ in HERA–p calculated with `SPRINT` *on* the 4π mm mrad (left) and the 64π mm mrad (right) ellipses at 800 GeV.

for the invariant spin field is obviously *not* equivalent to the closure condition $\vec{N}(\vec{u};s) = R_{3\times3}(\vec{u};s)\vec{N}(\vec{u};s)$ since in general a spin at $\vec{u}$ set parallel to $\hat{n}(\vec{u};s)$ is not a "closed spin solution" but has a new direction after one turn. In general $\vec{N}$ does not obey the T–BMT equation everywhere along an orbit since the orbital motion is not 1–turn periodic. Furthermore, in contrast to the loci in figure 1, the loci of $\vec{N}(\vec{u};s)$ are *not* invariant beyond the first turn. In fact, the calculation of the real $\hat{n}(\vec{u};s)$ is computationally nontrivial and requires "stroboscopic averaging" using the code `SPRINT` [2], or Fourier analysis as in `SODOM-II` [7].

Although we have concentrated on protons and have introduced $\hat{n}$ via invariant spin distributions it was first motivated in another way, namely by Derbenev and Kondratenko [8,9], as a $\vec{u}$ and s dependent semiclassical spin quantisation axis for calculating radiative spin flip for electrons. That leads to the Derbenev–Kondratenko–Mane formula for the equilibrium electron polarisation in a storage ring [9]. That formula needs the vector $\partial\hat{n}/\partial\delta$ (where δ is the fractional electron energy deviation) at each $\vec{u}$ at azimuths s inside dipole magnets. So to calculate the attainable electron polarisation the correct definition of $\hat{n}$ is required. In fact $\hat{n}$ is not only central to the understanding of equilibrium spin distributions for noiseless and dissipationless motion, but it is also an essential starting point for perturbative calculations of various depolarisation effects [10].

THE AMPLITUDE DEPENDENT SPIN TUNE

Another key quantity in spin dynamics is the "spin tune". On the closed orbit this is defined as the number of precessions per turn of an arbitrary spin around $\hat{n}_0(s)$. We denote it by ν_0. For particles executing synchrobetatron motion the definition of spin tune is more subtle. Once the invariant spin field $\hat{n}(\vec{u};s)$ has been set up, two other unit vectors $\hat{n}_1(\vec{u};s)$ and $\hat{n}_2(\vec{u};s)$ are attached to all $(\vec{u},s)$ such that the sets $(\hat{n}_1,\hat{n}_2,\hat{n})$ form local orthonormal coordinate systems at all points in phase space at each s. Like $\hat{n}$, $\hat{n}_1$ and $\hat{n}_2$ are 1–turn periodic in s: $\hat{n}_i(\vec{u};s+C)=\hat{n}_i(\vec{u};s)$ for $i\in\{1,2\}$. But unlike $\hat{n}$ they do not obey the T–BMT equation. If a spin $\vec{S}$ is followed along an orbit, the scalar product $\vec{S}\cdot\hat{n}$ of $\vec{S}$ and the local pre-established $\hat{n}$ is invariant since both vectors obey the T–BMT precession equation. Thus in the local pre-established $(\hat{n}_1,\hat{n}_2,\hat{n})$ coordinate system the motion of $\vec{S}$ is a precession around $\hat{n}$. Except for the uninteresting case of running on orbital resonance, the fields $\hat{n}_1(\vec{u};s)$ and $\hat{n}_2(\vec{u};s)$ can be chosen so that the rate of precession is constant and independent of the starting orbital phases [1–6]. The spin tune ν is the number of precessions per turn "measured" in this way. The spin tune depends only on the orbital amplitudes — a tune depending in some way on phases would hardly be a useful quantity since it would have to change as the phases advance! On the closed orbit ν reduces to ν_0 as required.

Spin motion is particularly strongly perturbed when the spin precession rate is near resonance with the orbital tunes: $\nu(J_1,J_2,J_3) = k_0+k_1Q_1+k_2Q_2+k_3Q_3$ where the Q's are the tunes of the orbital modes, the k's are integers and the J's are orbital amplitudes. Note that contrary to common practice we do not use ν_0 here. Indeed, that is the whole point of having a proper definition of spin tune as we now illustrate. Figure 2 (left) shows the dependence of the spin tune on orbital amplitude (= enclosed invariant emittance) for purely vertical betatron motion in HERA–p at 805 GeV with a suitable arrangement of snakes [3]. On the design orbit, i.e. at zero amplitude, ν is 1/2 as expected. But it deviates from 1/2 as the amplitude increases and at 27π mm mrad it jumps symmetrically across the resonant value $2Q_y$. After increasing further, ν then decreases and at an invariant emittance of 56π mm mrad it jumps back across the resonant value $2Q_y$. So ν never

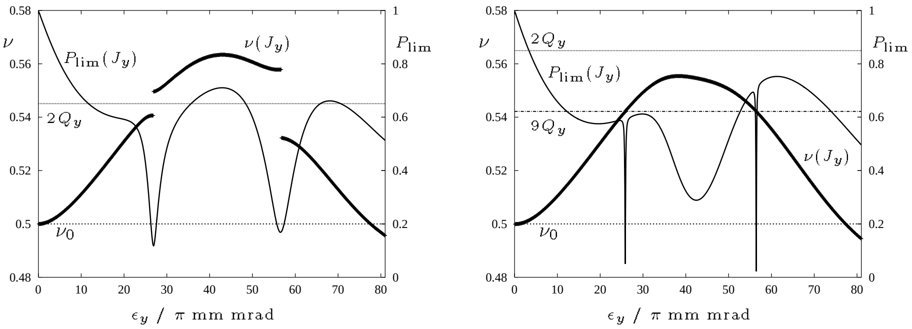

FIGURE 2. The amplitude dependent spin tune ν and $P_{\lim}$ *on* phase space ellipses with invariant vertical emittance ϵ_y as calculated with `SPRINT` for HERA–p at 805 GeV. Left: vertical tune $Q_y = 32.2725$, right: $Q_y = 32.2825$.

actually hits the resonant value but as one can see $P_{\lim}$ becomes small around the resonant amplitudes owing to the expected opening out of $\hat{n}$. Thus the behaviour of ν and $P_{\lim}$ are mutually consistent. Figure 2 (right) shows the behaviour of ν when Q_y is increased. The second order resonance can no longer be crossed but 9th order resonant behaviour occurs instead. These curves illustrate just how complicated spin motion can be at very high energy. Such phenomena could obviously not be seen without a properly defined spin tune. For example, a "fake spin tune" erroneously extracted from the complex eigenvalues of $R_{3\times 3}$ shows no correlation with dips in $P_{\lim}$. That is no surprise since that "tune" depends on the orbital phase and is therefore unsuitable for describing long term spin–orbit coherence. With the properly defined ν, the changes in orbital tunes needed to avoid resonances can be properly estimated. Moreover, the size of a resonant jump in ν, $\Delta\nu$, for a high order resonance, is a measure of the strength of the resonance and it has been possible to parametrise polarisation loss, when varying machine parameters *dynamically* through such resonances, in terms of a generalised Froissart–Stora formula [5,6,11], containing $\Delta\nu$.

SUMMARY AND CONCLUSION

The invariant spin field and the properly defined spin tune are indispensible for a clear understanding of spin–orbit resonant behaviour in storage rings. Their use allows high order resonances to be cleanly identified and their strengths to be determined. Moreover, misconceptions about depolarisation mechanisms based on a false understanding of the concept of spin tune can be swept away.

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