A FOKKER–PLANCK DESCRIPTION OF SPIN DIFFUSION

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We show how to construct an equation of the Fokker–Planck type for the spin motion of electrons in a storage ring.

1 Introduction and motivation

Relativistic electrons circulating in a storage ring can become spin polarized by the emission of spin–flip synchrotron radiation. This is the so–called Sokolov–Ternov (ST) effect [1].

In the absence of spin flip, spin motion for electrons moving in electric and magnetic fields is described by the T–BMT equation [2, 3] $d\vec{S}/ds = \vec{\Omega} \wedge \vec{S}$ where $\vec{S}$ is the rest frame spin expectation value of the electron and s is the distance around the ring. $\vec{\Omega}$ depends on the electric and magnetic fields, the velocity and the energy. Thus $\vec{\Omega}$ is a function of s and the vector $\vec{u}$ of the six canonical phase space coordinates [4, 5].

The stochastic element of photon emission together with the accompanying damping determines the equilibrium phase space density distribution and the beam can be described by a Fokker–Planck (FP) equation. This is traditionally derived by simulating the stochastic photon emission with Gaussian white noise [6, 7, 8, 9]. The same photon emission also imparts a stochastic element to $\vec{\Omega}(\vec{u}; s)$ via its dependence on $\vec{u}$ and then, through the T-BMT equation, spin diffusion (and thus depolarization) can occur in the inhomogeneous fields of the ring [10, 4, 5]. Thus synchrotron radiation can create polarization but can also lead to its destruction! In practice the polarization comes to equilibrium at a value given by the formula of Derbenev and Kondratenko and of Mane [11, 12]. Further details on this formalism can be found in [4, 5].

However, if we are just interested in spin diffusion it would be useful to have a kind of spin–orbit FP equation which would allow non–equilibrium spin–orbit systems to be studied.

We now show how to obtain such an equation following the traditional route based on Gaussian white noise in analogy with the description of orbital motion. From now on we will treat $\vec{S}$ as a classical spin vector. This account will be very brief but more details can be found in [13, 4].

2 Spin–orbit transport with radiation

We model the photon emission as a Gaussian white noise process overlaid onto smooth radiation damping. Then the orbital phase space density $W_{\rm orb}$ evolves according to a FP equation:

$$\frac{\partial W_{\rm orb}}{\partial s} = \mathcal{L}_{\rm FP,orb}\ W_{\rm orb}\ , \tag{1}$$

where the orbital FP operator $\mathcal{L}_{\rm FP,orb}$ accounts for "Hamiltonian" flow, damping and noise and contains zeroth, first and second derivatives w.r.t. the components of $\vec{u}$. The detailed form for $\mathcal{L}_{\rm FP,orb}$ can be found in [6, 7, 8] but is not important for the argument that follows. After a few damping times $W_{\rm orb}$ approaches an equilibrium form.

The analogue of Eq. (1) for spin will be an equation for the *polarization density* $\vec{\mathcal{P}} = 2/\hbar\ \vec{\mathcal{S}}$ where $\vec{\mathcal{S}}$ is the *spin angular momentum density per particle.* In terms of the local polarization, $\vec{P}_{\rm loc}(\vec{u};s)$, at $(\vec{u};s)$ we have $\vec{\mathcal{P}}(\vec{u};s) = \vec{P}_{\rm loc}(\vec{u};s)\ W_{\rm orb}(\vec{u};s)$.

We now introduce the joint spin–orbit density $W(\vec{u},\vec{S};s)$. This contains a factor $\delta(\hbar/2 - |\vec{S}|)$ accounting for the fact that we are describing processes for which $|\vec{S}| = \hbar/2$. We normalise W to unity: $\int d^6u\ d^3S\ W(\vec{u},\vec{S};s) = 1$. Moreover $\int d^3S\ W(\vec{u},\vec{S};s) = W_{\rm orb}(\vec{u};s)$. The polarization density can now be written as

$$\vec{\mathcal{P}}(\vec{u};s) = \frac{2}{\hbar}\int d^3S\ \vec{S}\ W(\vec{u},\vec{S};s)\ . \tag{2}$$

The polarization of the whole beam as measured by a polarimeter at azimuth s is $\int d^6u\ \vec{\mathcal{P}}(\vec{u};s)$.

Since here, spin is a spectator, being only indirectly affected by the radiation through the orbital motion, the FP equation for the combined orbit and spin density is

$$\frac{\partial W}{\partial s} = \mathcal{L}_{\rm FP,orb}W - (\vec{\Omega}\wedge\vec{S})\cdot(\vec{\nabla}_{\vec{S}}W) \tag{3}$$

where $\vec{\nabla}_{\vec{S}}W$ is the gradient of W w.r.t. the three components of spin.

Using Eq. (3) we can write

$$\frac{2}{\hbar}\int d^3S\ \vec{S}\ \frac{\partial W}{\partial s} = \frac{2}{\hbar}\int d^3S\ \vec{S}\Big(\mathcal{L}_{\rm FP,orb}W - (\vec{\Omega}\wedge\vec{S})\cdot(\vec{\nabla}_{\vec{S}}W)\Big) \tag{4}$$

and then by Eq. (2) we obtain

$$\frac{\partial \vec{\mathcal{P}}}{\partial s} = \mathcal{L}_{\rm FP,orb} \vec{\mathcal{P}} + \vec{\Omega} \wedge \vec{\mathcal{P}} \; . \quad (5)$$

This is the equation of the Fokker–Planck type for spin that we have been seeking. We call Eq. (5) a "Bloch" equation following the usage for equations of this general form in the nuclear magnetic resonance literature. Concrete examples of this equation for simple exactly solvable models can be found in [14].

3 Discussion and conclusion

The derivation of the Bloch equation for $\vec{\mathcal{P}}$ given here is independent of the source of noise and damping and as soon as we have the $\mathcal{L}_{\rm FP,orb}$ for a process we can write down the corresponding Bloch equation for $\vec{\mathcal{P}}$. The Bloch equation is valid far from spin–orbit equilibrium and it is linear in $\vec{\mathcal{P}}$. It is also universal in that it does not explicitly contain the orbital density $W_{\rm orb}$.

The corresponding evolution equation for $\vec{P}_{\rm loc}$ can be found by putting the relation $\vec{\mathcal{P}} = \vec{P}_{\rm loc}\, W_{\rm orb}$ into Eq. (5) and using Eq. (1) but it is complicated and it is *not* universal since it contains $W_{\rm orb}$. So to extract $\vec{P}_{\rm loc}$ one should first solve Eqs. (1) and (5) separately and then use the relation $\vec{P}_{\rm loc} = \vec{\mathcal{P}}/W_{\rm orb}$.

The first FP–like treatment of spin motion in storage rings was given in [15]. This was a semiclassical calculation of the effect of synchrotron radiation on the evolution of the spin–orbit density operator, $\rho = \frac{1}{2}(\rho_{\rm orb} + \vec{\sigma} \cdot \vec{\xi})$, where $\vec{\sigma}$ is the spin operator, $\rho_{\rm orb}$ is the density operator of the orbital motion and where the operator $\vec{\xi}$, which encodes information about the polarization, is equivalent to $\vec{\mathcal{P}}$. The resulting evolution equation for the Weyl transform of $\vec{\xi}$ contains terms equivalent to those on the r.h.s. of Eq. (5), which are due to pure spin diffusion, together with terms due to the ST effect. There are also the "cross terms". So starting with Eq. (5) one could, using physical intuition, add in the ST terms by hand. But the cross terms would be missed. So to obtain a complete description of spin motion a full quantum mechanical, or at least semiclassical, treatment of combined spin and orbital motion is unavoidable. Our work is a classical reconstruction of the pure noise and damping part of Eq. (2) in [15] and it helps to illuminate the meaning of the latter. Since the evolution equation for the orbital phase space density in [15] is the usual FP equation, one sees that the calculation in [15] provides a physical justification for using Gaussian white noise models for orbital motion.

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