Uniqueness

$$R_{\lambda\mu\rho\sigma} = -k\left(g_{\mu\sigma}g_{\lambda\rho} - g_{\mu\rho}g_{\lambda\sigma}\right)$$

$$R'_{\dots\dots} = -k\left(g'_{\dots}g'_{\dots} - g'_{\dots}g'_{\dots}\right)$$

'Sylvester's' law of inertia | g and g' have the same # of positive & negative eigenvalues

$$\rightsquigarrow \quad g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}\, g_{\rho\sigma}(x)$$

$\rightsquigarrow$ transformation to <u>locally</u> flat metric (at x^*):

$$g(x) = D^T(x,x^*)\,\eta\, D(x,x^*)$$

globally: plane or torus ...

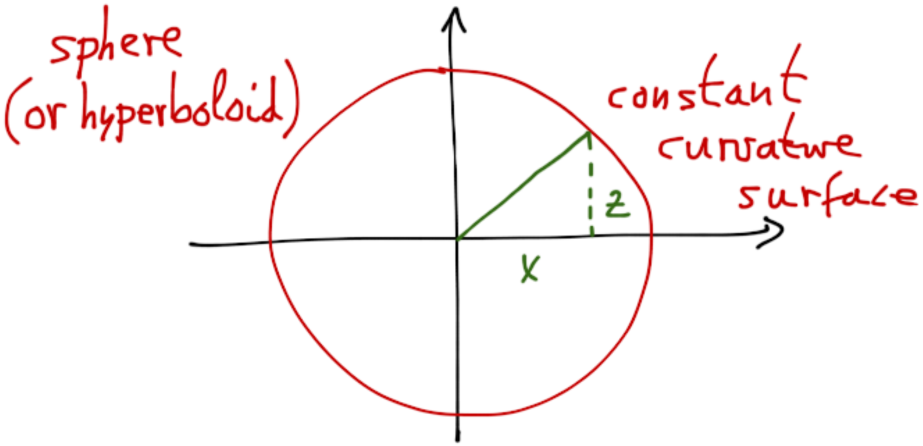
$$\eta = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \qquad \begin{pmatrix} -1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \qquad \begin{pmatrix} -1 &&& \\ & -1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$$

$k>0$ → $k=0$ → $k<0$

'de Sitter' 'Minkowski' 'anti-de Sitter'

Construction

MSS is unique; convenient construction of N-dim metric by embedding into flat $(N+1)$-dim space:



$$ds^2 = g_{AB}\, dx^A dx^B = C_{\mu\nu}\, dx^\mu dx^\nu + \frac{1}{k}\, dz^2$$

$C_{\mu\nu}$: constant $N \times N$ - matrix

$$C_{\mu\nu}\, x^\mu x^\nu + \frac{1}{k} z^2 = \frac{1}{k}$$

on the surface :

$$dz = -\frac{k}{z} \cdot C_{\mu\nu}\, x^\mu dx^\nu$$

$\curvearrowright$

$$ds^2 = C_{\mu\nu}\, dx^\mu dx^\nu + \frac{k}{z^2} \left(C_{\mu\nu}\, x^\mu dx^\nu \right)$$

$$\equiv g_{\mu\nu}\, dx^\mu dx^\nu$$

$\curvearrowright$

$$\boxed{g_{\mu\nu} = c_{\mu\nu} + \frac{k}{1 - k \cdot c_{\alpha\beta} x^\alpha x^\beta} c_{\mu\lambda} x^\lambda c_{\nu\sigma} x^\sigma}$$

$$c_{\mu\nu} = \left.\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \ldots \right\} N$$

explicit check:

$$R_{\mu\nu\rho\sigma} = -k \left(g_{\nu\rho} g_{\mu\sigma} - g_{\nu\sigma} g_{\mu\rho} \right)$$

O.K.

Einstein equations with cosmological constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda \cdot g_{\mu\nu} = 8\pi G \cdot T_{\mu\nu}$$

'vacuum' solution:

$$T_{\mu\nu} = 0$$

$$R = 4\Lambda = 12 \cdot k \quad (MSS)$$

MSS is vacuum solution to Einstein's equations for a cosmological constant: $\boxed{\Lambda = 3k}$

$k > 0$ 'de Sitter'

$k < 0$ 'anti-deSitter'

<u>Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:</u>

expanding universe:

$$\mathcal{M}_{3+1} = \mathbb{R} \times \Sigma$$

Σ : 3-dim MSS, Euclidean signature

$$ds^2 = dt^2 - a^2(t) \cdot d\sigma^2 , \; d\sigma^2 = \gamma_{ij}(u)\, du^i du^j$$

$$i,j = 1 \ldots 3$$

$${}^{(3)}R_{ijkl} = k \cdot (g_{ik} g_{jl} - g_{il} g_{jk})$$

$${}^{(3)}R = 6k$$

'construction' : $c_{ij} = \delta_{ij}$

$$d\sigma^2 = d\vec{u}^2 + \frac{1}{1 - k \cdot \vec{u}^2} (\vec{u} \cdot d\vec{u})^2$$

$u^1 = r \cdot \sin\theta \cdot \cos\varphi$, $u^2 = r \cdot \sin\theta \cdot \sin\varphi$

$u^3 = r \cdot \cos\theta$

$$\Rightarrow \boxed{d\sigma^2 = \frac{dr^2}{1-k\cdot r^2} + r^2\cdot d\Omega_2^2 \qquad d\Omega_2^2 = d\theta^2 + \sin^2\theta\cdot d\varphi^2}$$

$k = +1, 0, -1$: closed, flat, open universe

for <u>compact</u> space:

$\chi = \arcsin r$, $0 \le \chi \le \pi$

(coord. degeneracy)

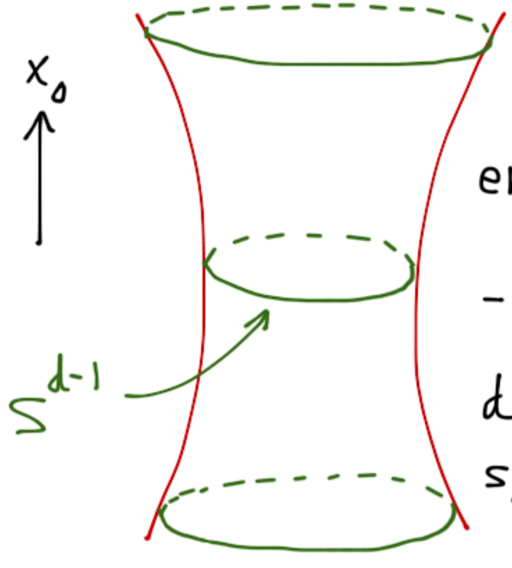
$$d\sigma^2 = d\chi^2 + \sin^2\chi\cdot d\Omega_2^2 = d\Omega_3^2$$

II.3 de Sitter space

Carroll: Space-time & Geometry (2004)

Mukhanov: Cosmology (2005)

Strominger et al.: hep-th/0110007



$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

embedding in $\mathcal{M}^{1,d}$:

$$-x_0^2 + x_1^2 + \ldots x_d^2 = \ell^2 = \frac{1}{k}$$

defines d-dim de Sitter space:

$$\mathcal{M} = dS_d$$

angular coordinates of the sphere S^{d-1}:

$$\omega_1 = \cos\theta_1$$

$$\omega_2 = \sin\theta_1 \cos\theta_2$$

$$\vdots$$

$$\omega_{d-1} = \sin\theta_1 \cdot \ldots \cdot \sin\theta_{d-2} \cdot \cos\theta_{d-1}$$

$$\omega_d = \sin\theta_1 \cdot \ldots \cdot \sin\theta_{d-2} \cdot \sin\theta_{d-1}$$

$$0 \le \theta_i \le \pi\ , \quad i = 0 \ldots d-2$$

$$0 \le \theta_{d-1} \le 2\pi\ , \quad \sum_i \omega_i^2 = 1$$

$$d\Omega_{d-1}^2 = \sum_i d\omega_i^2 = d\theta_1^2 + \ldots + \sin^2\theta_1 \cdot \ldots \cdot \sin^2\theta_{d-2} \cdot d\theta_{d-1}^2$$

<u>Global coordinates (τ, θ_i):</u>

$$x_0 = \ell \cdot \sinh\left(\frac{\tau}{\ell}\right), \quad x_i = \omega_i \cdot \ell \cdot \cosh\left(\frac{\tau}{\ell}\right)$$

$$i = 1 \ldots d$$

$$-\infty < \tau < \infty$$

from flat metric in $\mathcal{M}^{1,d}$:

$$ds^2 = -dx_0^2 + \sum_i dx_i^2$$

$$= -\cosh^2\left(\frac{\tau}{\ell}\right) \cdot d\tau^2 + \sinh^2\left(\frac{\tau}{\ell}\right) \cdot d\tau^2 \cdot \sum_i \omega_i^2 + \ell^2 \cdot \cosh^2\left(\frac{\tau}{\ell}\right) \cdot \sum_i d\omega_i^2$$

$$= -d\tau^2 + \ell^2 \cdot \cosh^2\left(\frac{\tau}{\ell}\right) \cdot d\Omega_{d-1}^2$$

geometry: S^{d-1}, infinite size at $\tau = -\infty$

grows to infinite size at $\tau = \infty$