DESY Summer Student Report

Quantisation of the SU(2)-WZNW-Model Using Its Symmetries

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Abstract

As a summer student in the DESY Theory Group, I studied the SU(2)-WZNW-model and the method of quantising it. This report summarises what I learned regarding this. The classical SU(2)-WZNW-model is formulated in the context of string theory, and the concept of conformal symmetry is introduced as a requirement for any such theory. By performing canonical quantisation, the current algebra of the model is found to be composed of two commuting Kac-Moody algebras. From these, we use the Sugawara construction to show that the model must possess quantum conformal symmetry and finally, the Hilbert space of the model is determined.

Contents

1 Introduction

My project this summer has been to learn how, using symmetry principles, the SU(2)-WZNW-model is quantised. The model, whose conformal invariance was first shown by Witten in 1984[1], is a simple example of a *conformal field theory* in a curved background, and so is relevant to the study of string theory.

This report summarises the classical theory (sections 2-3) and details the quantisation procedure (sections 4-5), including explicit calculations showing the symmetry properties of the quantised theory and the structure of the resulting Hilbert space.

2 Classical Conformal Field Theory

Before introducing the model itself, it is helpful to first understand the motivation for studying it.

2.1 String Theory

A topic of huge current interest in theoretical particle physics is *string theory*, which is based upon the assumption that the fundamental constituents of matter are not point-like but are one-dimensional objects (strings). Consider a *closed* string propagating in a background M which is parameterised by co-ordinates X^{μ}, as shown below in Figure 1.

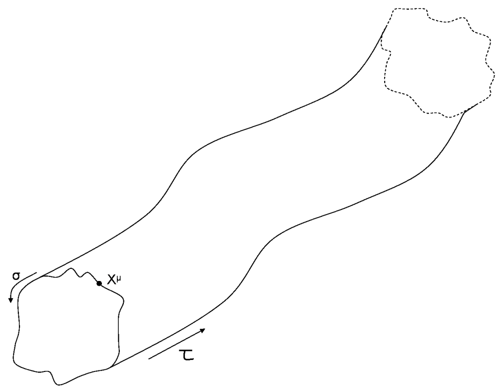

Figure 1: The string world-sheet.

This string will sweep out a two-dimensional *world-sheet* Σ in space-time which we can parameterise using the spatial and temporal co-ordinates (σ, τ). As in particle theory, the propagation of a string is determined by extremising its *action*, yielding equations of motion for $X^{\mu}(\sigma, \tau)$ - the map between parameter space and space-time. For a closed string, we require these functions to be periodic in σ:

$$X^{\mu}(\sigma + 2\pi, \tau) = X^{\mu}(\sigma, \tau) . \tag{2.1}$$

The action of a point particle is proportional to the invariant length of its world-line and so the action of the string is taken to be simply proportional to the invariant area of its world-sheet. A convenient form of the string action is

$$S[X] = \frac{1}{4\pi l_s^2} \int_{\Sigma} d^2x \left(G_{\mu\nu}(X) + B_{\mu\nu}(X)\right) \partial_+ X^{\mu} \partial_- X^{\nu}, \tag{2.2}$$

where we have reparameterised the world-sheet in terms of *light-cone co-ordinates*

$$x_{\pm} = \tau \pm \sigma, \tag{2.3}$$

$$\partial_{\pm} = \frac{1}{2}\left(\partial_{\tau} \pm \partial_{\sigma}\right) . \tag{2.4}$$

$G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ are the symmetric and anti-symmetric metric tensors of M, and l_s is the *string length*.

The choice of co-ordinates (x_+, x_-) is made here for convenience but it is crucial to note that a physically sensible theory must give the same results using *any* world-sheet parameterisation we choose. This property, which is required also of any proposed two-dimensional *quantum* field theory such as string theory, is called *conformal symmetry* and has very important consequences. For example, conformal invariance of the quantised string action constrains the possible forms of $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ - leading to a generalisation of the Einstein equation.

2.2 Conformal Symmetry

Classical conformal symmetry is defined as invariance under reparameterisations of the light-cone co-ordinates

$$x_+ \to f(x_+) \equiv x'_+, \tag{2.5}$$

$$x_- \to f(x_-) \equiv x'_-. \tag{2.6}$$

Recalling that σ are periodic, the co-ordinates $f(x_\pm)$ must also be periodic. Consider the co-ordinate transformation $X(x_+, x_-) \to X(f(x_+), x_-)$. Because of the σ periodicity, the reparameterisation function must satisfy $f(x_+ + 2\pi) = f(x_+) + 2\pi$. Therefore the set of possible reparameterised co-ordinates are the smooth, monotonous functions $f(x_+)$ given by

$$f(\sigma) = \begin{cases} f(\sigma) & \text{for } \; 0 \leq f(\sigma) < 2\pi, \\ f(\sigma) - 2\pi & \text{for } \qquad f(\sigma) \geq 2\pi, \end{cases} \tag{2.7}$$

and similarly for $f(x_-)$. These are simply the elements of the group of diffeomorphisms of the unit circle, Diff(S^1) - the conformal symmetry group of the closed string.

An infinitesimal conformal transformation can be expressed as

$$f(x_+) = x_+ + \epsilon\eta(x_+), \tag{2.8}$$

where $\eta(x_+ + 2\pi) = \eta(x_+)$. The corresponding change in $X^\mu(x_+, x_-)$ is

$$\delta_\eta X^\mu(x_+, x_-) = \eta(x_+)\,\partial_+ X^\mu(x_+, x_-). \tag{2.9}$$

Using the periodicity requirement to expand $\eta\left(x_{+}\right)$ as a Fourier series

$$\eta\left(x_{+}\right)=\sum_{n=-\infty}^{\infty} e^{inx_{+}}\eta_{n}, \tag{2.10}$$

we can define the generators of the Lie group

$$\delta_{n}X^{\mu}\left(x_{+},x_{-}\right)\equiv -ie^{inx_{+}}\partial_{+}X^{\mu}\left(x_{+}\right), \tag{2.11}$$

and calculate their infinite-dimensional Lie algebra

$$\left[\delta_{n},\delta_{m}\right]=\left(n-m\right)\delta_{n+m}. \tag{2.12}$$

This is the algebra of *classical* conformal symmetry. A field theory with this symmetry is called a *conformal field theory*. In the case of quantised theories, an analogous *quantum conformal symmetry* is required to ensure the theory is independent of the co-ordinate choice. We will see in section 4.3 that the corresponding quantum algebra is the *Virasoro algebra*, which differs from (2.12) by a central extension term.

3 The Classical SU(2)-WZNW-Model

The WZNW (Wess-Zumino-Novikov-Witten) model is an example of a conformal field theory on a Lie group manifold, and so is a model of a closed string moving in a curved background.

3.1 The SU(2) Lie Group Manifold

Suppose we choose the background M to be the 3-sphere S^3. We can label each point in M with the *hyperspherical co-ordinates* (ψ,θ,ϕ), where

$$\begin{aligned} x_0 &= \cos\psi, \\ x_1 &= \cos\phi\sin\theta\sin\psi, \\ x_2 &= \sin\phi\sin\theta\sin\psi, \\ x_3 &= \cos\theta\sin\psi, \end{aligned} \tag{3.1}$$

and $\psi\in[0,2\pi]$, $\theta,\phi\in[0,\pi]$. The metric of this space is

$$ds^2\equiv G_{\mu\nu}\left(X\right)dx^{\mu}dx^{\nu}=\kappa\alpha'\left(d\psi^2+\sin^2\psi\left(d\theta^2+\sin^2\theta d\phi^2\right)\right), \tag{3.2}$$

where $\alpha'\propto l_s^2$. We note here that the limit $\kappa\to\infty$ corresponds to taking the limit $l_s\to 0$ in the string action (2.2), and hence in this limit the strings become point particles.

Defining

$$\vec{v} = \psi \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix} \tag{3.3}$$

allows us to represent each point in M as a 2x2 unitary matrix

$$g = \exp\left(i\vec{v}.\vec{\sigma}\right) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}, \tag{3.4}$$

where σ_i are the Pauli matrices.

The elements comprising the space S^3 are then clearly isomorphic to the elements of the well-known SU(2) Lie group and so S^3 is the Lie group manifold associated with SU(2).

From now on we will be studying the WZNW model defined on this background but it should be noted that the results can be generalised to more general Lie group manifolds.

3.2 Equations of Motion

The equations of motion can be derived from the action in the usual way[1]:

$$\partial_- \left\{ \left(\partial_+ g\right) g^{-1} \right\} = 0, \tag{3.5}$$

$$\partial_+ \left\{ g^{-1} \partial_- g \right\} = 0. \tag{3.6}$$

The general solution of these is separable in $x_\pm$

$$g\left(x_+, x_-\right) = g_L\left(x_+\right) A g_R\left(x_-\right), \tag{3.7}$$

where A is a constant 2x2 matrix. To ensure 2π-periodicity of the solutions, A must satisfy

$$A = N_L A N_R, \tag{3.8}$$

where

$$\begin{aligned} g_L\left(x_+ + 2\pi\right) &= g_L\left(x_+\right) N_L, \\ g_R\left(x_- - 2\pi\right) &= N_R g\left(x_-\right). \end{aligned} \tag{3.9}$$

The theory exhibits two dynamic symmetries (operations which map the set of solutions to themselves), which will be crucial when we come to quantise it. Firstly, it has conformal symmetry - invariance under the mapping

$$g \to g_L\left(f_+\left(x_+\right)\right) A g_R\left(f_-\left(x_-\right)\right), \tag{3.10}$$

as can be seen from (3.5) and (3.6).

Secondly, it is invariant under the action of the SU(2) loop group, $\widehat{\mathrm{SU}(2)}$:

$$g \to h_L(x_+)\, g h_R^{-1}(x_-)\,, \tag{3.11}$$

where $h_{L,R}$ are elements of SU(2) satisfying $h_{L,R}(x_+ + 2\pi) = h_{L,R}(x_+)$.

We have already encountered the generators of conformal symmetry in section 2.2. An infinitesimal element of $\widehat{\mathrm{SU}(2)}$ can be written as

$$h_L(x_+) = 1 + \epsilon\eta(x_+)\,, \tag{3.12}$$

where we can expand $\eta(x_+)$:

$$\eta(x_+) = \sum_{b=1}^{3} \sum_{m=-\infty}^{\infty} \eta_m^b e^{imx_+} \tau^b. \tag{3.13}$$

where $\tau^a = \frac{\sigma^a}{2}$ are half the Pauli matrices. Thus we can define the group's generators

$$\delta_m^b \equiv -ie^{imx_+}\tau^b, \tag{3.14}$$

and their Lie algebra

$$\left[\delta_n^a, \delta_m^b\right] = -F^{abc}\delta_{n+m}^c, \tag{3.15}$$

where $F^{abc} = i\epsilon^{abc}$, and ϵ^{abc} is the completely antisymmetric Levi-Civita tensor.

As was the case with conformal symmetry, the quantised SU(2)-WZNW-model will retain a *quantum* loop group symmetry, whose Lie algebra differs from (3.15) by a central extension. These two symmetries will turn out to be intricately linked in the quantised model.

3.3 The Hamiltonian Formalism

From the equations of motion (3.5) and (3.6) we can define 'momenta'

$$K(x_+) \equiv (\partial_+ g)\, g^{-1}, \tag{3.16}$$

$$\bar{K}(x_-) \equiv g^{-1}\partial_- g. \tag{3.17}$$

These are not the usual canonical momenta but are useful quantities which are momenta-like (they contain $\dot{g}$ terms) and are functions of a single variable.

We can again use periodicity conditions to expand $K(x_+)$ and $\bar{K}(x_-)$ as Fourier series, this time in terms of the Pauli matrices σ^a:

$$K(x_+) = K^a(x_+)\,\tau^a, \tag{3.18}$$

where

$$K^a(x_+) = \sum_{n=-\infty}^{\infty} e^{inx_+} K_n^a, \tag{3.19}$$

and similarly for $\bar{K}(x_-)$.

To quantise the SU(2)-WZNW-model we will need to know the Poisson brackets of the co-ordinates and the Fourier modes of the momenta. These are[1]:

$$-i\left\{K_n^a, K_m^b\right\} = f^{abc}K_{n+m}^c + n\frac{\beta}{2}\delta^{ab}\delta_{n+m,0}, \tag{3.20}$$

$$-i\left\{\bar{K}_n^a, \bar{K}_m^b\right\} = f^{abc}\bar{K}_{n+m}^c + n\frac{\beta}{2}\delta^{ab}\delta_{n+m,0}, \tag{3.21}$$

$$-i\left\{K_n^a, g(x_+, x_-)\right\} = e^{inx_+}\tau^a g(x_+, x_-), \tag{3.22}$$

$$-i\left\{\bar{K}_n^a, g(x_+, x_-)\right\} = e^{inx_-} g(x_+, x_-)\,\tau^a, \tag{3.23}$$

$$\left\{g_{m\bar{m}}, g_{m'\bar{m}'}\right\} = 0, \tag{3.24}$$

where f^{abc} are related to the structure factors F^{abc} defined previously, as we shall see.

4 Quantisation of the Model

4.1 Canonical Quantisation

The first step in quantising the theory is to elevate the dynamical variables to operators $g \to \hat{g}(x_+, x_-)$, $K_n^a \to \hbar\hat{J}_n^a$ and $\bar{K}_n^a \to \hat{\bar{J}}_n^a$, and to calculate their commutation relations from the Poisson brackets (3.20)-(3.24) in the standard way. Defining $\hat{J}_n^a \equiv \frac{1}{\hbar}\hat{K}_n^a$ and $\hat{\bar{J}}_n^a \equiv \frac{1}{\hbar}\hat{\bar{K}}_n^a$, this yields:

$$\left[\hat{J}_a^n, \hat{J}_b^m\right] = f^{abc}\hat{J}_{n+m}^c + n\frac{\kappa}{2}\delta_{n+m,0}\delta^{ab}, \tag{4.1}$$

$$\left[\hat{\bar{J}}_a^n, \hat{\bar{J}}_b^m\right] = f^{abc}\hat{\bar{J}}_{n+m}^c + n\frac{\kappa}{2}\delta_{n+m,0}\delta^{ab}, \tag{4.2}$$

$$\left[\hat{J}_a^n, \hat{g}(x_+, x_-)\right] = e^{inx_+}\tau^a\hat{g}(x_+, x_-), \tag{4.3}$$

$$\left[\hat{\bar{J}}_a^n, \hat{g}\left(x_+, x_-\right)\right] = e^{inx_-}\hat{g}\left(x_+, x_-\right)\tau^a, \tag{4.4}$$

$$\left[\hat{g}_{m\bar{m}}\left(\sigma\right), \hat{g}_{m'\bar{m}'}\left(\sigma'\right)\right] = 0, \tag{4.5}$$

where $\kappa \equiv \frac{\beta}{\hbar}$.

The operators $\hat{J}_a^n$ and $\hat{\bar{J}}_a^n$ are generalisations of the creation and annihilation operators for flat-space closed strings (see, for example, [4]) to curved backgrounds. Their algebras (4.1) and (4.2), named the *current algebras*, are a pair of commuting *Kac-Moody algebras*.

Any operator in the theory can be constructed from these three basic constituents and hence from these commutation relations we could proceed to calculate the algebra of *all* operators in the theory.

4.2 Quantum Loop Group Symmetry

The current algebras (4.1) and (4.2) bear a striking resemblance to the Lie algebra of classical loop group symmetry (3.15), but with an additional central extension term. Thus we identify the operators $\hat{J}_n^a$ and $\hat{\bar{J}}_n^a$ as the generators of quantum loop group symmetry. Comparing these relations also suggests that the factors $f^{abc} = -F^{abc}$. This can be shown more formally by using the second pair of commutators (4.3) and (4.4):

$$\begin{aligned}
e^{i(m+n)x_+}f^{abc}\tau^c\hat{g} &= \left[\left[\hat{J}_m^a, \hat{J}_n^b\right], \hat{g}\right] \\
&= \left[\left[\hat{J}_n^a, \hat{g}\right], \hat{J}_m^b\right] + \left[\hat{J}_n^a, \left[\hat{J}_m^b, \hat{g}\right]\right] \\
&= e^{inx_+}\left[\tau^a\hat{g}, \hat{J}_m^b\right] + e^{imx_+}\left[\hat{J}_n^a, \tau^b\hat{g}\right] \\
&= -e^{i(m+n)x_+}\left[\tau^a, \tau^b\right]\hat{g} \\
&= -e^{i(m+n)x_+}F^{abc}\tau^c\hat{g}.
\end{aligned} \tag{4.6}$$

Thus we can identify $f^{abc} = -i\epsilon^{abc}$.

We can use the requirement of loop group symmetry to identify a quantum analogue to the relation

$$\partial_+ g = K\left(x_+\right)g. \tag{4.7}$$

The naive approach would be simply to preserve the form of (4.7), letting $K\left(x_+\right) \to \hat{J}\left(x_+\right)$, but as always when quantising a system, there is freedom to choose the *ordering* of a product of two operators to ensure we obtain physically sensible results. In this case, we define the *normal-ordered product*

$$: \hat{J}(x_+)\,\hat{g}: \; \equiv \sum_{m<0} e^{-imx_+} \hat{J}^b_m \tau^b \hat{g} + \sum_{m>0} e^{-imx_+} \tau^b \hat{g} \hat{J}^b_m + \frac{1}{2}\left(\hat{J}^b_0 \tau^b \hat{g} + \tau^b \hat{g} \hat{J}^b_0\right), \tag{4.8}$$

and the relation

$$-i\partial_+ \hat{g} \equiv \frac{2}{(\kappa + h^\vee)} : \hat{J}(x_+)\,\hat{g}: , \tag{4.9}$$

where $h^\vee$ is a constant 'quantum correction'.

Although it is not intuitively obvious why this is the correct form, it is in fact covariant under the action of $\hat{J}^a_n$ and so possesses the quantum loop group symmetry required.

Proof. We begin by calculating:

$$\begin{aligned}\left[\hat{J}^a_n, : \hat{J}(x_+)\,\hat{g}: \right] = &\sum_{m<0} e^{-imx_+} f^{abc} \hat{J}^c_{n+m} \tau^b \hat{g} + \sum_{m>0} e^{-imx_+} f^{abc} \tau^b \hat{g} \hat{J}^c_{n+m} \\ &+ \frac{1}{2} f^{abc}\left(\hat{J}^b_n \tau^b \hat{g} + \tau^b \hat{g} \hat{J}^b_n\right) + e^{inx_+} : \hat{J}^b(x_+)\,\tau^b \tau^a \hat{g}: \\ &+ e^{inx_+} n \frac{\kappa}{2} \delta^{ab} \tau^b \hat{g}.\end{aligned}$$

The first three terms resemble the normal-ordered product $: \hat{J}(x_+)\,\hat{g}:$ and we rewrite them in terms of it to give:

$$\begin{aligned}\left[\hat{J}^a_n, : \hat{J}(x_+)\,\hat{g}: \right] = \; &e^{inx_+} f^{abc} \tau^b : \hat{J}^c(x_+)\,\hat{g}: \; + \; e^{inx_+} n f^{abc} \tau^b \tau^c \hat{g} \\ &+ e^{inx_+} : \hat{J}^b(x_+)\,\tau^b \tau^a \hat{g}: \; + \; e^{inx_+} n \frac{\kappa}{2} \delta^{ab} \tau^b \hat{g}.\end{aligned}$$

Finally, we use the SU(2) algebra $\left[\tau^a, \tau^b\right] = -f^{abc}\tau^c$ and the subsidiary relation $f^{abc}\tau^b\tau^c = -\frac{1}{2} f^{abc} f^{bcd} \tau^d$ to give:

$$\left[\hat{J}^a_n, : \hat{J}(x_+)\,\hat{g}: \right] = e^{inx_+} \tau^a \left(: \hat{J}(x_+)\,\hat{g}: \; + \frac{n}{2}\left(\kappa - f^{abc} f^{bcd} \delta^{ad}\right) \hat{g}\right). \tag{4.10}$$

Making the identification

$$h^\vee \delta^{ad} \equiv f^{abc} f^{cbd}, \tag{4.11}$$

implies $h^\vee = 2$ (since $f^{abc} = -i\epsilon^{abc}$) and gives

$$\left[\hat{J}^a_n, \frac{2}{(\kappa + h^\vee)} : \hat{J}(x_+)\,\hat{g}: \right] = e^{inx_+} \tau^a \left(\frac{2}{(\kappa + h^\vee)} : \hat{J}(x_+)\,\hat{g}: \; + n\hat{g}\right). \tag{4.12}$$

Using (4.3) we can calculate

$$\left[\hat{J}^a_n, -i\partial_+\hat{g}\right] = e^{inx_+}\tau^a\left(n - i\partial_+\right)\hat{g}, \tag{4.13}$$

and hence we see that

$$\left[\hat{J}^a_n, -i\partial_+\hat{g} - \frac{2}{(\kappa + h^\vee)} : \hat{J}\left(x_+\right)\hat{g} : \right] = e^{inx_+}\tau^a\left(-i\partial_+\hat{g} - \frac{2}{(\kappa + h^\vee)} : \hat{J}\left(x_+\right)\hat{g} : \right), \tag{4.14}$$

which vanishes given (4.9). □

There also exists the analogous relation

$$\left[\hat{\bar{J}}^a_n, -i\partial_-\hat{g} - \frac{2}{(\kappa + h^\vee)} : \hat{g}\hat{\bar{J}}\left(x_-\right) : \right] = 0, \tag{4.15}$$

given that

$$-i\partial_-\hat{g} \equiv \frac{2}{(\kappa + h^\vee)} : \hat{g}\hat{\bar{J}}\left(x_-\right) : , \tag{4.16}$$

where

$$: \hat{g}\hat{\bar{J}}\left(x_-\right) : \; \equiv \sum_{m<0} e^{-imx_-}\hat{\bar{J}}^b_m\hat{g}\tau^b + \sum_{m>0} e^{-imx_-}\hat{g}\hat{\bar{J}}^b_m\tau^b + \frac{1}{2}\left(\hat{\bar{J}}^b_0\hat{g}\tau^b + \hat{g}\hat{\bar{J}}^b_0\tau^b\right). \tag{4.17}$$

Additionally, it is simple to prove that

$$\left[\hat{\bar{J}}^a_n, -i\partial_+\hat{g} - \frac{2}{(\kappa + h^\vee)} : \hat{J}\left(x_+\right)\hat{g} : \right] = 0, \tag{4.18}$$

and

$$\left[\hat{J}^a_n, -i\partial_-\hat{g} - \frac{2}{(\kappa + h^\vee)} : \hat{g}\hat{J}\left(x_-\right) : \right] = 0, \tag{4.19}$$

and thus it is clear that the chosen normal orderings and quantum correction $h^\vee$ lead to definitions of $\partial_+\hat{g}$ and $\partial_-\hat{g}$ which are covariant under the action of the loop group symmetry generators $\hat{J}^a_n$ and $\hat{\bar{J}}^a_n$. These definitions in fact fully define the dynamics of the quantum model.

4.3 Quantum Conformal Symmetry: The Sugawara Construction

One of the most remarkable features of the WZNW-model is that having established the presence of quantum loop group symmetry, this necessitates that the theory also possesses quantum conformal symmetry, and so is independent of the co-ordinates used to parameterise the world-sheet. We show this using the *Sugawara construction* which is based on the *Virasoro operators* $\hat{L}_n$:

$$\hat{L}_n \equiv \frac{1}{(\kappa + h^\vee)} \sum_{k=-\infty}^{\infty} \hat{J}^a_{-k} \hat{J}^a_{k+n}, \qquad n \neq 0, \tag{4.20}$$

$$\hat{L}_0 \equiv \frac{1}{(\kappa + h^\vee)} \left(\hat{J}^a_0 \hat{J}^a_0 + 2 \sum_{k>0} \hat{J}^a_{-k} \hat{J}^a_k \right). \tag{4.21}$$

In string theory, these correspond to the Fourier modes of the string's stress-energy tensor.

To demonstrate that these generate quantum conformal symmetry, we begin by showing that

$$\left[\hat{L}_n, \hat{J}^a_m \right] = -m \hat{J}^a_{m+n}. \tag{4.22}$$

Proof. To ensure all of our operator products are well-defined (i.e. normal-ordered), we initially consider the related operator

$$\hat{\chi}_{n,\epsilon} \equiv \sum_k \hat{J}^a_{-k} \hat{J}^a_{k+n} \zeta(k\epsilon), \tag{4.23}$$

where

$$\zeta(x) \equiv \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}. \tag{4.24}$$

Using the commutators (4.1) and (4.3) gives

$$\begin{aligned} \left[\hat{J}^a_m, \hat{\chi}_{n,\epsilon} \right] = & \sum_k f^{abc} \hat{J}^c_{m-k} \hat{J}^b_{k+n} \zeta(k\epsilon) + \sum_k f^{abc} \hat{J}^b_{-k} \hat{J}^c_{k+n+m} \zeta(k\epsilon) \\ & + \frac{1}{2} m \hat{J}^a_{m+n} \kappa \zeta(m\epsilon) + \frac{1}{2} m \hat{J}^a_{m+n} \kappa \zeta((n+m)\epsilon). \end{aligned}$$

We cannot take the limit $\epsilon \to 0$ as the first two terms are not normal-ordered. Using the standard definition

$$: \hat{J}^a_n \hat{J}^b_m : \; = \begin{cases} \hat{J}^a_n \hat{J}^b_m, & m \geq n \\ \hat{J}^b_m \hat{J}^a_n, & m < n \end{cases}, \tag{4.25}$$

we can normal-order the first two terms to give

$$\left[\hat{J}^a_m, \hat{\chi}_{n,\epsilon}\right] = f^{abc}\zeta\left(k\epsilon\right)\sum_k\left(:\hat{J}^c_{m-k}\hat{J}^b_{k+n}: + :\hat{J}^b_{-k}\hat{J}^c_{k+n+m}:\right)$$
$$+\left(\sum_{k<\frac{m-n}{2}} - \sum_{k<-\frac{m+n}{2}}\right)f^{abc}f^{bcd}\zeta\left(k\epsilon\right)\hat{J}^d_{n+m}$$
$$+\frac{1}{2}m\hat{J}^a_{n+m}\kappa\left(\zeta\left(m\epsilon\right)+\zeta\left(\left(n+m\right)\epsilon\right)\right).$$

Now, on taking the limit $\epsilon \to 0$ (and shifting $k \to k-m$ in the second term), the first pair of terms vanish due to the antisymmetry of f^{abc} leaving

$$\left[\hat{J}^a_m, \hat{\chi}_n\right] = \left(\kappa + h^\vee\right)m\hat{J}^a_{n+m}, \tag{4.26}$$

where we have used the definition of $h^\vee$ (4.11). Thus we obtain the result

$$\left[\hat{L}_n, \hat{J}^a_m\right] = -m\hat{J}^a_{n+m}. \tag{4.27}$$

□

With this result we can now show that that the Virasoro operators obey

$$\left[\hat{L}_n, \hat{L}_m\right] = \left(n-m\right)\hat{L}_{n+m} + \frac{C_\kappa}{12}n\left(n^2-1\right)\delta_{n+m,0}, \tag{4.28}$$

where

$$C_\kappa = \frac{3\kappa}{\kappa+2}. \tag{4.29}$$

This *Virasoro algebra* is equivalent to the classical conformal symmetry algebra (2.12) except for the central extension term, and is the algebra of quantum conformal symmetry.

Proof. Using the commutator derived above, we obtain

$$\left[\hat{L}_n, \hat{\chi}_{m,\epsilon}\right] = \sum_k\left(k\hat{J}^a_{n-k}\hat{J}^a_{k+m} - \left(k+m\right)\hat{J}^a_{-k}\hat{J}^a_{k+n+m}\right)\zeta\left(k\epsilon\right), \tag{4.30}$$

where we again use $\hat{\tau}_{m,\epsilon}$ until the terms are normal-ordered. Performing this normal ordering gives

$$\left[\hat{L}_n, \hat{\chi}_{m,\epsilon}\right] = \zeta\left(k\epsilon\right)\sum_k\left(k:\hat{J}^a_{n-k}\hat{J}^a_{k+m}: - \left(k+m\right):\hat{J}^a_{-k}\hat{J}^a_{k+m+n}:\right)$$
$$+\frac{\kappa}{2}\zeta\left(k\epsilon\right)\delta_{n+m,0}\delta^{aa}\left(\sum_{k<\frac{n-m}{2}}k\left(n-k\right) + \sum_{k<-\frac{n+m}{2}}k\left(k+m\right)\right),$$

and taking the limit $\epsilon \to 0$ (as well as shifting $k \to k+n$ in the first term) gives

$$\left[\hat{L}_n, \hat{\chi}_m\right] = \sum_{k=-\infty}^{\infty} (n-m) : \hat{J}^a_{-k} \hat{J}^a_{k+n+m} : + \frac{3\kappa}{2} \sum_{k=0}^{n-1} k\,(n-k)\,. \tag{4.31}$$

Using the definition of $\hat{L}_n$ and the identities

$$\sum_{k=0}^{n} k = \frac{1}{2} n\,(n+1)\,, \tag{4.32}$$

$$\sum_{k=0}^{n} k^2 = \frac{1}{6} n\,(n+1)\,(2n+1)\,, \tag{4.33}$$

we obtain

$$\left[\hat{L}_n, \hat{L}_m\right] = (n-m)\,\hat{L}_{n+m} + \frac{3\kappa}{12\,(\kappa+2)} n\left(n^2-1\right)\delta_{n+m,0}. \tag{4.34}$$

□

The appearance of the central extension terms in the symmetry algebras of the quantised theory is related to the fact that in a quantum theory, the most general combination of two transformations with transformation parameters (λ, λ') is

$$U_\lambda U_{\lambda'} = e^{i\eta(\lambda,\lambda')} U_{\lambda.\lambda'}, \tag{4.35}$$

since the introduction of an overall phase factor does affect the resulting physics. To illustrate this, consider the expectation value of some operator $\hat{A}$:

$$\left\langle \hat{A} \right\rangle = \left\langle \psi \left| \hat{A} \right| \psi \right\rangle. \tag{4.36}$$

The change in this expectation value due to some infinitesimal transformation generated by the operator $\hat{J}_n$ is

$$\delta_n \left\langle \hat{A} \right\rangle = \left\langle \psi \left| \left[\hat{J}_n, \hat{A}\right] \right| \psi \right\rangle. \tag{4.37}$$

If we apply two transformations, the difference in $\left\langle \hat{A} \right\rangle$ will depend on the order we apply the transformations:

$$\left[\delta_n, \delta_m\right] \left\langle \hat{A} \right\rangle = \left\langle \psi \left| \left[\left[\hat{J}_n, \hat{J}_m\right], \hat{A}\right] \right| \psi \right\rangle, \tag{4.38}$$

but since a complex number central extension term commutes with all $\left\langle \hat{A} \right\rangle$, we see that its presence in the quantum algebra does not affect the transformation properties - it is just a generalisation of the classical algebra.

Having established the presence of quantum conformal symmetry, it is useful for later to calculate one final commutator:

$$\left[\hat{L}_n, \hat{g}\right] = e^{inx_+} \left(-i\partial_+ + \Delta_\kappa n\right) \hat{g}, \tag{4.39}$$

where

$$\Delta_\kappa = \frac{3}{4\left(\kappa+2\right)}. \tag{4.40}$$

Proof. Again, we begin by considering the commutator

$$\left[\hat{\chi}_{n,\epsilon}, \hat{g}\right] = e^{inx_+} \zeta\left(k\epsilon\right) \sum_k \left(e^{inx_+} \hat{J}^a_{-k} \tau^a \hat{g} + e^{-i(k+n)x_+} \tau^a \hat{g} \hat{J}^a_{k+n}\right), \tag{4.41}$$

and normal order the right-hand side as in (4.8) to produce

$$\left[\hat{\chi}_{n,\epsilon}, \hat{g}\right] = e^{inx_+} \zeta\left(k\epsilon\right) \left(2 \colon \hat{J}\left(x_+\right) \hat{g} \colon + \left(\sum_{k<0} - \sum_{k<-n}\right) \frac{3}{4} \hat{g}\right), \tag{4.42}$$

recalling that $\tau^a \tau^a = \frac{3}{4}$. Now, taking the limit $\epsilon \to 0$ and using (4.9):

$$\left[\hat{\chi}_n, \hat{g}\right] = e^{inx_+} \left(-i\left(\kappa+2\right) \partial_+ \hat{g} + \frac{3n}{4} \hat{g}\right). \tag{4.43}$$

Finally, divide through by $(\kappa+2)$ to obtain

$$\left[\hat{L}_n, \hat{g}\right] = e^{inx_+} \left(-i\partial_+ + \frac{3n}{4\left(\kappa+2\right)}\right) \hat{g}. \tag{4.44}$$

□

This shows that $\hat{g}$ is a *Virasoro primary field* with conformal dimension Δ_κ, i.e. it transforms under reparameterisations as

$$\hat{g}\left(x_+, x_-\right) \to \left(f'\left(x_+\right)\right)^{\Delta_\kappa} \hat{g}\left(f\left(x_+\right), x_-\right). \tag{4.45}$$

To summarise, we have constructed from $\hat{J}^a_n$ the Virasoro operators $\hat{L}_n$ which are the generators of quantum conformal symmetry. These obey the relations

$$\left[\hat{L}_n, \hat{J}^a_m\right] = -m \hat{J}^a_{m+n}, \tag{4.46}$$

$$\left[\hat{L}_n, \hat{L}_m\right] = \left(n-m\right) \hat{L}_{n+m} + \frac{\kappa}{4\left(\kappa+2\right)} n\left(n^2-1\right) \delta_{n+m,0}, \tag{4.47}$$

$$\left[\hat{L}_n, \hat{g}\right] = e^{inx_+} \left(-i\partial_+ + \frac{3n}{4\left(\kappa+2\right)}\right) \hat{g}. \tag{4.48}$$

4.4 The Hamiltonian

We can also define the Virasoro operators corresponding to reparameterisations of the x_- co-ordinate:

$$\hat{\bar{L}}_n \equiv \frac{1}{(\kappa + h^v)} \sum_{k=-\infty}^{\infty} \hat{\bar{J}}^a_{-k} \hat{\bar{J}}^a_{k+n}, \qquad n \neq 0, \tag{4.49}$$

$$\hat{\bar{L}}_0 \equiv \frac{1}{(\kappa + h^v)} \left(\hat{\bar{J}}^a_0 \hat{\bar{J}}^a_0 + 2 \sum_{k>0} \hat{\bar{J}}^a_{-k} \hat{\bar{J}}^a_k \right). \tag{4.50}$$

Their commutation relations can derived in a similar way as above:

$$\left[\hat{\bar{L}}_n, \hat{\bar{J}}^a_m \right] = -m \hat{\bar{J}}^a_{m+n}, \tag{4.51}$$

$$\left[\hat{\bar{L}}_n, \hat{\bar{L}}_m \right] = (n-m) \hat{\bar{L}}_{n+m} + \frac{\kappa}{4(\kappa+2)} n \left(n^2 - 1 \right) \delta_{n+m,0}, \tag{4.52}$$

$$\left[\hat{\bar{L}}_n, \hat{g} \right] = e^{inx_-} \left(-i\partial_- + \frac{3n}{4(\kappa+2)} \right) \hat{g}. \tag{4.53}$$

If we now consider the operator $\hat{H} \equiv \hat{L}_0 + \hat{\bar{L}}_0$, we obtain the commutator

$$\begin{aligned} \left[\hat{L}_0 + \hat{\bar{L}}_0, \hat{g}(x_+, x_-) \right] = &- i \left(\partial_+ + \partial_- \right) \hat{g}(x_+, x_-) \\ &- i \partial_\tau \hat{g}(\sigma, \tau). \end{aligned} \tag{4.54}$$

This relation demonstrates that $\hat{H}$ generates the time-evolution of the system and so we identify $\hat{H}$ as the *Hamiltonian* of the SU(2)-WZNW-model.

5 The Hilbert Space

Having established the system's Hamiltonian, we are now in a position to consider the Hilbert space $\mathfrak{H}$ of the theory. Using the Virasoro algebra and some simple physical principles, we will be able to identify the Hilbert space and hence solve the theory.

5.1 Two Initial Requirements

To begin, we make two assumptions regarding the properties of the Hilbert space. Firstly, it should be a *unitary representation* of the algebra of $\hat{J}^a_n$ and $\hat{\bar{J}}^a_n$. For a finite transformation $U_g = e^{i\alpha K}$, this would mean that

$$\left(U_g \right)^\dagger = \left(U_g \right)^{-1}, \tag{5.1}$$

and hence the Lie algebra generators satisfy $K^\dagger = K$. In our case, this corresponds to

$$\left(\hat{J}^a_n\right)^\dagger = \hat{J}^a_{-n}, \tag{5.2}$$

and similarly

$$\left(\hat{\bar{J}}^a_n\right)^\dagger = \hat{\bar{J}}^a_{-n}, \tag{5.3}$$

where we are working in the standard Pauli matrix basis $a = 1, 2, 3$. If we use the basis $\alpha = +, -, 3$ where $\sigma^\pm = \sigma^1 \pm i\sigma^2$, then the unitarity requirement becomes

$$\left(\hat{J}^+_n\right)^\dagger = \hat{J}^-_{-n}, \tag{5.4}$$

and

$$\left(\hat{J}^3_n\right)^\dagger = \hat{J}^3_{-n}, \tag{5.5}$$

with identical relations for the $\hat{\bar{J}}^\alpha_n$ operators.

The motivation for the unitarity requirement is illustrated by noting that it implies that the operators $\hat{J}^a\left(x_+\right)$ and $\hat{\bar{J}}^a\left(x_-\right)$ are Hermitian:

$$\left(\hat{J}^a\left(x_+\right)\right)^\dagger = \hat{J}^a\left(x_+\right), \tag{5.6}$$

and

$$\left(\hat{\bar{J}}^a\left(x_-\right)\right)^\dagger = \hat{\bar{J}}^a\left(x_-\right), \tag{5.7}$$

as we would expect for the quantum mechanical operators corresponding to the *observables* $K^a\left(x_+\right)$ and $\bar{K}^a\left(x_-\right)$.

In the $\alpha = +, -, 3$ basis, the corresponding expressions are

$$\left(\hat{\bar{J}}^+\left(x_+\right)\right)^\dagger = \hat{\bar{J}}^-\left(x_+\right), \tag{5.8}$$

and

$$\left(\hat{\bar{J}}^3\left(x_+\right)\right)^\dagger = \hat{\bar{J}}^3\left(x_+\right), \tag{5.9}$$

with identical expressions for $\hat{\bar{J}}^\alpha\left(x_-\right)$.

The second constraint we impose on the Hilbert space is the familiar requirement that the Hamiltonian $\hat{H}$ must be bounded from below, so that the ground state has finite energy.

5.2 The Hilbert Space Basis

To begin our identification of a basis for $\mathfrak{H}$, we require the commutator

$$\left[\hat{H}, \hat{J}_n^a\right] = -n\hat{J}_n^a, \tag{5.10}$$

which follows trivially from (4.22) and (4.47). Consider now $|E\rangle$, an eigenstate of the Hamiltonian with energy E. Then it follows from (5.10) that $\hat{J}_n^a$ is also an eigenstate of the Hamiltonian, with energy $E - n$:

$$\hat{H}\hat{J}_n^a |E\rangle = (E - n)\,\hat{J}_n^a |E\rangle\,. \tag{5.11}$$

Thus we identify $\hat{J}_n^a$ as raising operators for $n < 0$ and lowering operators for $n > 0$.

Since $\hat{H}$ must be bounded from below, there must exist a ground state, denoted by vectors $|\psi_0\rangle$, which is annihilated by all lowering operators:

$$\hat{J}_n^a |\psi_0\rangle = 0, \qquad \text{for } n > 0. \tag{5.12}$$

Recalling (4.1), we note that

$$\left[\hat{J}_0^a, \hat{J}_n^b\right] = f^{abc}\hat{J}_n^c, \tag{5.13}$$

and hence the vectors $|\psi_0\rangle$ are common eigenstates of the operators $\hat{J}_n^a$ *and* $\hat{J}_0^a$. The operators $\hat{J}_0^a$ obey the SU(2) algebra and thus we take as the ground state vectors $|\psi_o\rangle$ the well-known unitary representations of SU(2) e_m^j, where $j = 0, \frac{1}{2}, 1 \ldots$, $m = -j, \ldots, j$, and

$$\hat{J}_0^+ e_m^j = \sqrt{(j+m+1)\,(j-m)}e_{m+1}^j, \tag{5.14}$$

$$\hat{J}_0^- e_m^j = \sqrt{(j-m+1)\,(j+m)}e_{m-1}^j, \tag{5.15}$$

$$\hat{J}_0^3 e_m^j = m e_m^j. \tag{5.16}$$

From these ground states we can generate excited states using the raising operators and so a possible generating set for $\mathfrak{H}$ is the set of vectors of the form

$$\prod_r \left(\hat{J}_{-r}^+\right)^{k_r} \prod_s \left(\hat{J}_{-s}^3\right)^{l_s} \prod_t \left(\hat{J}_{-t}^-\right)^{m_t} e_m^j. \tag{5.17}$$

We denote the space with this basis $\mathfrak{M}_{j,\kappa}$, which is a representation of $\widehat{SU(2)}$. Figure 2 overleaf shows schematically the vectors of $\mathfrak{M}_{j,\kappa}$:

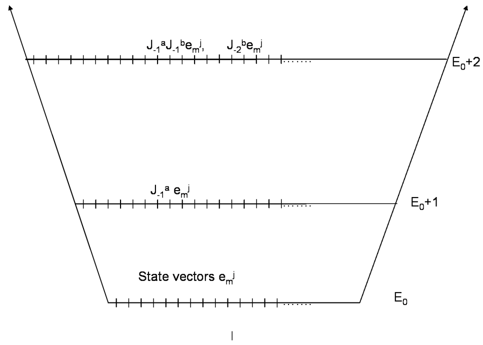

Figure 2: Schematic representation of $\mathfrak{M}_{j,\kappa}$.

It may appear that we have now succeeded in identifying $\mathfrak{H}$ but this is not the case. Consider the vector $n_j \equiv \hat{J}^+_{-1} e^j_j$:

$$\begin{aligned}
\|N_j\|^2 &= \left\langle \hat{J}^+_{-1} e^j_j | \hat{J}^+_{-1} e^j_j \right\rangle \\
&= \left\langle e^j_j | \hat{J}^-_1 \hat{J}^+_{-1} | e^j_j \right\rangle \\
&= \left\langle e^j_j | \left[\hat{J}^-_1, \hat{J}^+_{-1} \right] | e^j_j \right\rangle \\
&= \left\langle e^j_j | -2\hat{J}^3_0 + \kappa | e^j_j \right\rangle \\
&= \kappa - 2j.
\end{aligned} \tag{5.18}$$

Hence, in the case $\kappa \leq 2j$, there exist vectors of zero and negative norm - the space $\mathfrak{M}_{j,\kappa}$ is clearly *not* unitary for all j. This is due to the central extension term in the commutator (4.1).

5.3 Removing the Null Vectors

A null-vector, such as N_j above, is defined by the property

$$\hat{J}^a_n N_j = 0, \qquad \forall n > 0, \quad \forall a = 1, 2, 3, \tag{5.19}$$

where $N_j \neq e^j_m$. The null-vectors generate a subrepresentation $\mathfrak{N}_{j,\kappa}$ within $\mathfrak{M}_{j,\kappa}$, shown schematically in Figure 3, spanned by the basis

$$\prod_r \left(\hat{J}^+_{-r}\right)^{k_r} \prod_s \left(\hat{J}^3_{-s}\right)^{l_s} \prod_t \left(\hat{J}^-_{-t}\right)^{m_t} N_j. \tag{5.20}$$

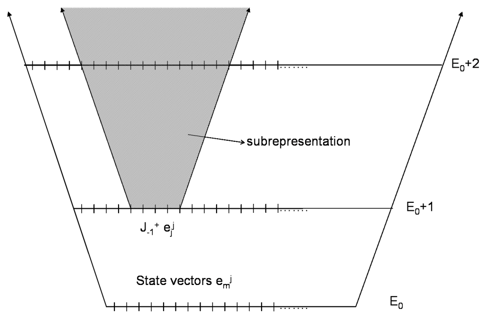

Figure 3: Schematic representation of $\mathfrak{N}_{j,\kappa}$.

Hence the presence of these null vectors violates both the unitarity *and* the irreducibility of $\mathfrak{M}_{j,\kappa}$.

Thus we define the representation

$$\mathfrak{V}_{j,\kappa} \equiv \mathfrak{M}_{j,\kappa}/\mathfrak{N}_{j,\kappa}, \tag{5.21}$$

which is *irreducible*.

Proof. Suppose $\mathfrak{V}_{j,\kappa}$ is reducible and hence there exists a subrepresentation $\mathfrak{W}_{j,\kappa} \subsetneq \mathfrak{V}_{j,\kappa}$ such that $\hat{J}^a_n \mathfrak{W}_{j,\kappa} \subset \mathfrak{W}_{j,\kappa}$. Denote the lowest energy states of this by $\mathfrak{W}^0_{j,\kappa} \subset \mathfrak{W}_{j,\kappa}$. For $n > 0$, $\hat{J}^a_n \omega_0 = 0 \quad \forall \omega_0 \in \mathfrak{W}^0_{j,\kappa}$. However, since $\mathfrak{V}_{j,\kappa}$ contains no zero-norm states, $\omega_0 = 0$ and thus $\mathfrak{W}_{j,\kappa} = \emptyset$. $\square$

We saw previously in (5.18) that the condition $j \leq \frac{\kappa}{2}$ is necessary for unitarity. This is also a sufficient condition for unitarity[2] and hence $\mathfrak{V}_{j,\kappa}$ is unitary if and only if $j \leq \frac{\kappa}{2}$. This is completely different from the familiar representation theory of angular momentum where j is not bounded from above.

Recalling that for each operator $\hat{J}_n^a$, there is a corresponding operator $\hat{\bar{J}}_n^a$, we can write the Hilbert space $\mathfrak{H}$ as a subspace of the direct product of two independent spaces:

$$\mathfrak{H} \subset \bigoplus_{j,\bar{j}=0,\frac{1}{2}\ldots}^{\frac{\kappa}{2}} \mathfrak{V}_{j,\kappa} \otimes \mathfrak{V}_{\bar{j},\kappa}. \tag{5.22}$$

5.4 Periodicity Requirements

Until now, $\mathfrak{H}$ has been constructed so that it is a unitary and irreducible representation of the symmetry algebra. However, we have not yet introduced the very basic periodicity constraint required of a closed string:

$$g\left(\sigma + 2\pi, \tau\right) = g\left(\sigma, \tau\right). \tag{5.23}$$

The quantum mechanical equivalent of this requirement is that the matrix elements of $\hat{g}$ must all be periodic in σ:

$$\left\langle \psi_2 \left| \hat{g}\left(\sigma + 2\pi, \tau\right) \right| \psi_1 \right\rangle = \left\langle \psi_2 \left| \hat{g}\left(\sigma, \tau\right) \right| \psi_1 \right\rangle. \tag{5.24}$$

Hence we must attempt to the matrix elements corresponding to *all* possible states ψ_1 and ψ_2.

We begin by considering the elementary matrix elements

$$\left\langle E_{M_2}^{J_2} \left| \hat{g}\left(x_+, x_-\right) \right| E_{M_1}^{J_1} \right\rangle, \tag{5.25}$$

where $E_M^J = e_m^j \otimes e_{\bar{m}}^{\bar{j}}$ are the basis vectors, and the labels $M = (m, \bar{m})$ and $J = (j, \bar{j})$ each denote a pair of values.

Firstly, note that for each fixed (x_+, x_-), the 2x2 matrix $\hat{g} \in \mathfrak{V}_{\frac{1}{2}} \otimes \mathfrak{V}_{\frac{1}{2}}$. We can therefore use the *Wigner-Eckart Theorem* (see, for example, [3]) for such tensor operators, which states that

$$\begin{aligned} \left\langle E_{M_2}^{J_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E_{M_1}^{J_1} \right\rangle \propto C \begin{pmatrix} j_2 & \frac{1}{2} & j_1 \\ m_2 & m & m_1 \end{pmatrix} C \begin{pmatrix} \bar{j}_2 & \frac{1}{2} & \bar{j}_1 \\ \bar{m}_2 & \bar{m} & \bar{m}_1 \end{pmatrix} \\ \times D\left(J_2, J_1 | x_+, x_-\right) \end{aligned} \tag{5.26}$$

where the indices $m, \bar{m} = \pm\frac{1}{2}$. The C symbols are the familiar *Clebsch-Gordan coefficients*, and D is an unknown function of J_1 and J_2 but *not* of M_1 or M_2.

To enforce periodicity requirements, we must know the specific dependence of D on (x_+, x_-). From the definition of the basis vectors,

$$\left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \hat{L}_0 \right| E^{J_1}_{M_1} \right\rangle = \Delta_{j_1} \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle , \qquad (5.27)$$

where

$$\Delta_{j_1} \equiv \frac{1}{\kappa+2} j_1 \left(j_1 + 1\right), \qquad (5.28)$$

since e^j_m is an eigenvector of the Casimir operator J_0^2 with eigenvalue $j\left(j+1\right)$.

Alternatively, we can can calculate this matrix element using commutation relation (4.39):

$$\begin{aligned} \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \hat{L}_0 \right| E^{J_1}_{M_1} \right\rangle &= -\left\langle E^{J_2}_{M_2} \left| \left[\hat{L}_0, \hat{g}_{m\bar{m}}\right] \right| E^{J_1}_{M_1} \right\rangle \\ &\quad + \left\langle \hat{L}_0 E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}} \right| E^{J_1}_{M_1} \right\rangle \\ &= \left(i\partial_+ + \Delta_{j_2}\right) \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}} \right| E^{J_1}_{M_1} \right\rangle . \end{aligned} \qquad (5.29)$$

Comparison of (5.27) and (5.29) yields

$$i\partial_+ \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle = \left(\Delta_{j_1} - \Delta_{j_2}\right) \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle , \qquad (5.30)$$

and so

$$\left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle \propto e^{i\left(\Delta_{j_2} - \Delta_{j_1}\right)x_+}. \qquad (5.31)$$

Performing a similar analysis for $\hat{\bar{L}}_0$ yields

$$\begin{aligned} \left\langle E^{J_2}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle &= e^{i\left(\Delta_{j_2} - \Delta_{j_1}\right)x_+} e^{i\left(\Delta_{\bar{j}_2} - \Delta_{\bar{j}_1}\right)x_-} \\ &\quad \times C\begin{pmatrix} j_2 & \frac{1}{2} & j_1 \\ m_2 & m & m_1 \end{pmatrix} C\begin{pmatrix} \bar{j}_2 & \frac{1}{2} & \bar{j}_1 \\ \bar{m}_2 & \bar{m} & \bar{m}_1 \end{pmatrix} \\ &\quad \times D'\left(J_2, J_1\right). \end{aligned} \qquad (5.32)$$

The Clebsch-Gordan coefficient $C\begin{pmatrix} j_2 & \frac{1}{2} & j_1 \\ m_2 & m & m_1 \end{pmatrix}$ is non-zero only if $j_2 = j_1 \pm \frac{1}{2}$ and $m_1 + m = m_2$, in which case $\Delta_{j_1} - \Delta_{j_2} = \frac{\pm 1}{\kappa+2}\left(j_1 + \frac{1}{2}\right)$. Thus the only non-zero elementary matrix elements are

$$\begin{aligned} \left\langle E^{J_1 \pm \frac{1}{2}}_{M_2} \left| \hat{g}_{m\bar{m}}\left(x_+, x_-\right) \right| E^{J_1}_{M_1} \right\rangle &= e^{\frac{i\tau}{\kappa+2}\left(j_1 + \bar{j}_1 + 1\right)} e^{\frac{i\sigma}{\kappa+2}\left(j_1 - \bar{j}_1\right)} \\ &\quad \times C\begin{pmatrix} j_1 \pm \frac{1}{2} & \frac{1}{2} & j_1 \\ m_1 + m & m & m_1 \end{pmatrix} C\begin{pmatrix} \bar{j}_1 \pm \frac{1}{2} & \frac{1}{2} & \bar{j}_1 \\ \bar{m}_1 + \bar{m} & \bar{m} & \bar{m}_1 \end{pmatrix} \\ &\quad \times D'\left(J_1\right). \end{aligned} \qquad (5.33)$$

Having established the form of the elementary matrix elements, it is clear that for the periodicity requirement (5.24) to hold, we must further restrict the Hilbert space $\mathfrak{H}$ such that $j_1 = \bar{j}_1$. This couples the left-moving x_+ modes to the right-moving x_- modes on the string in a way such that $\hat{g}(\sigma, \tau)$ is periodic.

This restriction implies that *all* matrix elements, not just those of the form (5.25), have the required periodicity. The matrix elements of any excited state can be calculated from the elementary ones using just the commutation relations of section 4.1. This is best illustrated using a simple example:

$$\begin{aligned}
\left\langle E^{J_2}_{M_2} \left|\hat{g}\right| \hat{J}^a_{-n} E^{J_1}_{M_1} \right\rangle &= -\left\langle E^{J_2}_{M_2} \left| \left[\hat{J}^a_{-n}, \hat{g}\right] \right| E^{J_1}_{M_1} \right\rangle \\
&\quad + \left\langle \left(\hat{J}^a_{-n}\right)^\dagger E^{J_2}_{M_2} \left|\hat{g}\right| E^{J_1}_{M_1} \right\rangle \\
&= -e^{-inx_+} \tau^a \left\langle E^{J_2}_{M_2} \left|\hat{g}\right| E^{J_1}_{M_1} \right\rangle \\
&\quad + \left\langle \left(\hat{J}^a_{n}\right) E^{J_2}_{M_2} \left|\hat{g}\right| E^{J_1}_{M_1} \right\rangle \\
&= -e^{-inx_+} \tau^a \left\langle E^{J_2}_{M_2} \left|\hat{g}\right| E^{J_1}_{M_1} \right\rangle,
\end{aligned} \tag{5.34}$$

since $\hat{J}^a_n$ annihilate the ground state vectors e^j_m. This is an example of the 'Wigner-Eckart Theorem for $\widehat{\mathrm{SU}(2)}$'.

5.5 Summary

We have seen that the Hilbert space $\mathfrak{H}$ can be written

$$\mathfrak{H} \subset \bigoplus_{j=0,\frac{1}{2}\ldots}^{\frac{\kappa}{2}} \mathfrak{V}_{j,\kappa} \otimes \mathfrak{V}_{j,\kappa}, \tag{5.35}$$

and that all of the matrix elements of $\hat{g}$ can be calculated from

$$\begin{aligned}
\left\langle E^{J_1 \pm \frac{1}{2}}_{M_2} \left|\hat{g}_{m\bar{m}}\left(x_+, x_-\right)\right| E^{J_1}_{M_1} \right\rangle &= e^{\frac{\pm i\tau}{\kappa+2}(2j_1+1)} \times C \begin{pmatrix} j_1 \pm \frac{1}{2} & \frac{1}{2} & j_1 \\ m_1 + m & m & m_1 \end{pmatrix} \\
&\quad \times C \begin{pmatrix} \bar{j}_1 \pm \frac{1}{2} & \frac{1}{2} & \bar{j}_1 \\ \bar{m}_1 + \bar{m} & \bar{m} & \bar{m}_1 \end{pmatrix} \times D_\pm\left(j_1\right).
\end{aligned} \tag{5.36}$$

To solve the theory, all we now require is the explicit form of the function $D_\pm\left(j_1\right)$. Now that we have enforced all symmetry and periodicity requirements on the model, we can work with the results obtained so far to calculate these functions.

The only commutator arising from canonical quantisation that we have yet to use is

$$[\hat{g}(\sigma), \hat{g}(\sigma')] = 0. \tag{5.37}$$

Although I did not have time to study the details of the calculation during the summer student program, it is possible to use the the previous result (5.36) along with (4.9) and (5.37) to obtain a differential equation for $D_{\pm}(j_1)$, which has a solution in terms of *hypergeometric functions*. This allows us to write the explicit form of any of the matrix elements in the Hilbert space, and so completes our study of it.

6 Conclusion

This report has detailed how the classical SU(2)-WZNW-model is quantised by requiring the resulting theory to exhibit loop group symmetry, conformal symmetry and periodicity. Following canonical quantisation, the presence of the symmetries was explicitly shown - in particular we learned that due to the presence of quantum loop group symmetry, the theory *must* also exhibit conformal invariance. By applying some simple, physically-sensible constraints, the Hilbert space of the theory was then determined.

Although the motivation for studying it was mainly due to its applications in string theory, this specific WZNW-model is clearly an unrealistic theory for our universe - it is a theory of a three dimensional space-time for a start! It should also be noted that we have restricted our study to the closed bosonic string and so the SU(2)-WZNW-model cannot be a theory of fermions. However, it has proved a very useful introduction to the general WZNW-model, which can be defined over various Lie group manifolds. The most physically interesting ones are naturally those corresponding to four or more space-time dimensions.

As a final remark, the WZNW-model provides no prediction at all of how the space-time background arises (as we would expect any fundamental theory such as string theory to do), but instead this information must be inserted initially by hand.

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