Integrability and the Zero Curvature Representation of Affine Toda Field Theory

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Abstract

We study a class of integrable systems known as affine Toda field theories. These theories are closely related to affine Lie algebras and can be reformulated as the condition for the vanishing of the curvature of a vector bundle connection. We show how this zero curvature representation leads to an infinite number of conserved quantities for these systems, and explicitly construct some of these quantities for $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ affine Toda field theory.

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1 Introduction

Integrable systems occur in a wide range of areas throughout modern physics and mathematics. In classical mechanics, we have such examples as the Euler top, the harmonic oscillator and the Coulomb potential. In the theory of non-linear partial differential equations we have famous examples such as the KdV equation. The Landau-Lifshitz equation and the Heisenberg model arise in the study of magnetism. In classical gauge field theory we have the (anti-)self-dual Yang-Mills equations and their dimensional reductions, which describe for instance instantons and magnetic monopoles.

Such systems could also be called "exactly solvable models." They are characterised by the ability to write down explicit solutions, often making use of inverse scattering methods. These solutions sometimes appear as solitons - localised, particle-like solutions. However the feature which forms the central theme of this report is the existence of conserved quantities.

We know from classical mechanics the importance of the conservation of quantities such as energy and angular momentum in solving systems like central force problem. It is possible to use conserved quantities to reduce the number of degrees of freedom of a system to a manageable amount, sometimes letting us solve by quadrature.

The essential concept of integrability that is to be followed in this report is exactly this: the existence of conserved quantities. In particular, to ensure integrability - which we still need to define properly - we need as many conserved quantities as degrees of freedom.

Our focus will be on infinite dimensional systems, and especially a class of field theories known as affine Toda field theories. The practical problem we want to overcome is then the construction of an infinite number of conserved quantities. The approach we will use for this task is the reformulation of the equations of motion as a zero curvature condition. A general reference for this method is [1]. We make use also of the papers [4], [5] and the lecture notes [3].

This report is at the level of a review: we explain some of the necessary background theory and explore in detail some non-trivial examples. By doing this, we hope to elucidate important common features of integrable systems, and gain an insight into this area of research.

Let us first give here a brief overview of integrability as it applies in classical mechanics [2]. Consider an n-dimensional Hamiltonian system, with coordinates q_i, momenta p_i ($i = 1, \ldots, n$) and Hamiltonian $H = H(q_i, p_i)$. We can describe the motion of the system using the canonical Poisson brackets: $\{p_i, q_j\} = \delta_{ij}$, $\{p_i, p_j\} = \{q_i, q_j\} = 0$. We then have

$$\dot{p}_i = \{H, p_i\} \qquad \dot{q}_i = \{H, q_i\} \tag{1.1}$$

where we note that the Poisson bracket acts as a derivation, $\{f, gh\} = \{f, g\}h + \{f, h\}g$, and is skew-symmetric $\{f, g\} = -\{g, f\}$.

For any function $f = f(p_i, q_i)$ which depends on time only through the coordinates and momenta we have that

$$\frac{d}{dt} f = \{H, f\} \tag{1.2}$$

It follows that a function f is conserved if its Poisson bracket with the Hamiltonian vanishes.

Such a system is said to be Liouville integrable if there exist n functions $I_j = I_j(q_i, p_i)$ which are conserved and mutually Poisson commuting, $\{I_j, I_k\} = 0\, \forall j, k$. In this case it is possible to canonically transform to a new set of coordinates known as action-angle variables. The action variables are our new momenta, which are functions of the I_k only, and so are constant. The angle variables are our new coordinates, which evolve periodically. Geometrically, we can view this as a torus fibration, with base space consisting of all possible values for the action variables and the fibre at each point being an n-dimensional torus, with the angle variables as fibre coordinates.

So we then expect a characteristic of infinite-dimensional integrable systems to be the existence of an infinite number of conserved quantities, which commute under some Poisson bracket.

In this report we show how this is the case for infinite-dimensional systems admitting a zero curvature representation, discussing mainly the case of affine Toda field theories. In sections 2 and 3 we present some of the necessary mathematical preliminaries, introducing the zero curvature condition and showing how it leads naturally to the time-independence of the trace of the monodromy matrix. We also discuss the Hamiltonian and r-matrix formalisms.

In section 4 we give the formulation of the general affine Toda field theory associated to an affine Lie algebra. (The background details on such algebras are given in an appendix.) Section 5 is then given over to applying the ideas discussed previously to the simplest affine Toda field theory, the $\mathfrak{sl}(2)$ affine Toda field theory, which corresponds to the sinh-Gordon equation. We give an explicit construction of a number of the conserved quantities for this system, as well as obtaining the r-matrix and discussing the theory in light-cone coordinates. Section 6 shows how to apply the same methods to $\mathfrak{sl}(3)$ affine Toda field theory.

We conclude in section 7 by showing how we can use any conserved quantity to define a Hamiltonian flow, yielding an infinite hierarchy of integrable systems. In particular we obtain the modified KdV equation from affine $\mathfrak{sl}(2)$ theory expressed in light-cone coordinates.

2 The Zero Curvature Condition

2.1 The condition

We assume we are studying scalar fields $\phi_i = \phi_i(x,t)$ defined over a $1+1$-dimensional spacetime, with periodic boundary conditions, $\phi_i(x,t) = \phi_i(x+R,t)$. We denote partial derivatives as $\frac{\partial}{\partial t}\phi \equiv \partial_t\phi \equiv \dot{\phi}$ and $\frac{\partial}{\partial x}\phi \equiv \partial_x\phi \equiv \phi'$.

Our starting point is to seek two $n \times n$ matrices $U(x,t,\lambda)$, $V(x,t,\lambda)$, depending on the fields $\phi_i(x,t)$ and an additional parameter λ, such that our equations of motion are satisfied iff

$$[\partial_t - V, \partial_x - U] = 0 \tag{2.1}$$

We can regard $U(x,t,\lambda)$ and $V(x,t,\lambda)$ as being the time and space components of a vector bundle connection. This vector bundle has fibre $\mathbb{C}^n$ and base space our $1+1$-dimensional spacetime.

The commutator $[\partial_t - V, \partial_x - U]$ gives the curvature of the connection, which vanishes if the equations of motion are satisfied.

Example: The sinh-Gordon equation for one scalar field ϕ

$$\ddot{\phi} - \phi'' = -4m^2 \sinh 4\phi$$

is equivalent to the zero curvature condition for

$$U(x,t,\lambda) = \begin{pmatrix} \dot{\phi} & m\lambda^{-1}e^{2\phi} + m\lambda e^{-2\phi} \\ m\lambda^{-1}e^{-2\phi} + m\lambda e^{2\phi} & -\dot{\phi} \end{pmatrix}$$

$$V(x,t,\lambda) = \begin{pmatrix} \phi' & -m\lambda^{-1}e^{2\phi} + m\lambda e^{-2\phi} \\ -m\lambda^{-1}e^{-2\phi} + m\lambda e^{2\phi} & -\phi' \end{pmatrix}$$

The connection is only unique up to gauge transformations. Such a transformation corresponds to a change of basis in the fibres of the vector bundle, and so is given by an invertible matrix $g = g(x,t)$ which acts by conjugation:

$$\partial_x - U \to g^{-1}(\partial_x - U)g = \partial_x + g^{-1}\partial_x g - g^{-1}Ug \tag{2.2}$$

and similarly for $\partial_t - V$.

2.2 Parallel transport and the monodromy matrix

Now, the main reason for defining a connection on a vector bundle is to allow us to parallel transport vectors from fibre to fibre. The idea is that we have a vector ψ in the fibre above a point $\gamma(0)$ and a path $\gamma(s)$ in our base space passing through $\gamma(0)$. As we move along the path γ we would like to have a way of also transporting the vector ψ through the corresponding fibres above the points $\gamma(s)$, such that ψ in some sense remains constant.

This is achieved once we specify a connection A_μ, with associated covariant derivative $\partial_\mu - A_\mu$. Then the parallel transport ψ' of ψ along the path $\gamma(s)$ is defined by the condition that the covariant derivative of ψ' along $\gamma(s)$ is zero.

If the curvature of the connection vanishes then parallel transport is well-defined, so that the vectors obtained by parallel transporting along different paths from a point $\gamma(0)$ to $\gamma(1)$ are identical. This can be seen by considering the infinitesimal parallel transport of a vector ψ, under which

$$\psi \to \psi + (\partial_\mu \psi - A_\mu \psi)\, dx^\mu$$

where we sum over μ (in our case $\mu = 0, 1$ with $x_0 \equiv t$ and $x_1 \equiv x$). If we denote by ψ_1 the vector obtained from the infinitesimal parallel transport $x \to x + dx_1^\mu \to x + dx_1^\mu + dx_2^\mu$ and ψ_2 the vector obtained by taking the route $x \to x + dx_2^\mu \to x + dx_2^\mu + dx_1^\mu$, then computing the difference between these vectors we find

$$\begin{aligned} \psi_2 - \psi_1 &= -\left(\partial_\mu A_\nu - \partial_\nu A_\mu - A_\mu A_\nu + A_\nu A_\mu]\right) \psi dx_1^\mu dx_2^\nu \\ &= [\partial_\mu - A_\mu, \partial_\nu - A_\nu]\psi dx_1^\mu dx_2^\nu \end{aligned}$$

where the commutator $[\partial_\mu - A_\mu, \partial_\nu - A_\nu]$ is exactly the curvature of the connection. Thus the zero curvature condition tells us that for any two paths between two points, $\psi_1 = \psi_2$. In particular, the parallel transport of a vector around a closed loop gives us back the same vector we started with.

We can define the operator of parallel transport along a path γ by

$$M_\gamma(x, t, \lambda) = \mathcal{P} \exp\left(\int_\gamma [dx\, U(x, t, \lambda) + dt\, V(x, t, \lambda)] \right) \tag{2.3}$$

where $\mathcal{P}$ is a path-ordering operator. If we consider dividing γ into N segments $\gamma_1, \ldots, \gamma_N$ and let

$$L_n = I + \int_{\gamma_n} (dx\, U + dt\, V)$$

and

$$M_N = L_N \ldots L_1$$

then we can view M_γ as being the $N \to \infty$ limit of M_N. The path-ordering operator ensures that the products $L_i L_j$ are ordered so that they are multiplied together in the correct order, with "earlier" terms appearing on the right. This is a complication due to the non-commutativity of matrix multiplication.

The matrix M_γ can also be thought of as being the solution to the parallel transport equations, $(\partial_x - U)M = 0$ and $(\partial_t - V)M = 0$ with initial condition $M_0 = I$ (here 0 represents a trivial path which is just a single point). In this context one can also view the path-ordering operator purely as notation denoting the solution to these equations. It is clear that given a vector ψ then $\psi' = M_\gamma \psi$ is the parallel transport of ψ along γ.

Let us now consider only the x-direction. The solution to the parallel transport equation $(\partial_x - U)T(x, y, \lambda) = 0$, $T(x, x, \lambda) = I$, is the transfer matrix:

$$T(x, y, \lambda) = \mathcal{P} \exp\left(\int_y^x dx\, U(x, t, \lambda) \right) \tag{2.4}$$

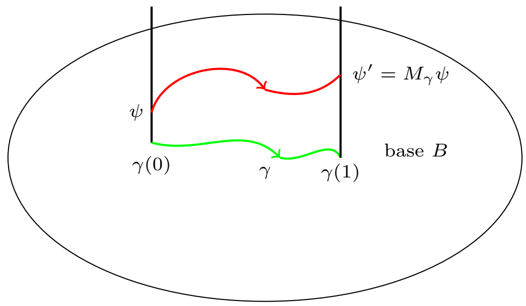

Figure 1: Parallel transport on a vector bundle; the operator M_γ gives the change in the vector ψ when we travel along the path γ in the base space. The straight lines represent individual fibres.

where here our path γ runs from y to x. A special case of the transfer matrix is the monodromy matrix

$$\mathcal{M}(\lambda) \equiv T(0, R, \lambda) = \mathcal{P} \exp \left(\int_0^R dx \, U(x, t, \lambda) \right) \tag{2.5}$$

which is the operator of parallel transport for the entire x domain.

2.3 A generating function for conserved quantities

We will now show that the trace of the monodromy matrix is time-independent, and so can be used as a generating function for conserved quantities. Consider a closed path defined as follows: starting at $x = 0$ and running to $x = R$ at time t_1, then going from time t_1 to time t_2 at $x = R$, and then back to $x = 0$ at time t_2, and finally back to $t = t_1$ at $x = 0$.

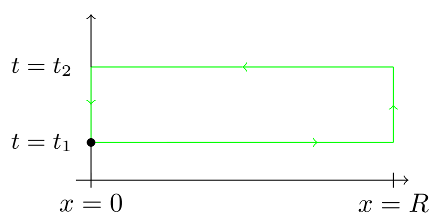

Figure 2: Path used to show that the trace of the monodromy matrix does not depend on time.

If we let

$$S(R) = \mathcal{P} \exp \left(\int_{t_1}^{t_2} dt \, V \right)$$

which is equal to $S(0)$ by periodicity, then the complete parallel transport operator for the closed loop is

$$\mathcal{M}(\lambda)|_{t_1} S(R) \mathcal{M}^{-1}(\lambda)|_{t_2} S^{-1}(R) = I$$

$$\Rightarrow \mathcal{M}(\lambda)|_{t_1} = S(R) \mathcal{M}(\lambda)|_{t_2} S^{-1}(R)$$

so that $\mathcal{M}(\lambda)$ at time t_1 is conjugate to $\mathcal{M}(\lambda)$ at time t_2. It follows that

$$\mathcal{T}(\lambda) = \operatorname{tr} \mathcal{M}(\lambda) \tag{2.6}$$

is constant in time.

We now want to see how we can use the time-independence of $\mathcal{T}(\lambda)$ to generate conserved quantities. The idea is to look at the asymptotic expansions of $\mathcal{T}(\lambda)$ as $\lambda \to 0$ and $\lambda \to \infty$. In these limits we find expansions of the form

$$\log \mathcal{T}(\lambda) \approx \lambda R - \sum_{r=1}^{\infty} H_r^+ \lambda^{-r} + O(e^{-\lambda R}) \tag{2.7}$$

for $\lambda \to \infty$, and

$$\log \mathcal{T}(\lambda) \approx \lambda^{-1} R - \sum_{r=1}^{\infty} H_r^- \lambda^{r} + O(e^{-\lambda^{-1} R}) \tag{2.8}$$

for $\lambda \to 0$. The $H_r^\pm$ are local conserved quantities, which is to say that they are of the form

$$H_r^\pm = \int_0^R dx \, I_r^\pm \tag{2.9}$$

where $I_r^\pm$ is a polynomial in the fields and their derivatives. (A global conserved quantity would consist of multiple integrals.)

This will be illustrated in later sections when we delve into the theory of the sinh-Gordon and $\mathfrak{sl}(3)$ affine Toda field equations in more detail. We give here a rough outline of the method [5] for obtaining the asymptotic expansions: first, we seek gauge transformations which allow us to write U in the form

$$U = \lambda E - U_0 - \sum_{k=1}^{\infty} \lambda^{-k} U_k$$

where the sum should end after a finite number of terms. The motivation for seeking this form of the potential is that for large λ we expect the leading behaviour to be proportional to λ. Here we assume we have chosen a gauge such that E is diagonal.

The next step is to use a gauge transformation

$$\omega = I + \sum_{k=1}^{\infty} \lambda^{-k} \omega_k$$

which transforms U into

$$U' = \lambda E - \sum_{k=0}^{\infty} \lambda^{-k} \tilde{I}_k \tag{2.10}$$

where the matrices $\tilde{I}_k$ are diagonal, proportional to some polynomial I_k in the fields and their derivatives. This gauge transformation cannot be written down immediately - however from

$$(\partial_x - U)\omega = \omega(\partial_x - U')$$

we can derive a recursion relation which at each step gives us $\tilde{I}_k, \omega_{k+1}$ in terms of known quantities. Thus we can explicitly construct the conserved quantities I_k.

3 The Hamiltonian formalism

3.1 Elements of classical field theory

Traditionally, field theory is studied from the Lagrangian viewpoint. The action of the system is given by

$$S = \int dt \, dx \, \mathcal{L}(\phi, \dot{\phi}, \phi') \tag{3.1}$$

and the equations of motion are the Euler-Lagrange equations

$$\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial \phi'} = \frac{\partial \mathcal{L}}{\partial \phi} \tag{3.2}$$

For the field ϕ we can define the conjugate momentum π by

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{3.3}$$

and the associated Hamiltonian is

$$H = \int dx \left(\pi\dot{\phi} - \mathcal{L}\right) \tag{3.4}$$

The equations of motion can then be described by choosing the correct Poisson bracket relationship. For the theories we will study in the next sections, this is

$$\{\pi(x), \phi(y)\} = \delta(x-y) \qquad \{\phi(x), \phi(y)\} = \{\phi(x), \phi(y)\} = 0$$

with the equations of motion taking the form

$$\dot{\pi} = \{H, \pi\} \qquad \dot{\phi} = \{H, \phi\}$$

Note that here we are only discussing the case of a single scalar field ϕ - the generalisation to multiple fields should hopefully be obvious.

3.2 The r-matrix

Recall that the tensor product of two matrices A, B (assumed to be both $n \times n$ for simplicity) is given by

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix} \tag{3.5}$$

where each entry shown is an $n \times n$ block. We can define a Poisson bracket encompassing every entry of two matrices A, B whose entries depend on the fields ϕ_i and their conjugate momenta by:

$$\{A \widetilde{\otimes} B\} = \begin{pmatrix} \{a_{11}, B\} & \dots & \{a_{1n}, B\} \\ \vdots & \ddots & \vdots \\ \{a_{n1}, B\} & \dots & \{a_{nn}, B\} \end{pmatrix} \tag{3.6}$$

where $\{a_{ij}, B\}$ is the matrix consisting of the Poisson brackets of a_{ij} with the entries of B.

Now we take the transition matrix $U(x, t, \lambda)$, and consider its Poisson bracket with $U(y, t, \mu)$. The goal is to find a matrix $r(\lambda, \mu)$ such that we can express this Poisson bracket as a certain commutation relation:

$$\{U(x, \lambda) \widetilde{\otimes} U(y, \mu)\} = [r(\lambda, \mu), U(x, \lambda) \otimes I + I \otimes U(x, \mu)]\delta(x - y) \tag{3.7}$$

This is an important step, as we can then show (see [1] for details) that this implies

$$\{T(x, y, \lambda) \widetilde{\otimes} T(x, y, \mu)\} = [r(\lambda, \mu), T(x, y, \lambda) \otimes T(x, y, \mu)] \tag{3.8}$$

where $T(x, y, \lambda)$ is the transfer matrix (2.4):

$$T(x, y, \lambda) = \mathcal{P} \exp\left(\int_y^x dx\, U(x, \lambda)\right)$$

Obviously this applies to the monodromy matrix $\mathcal{M}(\lambda) = T(0, R, \lambda)$:

$$\{\mathcal{M}(\lambda) \widetilde{\otimes} \mathcal{M}(\mu)\} = [r(\lambda, \mu), \mathcal{M}(\lambda) \otimes \mathcal{M}(\mu)]$$

We now take the trace of this equation. This gives zero on the right-hand side as the trace of a commutator vanishes, and on the left-hand side we use that $\mathrm{tr}\,(A \otimes B) = \mathrm{tr}\, A\, \mathrm{tr}\, B$ to find

$$\{\mathcal{T}(\lambda), \mathcal{T}(\mu)\} = 0$$

where $\mathcal{T}(\lambda) = \mathrm{tr}\, \mathcal{M}(\lambda)$ is the generating function for our conserved quantities.

If we now consider

$$\{\log T(\lambda), \log T(\mu)\} = \frac{2}{T(\lambda) T(\mu)} \{T(\lambda), T(\mu)\} = 0$$

and insert the asymptotic expansions (2.7), (2.8) for $\log T(\lambda)$ we find that all the Poisson brackets between the conserved quantities $H_k^\pm$ must vanish:

$$\{H_k^\zeta, H_m^\eta\} = 0 \qquad \zeta, \eta = \pm \tag{3.9}$$

This shows that we have an infinite number of Poisson commuting conserved quantities, and so guarantees the integrability of the system.

4 Affine Toda Field Theory

We give here the Lagrangian and Hamiltonian formulations of affine Toda field theory. The word "affine" in this case refers to an affine Lie algebra, whose simple roots are used to define the field theory. The necessary ideas from the theory of Lie algebras are contained in the appendix.

Our definition of affine Toda field theory is based on that given in [6]. As our Lagrangian we take

$$L = \int_0^R dx \left(\frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 - 2m^2 \sum_{i=0}^{r} \frac{1}{\alpha_i^2} e^{2\alpha_i \cdot \phi} \right) \tag{4.1}$$

Here ϕ is an r-component vector of scalar fields ϕ_i and the α_i are the simple roots of an affine Lie algebra $\hat{\mathfrak{g}}$. We have periodic boundary conditions, $\phi_i(x,t) = \phi_i(x+R,t)$.

The equations of motion are of the form

$$\partial_t^2 \phi_i - \partial_x^2 \phi_i = -2m^2 \frac{\partial}{\partial \phi_i} \sum_{j=0}^{r} \frac{1}{\alpha_j^2} e^{2\alpha_j \cdot \phi} \tag{4.2}$$

We assume that for each system the associated affine Lie algebra is written in the Cartan-Weyl basis so that for the simple roots α_i we have

$$[H^j, E^{\pm i}] = \pm (\alpha_i)^j E^{\pm i}$$

Here $i = 0, 1, \ldots, r$ with α_0 being minus the highest root of the simple Lie algebra $\mathfrak{g}$ that the affine Lie algebra is constructed from. The H^j, $j = 1, \ldots, r$ are the elements of the Cartan subalgebra of $\mathfrak{g}$ and E^i is the raising operator corresponding to the simple root α_i.

The connections for this system are[1]

$$U = \dot{\Phi} + \lambda^{-1} m e^{\Phi} I_+ e^{-\Phi} + \lambda m e^{-\Phi} I_- e^{\Phi} \tag{4.3}$$

$$V = \Phi' - \lambda^{-1} m e^{\Phi} I_+ e^{-\Phi} + \lambda m e^{-\Phi} I_- e^{\Phi} \tag{4.4}$$

[1]Note this is the same as in [6] but with $\lambda \leftrightarrow \lambda^{-1}$

where

$$\Phi = \sum_{i=1}^{r} \phi_i H^i \quad I_+ = \sum_{i=0}^{r} E^i \quad I_- = \sum_{i=0}^{r} E^{-i} \tag{4.5}$$

The theory may also be formulated using the Hamiltonian

$$H = \int_0^R dx \, \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + 2m^2 \sum_{i=0}^{r} \frac{1}{\alpha_i^2} e^{2\alpha_i \cdot \phi} \right) \tag{4.6}$$

where π is an r-component vector of the momenta π_i associated to the scalar fields ϕ_i. We have Poisson brackets

$$\{\pi_i(x), \phi_j(y)\} = \delta_{ij}\delta(x-y) \qquad \{\pi_i(x), \pi_j(y)\} = \{\phi_i(x), \phi_j(y)\} = 0 \tag{4.7}$$

and also

$$\pi_i = \dot{\phi}_i \tag{4.8}$$

5 Example: $\mathfrak{sl}(2)$

The Lie algebra $\mathfrak{sl}(n)$ consists of traceless real $n \times n$ matrices. A Cartan-Weyl basis for $\mathfrak{sl}(2)$ is given by

$$H^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad E^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad E^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5.1}$$

with

$$[H^1, E^{\pm 1}] = \pm 2E^{\pm 1} \tag{5.2}$$

so that there is just one simple root, $\alpha_1 = 2$, which is also the highest root of the algebra. It follows that the generators corresponding to the affine root $\alpha_0 = -\alpha$ are

$$H^0 = -H^1 \qquad E^0 = E^{-1} \qquad E^{-0} = E^1 \tag{5.3}$$

The Hamiltonian for this case is

$$H = \int_0^R dx \, \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_x \phi)^2 + m^2 \cosh 4\phi \right) \tag{5.4}$$

To illustrate how the Poisson bracket formulation works in practice, we derive the equations of motion:

$$\begin{aligned} \dot{\phi}(x) = \{H, \phi(x)\} &= \int_0^R dy \, \frac{1}{2}\{\pi(y)^2, \phi(x)\} \\ &= \int_0^R dy \, \pi(y)\{\pi(y), \phi(x)\} \\ &= \int_0^R dy \, \pi(y)\delta(y-x) \end{aligned}$$

hence

$$\dot{\phi} = \pi \tag{5.5}$$

and also

$$\begin{aligned}\dot{\pi}(x) = \{H, \pi(x)\} &= \int_0^R dy\, \left(\frac{1}{2}\{(\partial_y\phi(y))^2, \pi(x)\} + m^2\{\cosh 4\phi(y), \pi(x)\}\right) \\ &= \int_0^R dy\, \Big(\partial_y\phi(y)\partial_y\{\phi(y), \pi(x)\} + 4m^2\sinh 4\phi(y)\{\phi(y), \pi(x)\}\Big) \\ &= \int_0^R dy\, \Big(- \partial_y\phi(y)\partial_y\delta(x-y) - 4m^2\sinh 4\phi(y)\delta(x-y)\Big) \\ &= \int_0^R dy\, \partial_y^2\phi(y)\delta(x-y) - 4m^2\sinh 4\phi(x)\end{aligned}$$

having integrated by parts, and so

$$\dot{\pi} - \phi'' = -4m^2\sinh 4\phi \tag{5.6}$$

or

$$\ddot{\phi} - \phi'' = -4m^2\sinh 4\phi \tag{5.7}$$

This is known as the sinh-Gordon equation.

5.1 The connection

Constructing the connection terms as defined in (4.3), (4.4), (4.5) we find

$$U(x,t,\lambda) = \dot{\phi}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + m\begin{pmatrix}0 & \lambda^{-1}e^{2\phi} + \lambda e^{-2\phi}\\ \lambda^{-1}e^{-2\phi} + \lambda e^{2\phi} & 0\end{pmatrix} \tag{5.8}$$

$$V(x,t,\lambda) = \phi'\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + m\begin{pmatrix}0 & -\lambda^{-1}e^{2\phi} + \lambda e^{-2\phi}\\ -\lambda^{-1}e^{-2\phi} + \lambda e^{2\phi} & 0\end{pmatrix} \tag{5.9}$$

and it is straightforward to verify that the zero curvature condition gives the equation of motion

$$\ddot{\phi} - \phi'' = -4m^2\sinh 4\phi \tag{5.10}$$

which agrees with (5.7).

We now seek to gauge transform the connection $U(x,t,\lambda)$ into the form (2.10).This is achieved as follows: the non-constant transformation

$$g_1 = \begin{pmatrix}e^{-\phi} & 0\\ 0 & e^{\phi}\end{pmatrix} \tag{5.11}$$

sends $L \equiv \partial_x - U$ to

$$L' = \partial_x - (\dot{\phi}+\phi')\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \lambda m\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} - \lambda^{-1}m\begin{pmatrix}0 & e^{4\phi}\\ e^{-4\phi} & 0\end{pmatrix}$$

and then applying

$$g_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{5.12}$$

gives us

$$L'' = \partial_x - \lambda\sigma_3 - A - \lambda^{-1}B \tag{5.13}$$

where we have rescaled $\lambda \equiv \lambda m$,

$$A = (\dot{\phi}+\phi')\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = (\dot{\phi}+\phi')\sigma_1 \qquad B = m^2\begin{pmatrix}\cosh 4\phi & -\sinh 4\phi\\ \sinh 4\phi & -\cosh 4\phi\end{pmatrix} = m^2\cosh 4\phi\,\sigma_3 - m^2\sinh 4\phi\, i\sigma_2 \tag{5.14}$$

and the three Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.15}$$

Note that these satisfy $\sigma_i \sigma_j = I\delta_{ij} + i\varepsilon_{ijk}\sigma_k$.

We now claim that there exists a gauge transformation

$$\omega = I + \sum_{k=1}^{\infty} \lambda^{-k} \omega_k \tag{5.16}$$

such that

$$\omega^{-1} L'' \omega = \partial_x - \lambda\sigma_3 + \sum_{k=0}^{\infty} \lambda^{-k} \tilde{I}_k$$

where $\tilde{I}_k$ are diagonal and ω_k are off-diagonal. Now, this means

$$\left(I + \sum_{k=1}^{\infty} \lambda^{-k}\omega_k \right) \left(\partial_x - \lambda\sigma_3 + \sum_{k=0}^{\infty} \lambda^{-k}\tilde{I}_k \right) = \left(\partial_x - \lambda\sigma_3 - A - \lambda^{-1}B \right) \left(I + \sum_{k=1}^{\infty} \lambda^{-k}\omega_k \right)$$

$$\Rightarrow \sum_{k=0}^{\infty} \lambda^{-k}\tilde{I}_k + \sum_{k=1}^{\infty} \lambda^{-k+1}[\sigma_3, \omega_k] = \sum_{k=1}^{\infty} \lambda^{-k}\omega_k' - \sum_{k=1}^{\infty} \lambda^{-k} A\,\omega_k - \sum_{k=1}^{\infty} \lambda^{-k-1} B\,\omega_k - A - \lambda^{-1}B - \sum_{k=1}^{\infty} \lambda^{-k}\omega_k \sum_{r=0}^{\infty} \lambda^{-r}\tilde{I}_r \tag{5.17}$$

The first few terms in this sequence are given by

$$O(\lambda^0) \,:\, \tilde{I}_0 + [\sigma_3, \omega_1] = -A \tag{5.18}$$

$$O(\lambda^{-1}) \,:\, \tilde{I}_1 + [\sigma_3, \omega_2] = \omega_1' - A\,\omega_1 - B - \omega_1\tilde{I}_0 \tag{5.19}$$

$$O(\lambda^{-2}) \,:\, \tilde{I}_2 + [\sigma_3, \omega_3] = \omega_2' - A\,\omega_2 - B\omega_1 - \omega_1\tilde{I}_1 - \omega_2\tilde{I}_0 \tag{5.20}$$

and in general

$$\tilde{I}_k + [\sigma_3, \omega_{k+1}] = \omega_k' - A\omega_k - B\omega_{k-1} - \sum_{r=1}^{k} \omega_r \tilde{I}_{k-r} \tag{5.21}$$

Next we note that for an arbitrary two-by-two matrix $P = P_d + P_{od}$ we have $[\sigma_3, P] = 2\sigma_3 P_{od}$ (where P_d denotes the diagonal part of P and P_{od} denotes the off-diagonal part). Thus we have a unique solution for the recursion relations for $(\tilde{I}_k, \omega_{k+1})$ if we require ω_{k+1} to be off-diagonal and $\tilde{I}_k$ to be diagonal.

We work out the first few terms. Equations (5.18) and (5.14) give

$$\tilde{I}_0 + 2\sigma_3\omega_1 = -(\dot{\phi} + \phi')\sigma_1$$

hence

$$\tilde{I}_0 = 0 \qquad \omega_1 = -\frac{1}{2}(\dot{\phi} + \phi') i\sigma_2 \tag{5.22}$$

The next term is

$$\tilde{I}_1 + 2\sigma_3\omega_2 = -\frac{1}{2}(\dot{\phi} + \phi')' i\sigma_2 + \frac{1}{2}(\dot{\phi} + \phi')^2 i\sigma_1\sigma_2 - m^2 \cosh 4\phi\,\sigma_3 + m^2 \sinh 4\phi\, i\sigma_2$$

hence

$$\tilde{I}_1 = -\left(\frac{1}{2}(\dot{\phi} + \phi')^2 + m^2 \cosh 4\phi \right) \sigma_3 \qquad \omega_2 = -\left(\frac{1}{4}(\dot{\phi} + \phi')' - \frac{1}{2} m^2 \sinh 4\phi \right) \sigma_1 \tag{5.23}$$

Proceeding in this way we obtain

$$\tilde{I}_2 = \frac{1}{8}\partial_x(\dot\phi+\phi')^2 I \qquad \omega_3 = \frac{1}{4}i\sigma_2\left(-\frac{1}{2}(\dot\phi+\phi')'' + 4m^2\cosh 4\phi\phi' + \frac{1}{2}(\dot\phi+\phi')^3 + 2(\dot\phi+\phi')m^2\cosh 4\phi\right) \tag{5.24}$$

and also

$$\begin{aligned}\tilde{I}_3 = \Bigg[\frac{1}{4}(\dot\phi+\phi')\left(-\frac{1}{2}(\dot\phi+\phi')'' + 4m^2\cosh 4\phi\phi' + \frac{1}{2}(\dot\phi+\phi')^3 + 2(\dot\phi+\phi')m^2\cosh 4\phi\right)\\ -\frac{1}{4}(\dot\phi+\phi')'m^2\sinh 4\phi + \frac{1}{2}m^4\sinh^2 4\phi\Bigg]\sigma_3\end{aligned} \tag{5.25}$$

We have thus shown that

$$\partial_x - U \to \partial_x - \lambda\sigma_3 + \sum_{k=1}^{\infty}\lambda^{-k}\tilde{I}_k$$

so

$$U(x,t,\lambda) = \lambda\sigma_3 - \sum_{k=1}^{\infty}\lambda^{-k}\tilde{I}_k^+$$

where $\tilde{I}_k = I_k^+\sigma_3$ for k odd, and $\tilde{I}_k = I_k^+ I$ for k even. Let us now construct the monodromy matrix

$$\mathcal{M}(\lambda) = \mathcal{P}\exp\left(\int_0^R dx\, U(x,\lambda)\right)$$

As U is diagonal the path-ordering operator is not needed and we can easily write

$$\mathcal{M}(\lambda) = \begin{pmatrix}\exp\left(\int_0^R dx\left[\lambda - \sum_{k=1}^{\infty}\lambda^{-k}I_k^+\right]\right) & 0\\ 0 & \exp\left(\int_0^R dx\left[-\lambda + \sum_{k=1}^{\infty}(-1)^{k+1}\lambda^{-k}I_k^+\right]\right)\end{pmatrix} \tag{5.26}$$

We now take the logarithm of the trace:

$$\begin{aligned}\log\mathcal{T}(\lambda) &= \log\left[e^{\lambda R}\exp\left(-\int_0^R dx\sum_{k=1}^{\infty}\lambda^{-k}I_k^+\right)\right]\left[1 + e^{-2\lambda R}\exp\left(-2\int_0^R dx\sum_{k\, odd}\lambda^{-k}I_k^+\right)\right]\\ &\approx \lambda R - \sum_{k=1}^{\infty}\lambda^{-k}\int_0^R dx\, I_k^+ + O(e^{-2\lambda R})\end{aligned}$$

as $\lambda\to\infty$.

We conclude that the quantities

$$H_k^+ = \int_0^R dx\, I_k^+ \tag{5.27}$$

are local conserved quantities of the system. Note however that

$$H_2^+ = \frac{1}{8}\int_0^R dx\,\partial_x(\dot\phi+\phi')^2 = \frac{1}{8}(\dot\phi+\phi')^2\Big|_0^R$$

which vanishes by our periodic boundary conditions, and thus is a trivial conserved quantity. Indeed every even-indexed conserved quantity H_{2k}^+ is trivial.

The same procedure can be carried out with only minor changes to obtain the quantities H_k^-. In this case we start with the gauge transformation $g_1 = \mathrm{diag}\,(e^{\phi}, e^{-\phi})$ so as to obtain $\lambda^{-1}\sigma_3$ as our leading term; we also end up with

$$A = (\dot\phi - \phi')\sigma_1 \qquad B = m^2\cosh 4\phi\,\sigma_3 + m^2\sinh 4\phi\, i\sigma_2 \tag{5.28}$$

and obtain

$$I_0^- = 0 \qquad I_1^- = -\left(\frac{1}{2}(\dot{\phi} - \phi')^2 + m^2 \cosh 4\phi\right) \qquad I_2^- = \frac{1}{8}\partial_x(\dot{\phi} - \phi')^2 \tag{5.29}$$

It is interesting to note that

$$H_1^\pm = -\int_0^R dx \left(\frac{1}{2}(\dot{\phi} \pm \phi')^2 + m^2 \cosh 4\phi\right) \tag{5.30}$$

are related to the Hamiltonian (5.4) by

$$H = -\frac{1}{2}(H_1^+ + H_1^-) \tag{5.31}$$

and the momentum

$$P = \int_0^R dx \, \dot{\phi}\phi' \tag{5.32}$$

by

$$P = -\frac{1}{2}(H_1^+ - H_1^-) \tag{5.33}$$

5.2 Light-cone coordinates

We can also switch to light-cone coordinates, defined by

$$x_\pm = x \pm t \tag{5.34}$$

with the corresponding derivatives

$$\partial_\pm = \frac{1}{2}(\partial_x \pm \partial_t) \tag{5.35}$$

The light-cone connections are

$$U_\pm = \frac{1}{2}(U \pm V) \tag{5.36}$$

with U, V as given at the start of the previous section. Explicitly,

$$U_\pm = \pm\partial_\pm\phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + m\lambda^{\pm 1} \begin{pmatrix} 0 & e^{\mp 2\phi} \\ e^{\pm 2\phi} & 0 \end{pmatrix} \tag{5.37}$$

To express the monodromy matrix in terms of these new operators, we use the fact that the zero curvature condition allows us to deform our contour of integration at will, once the endpoints are fixed. Rather than integrate from $x = 0$ to $x = R$ we choose a two-segment path (figure 3):

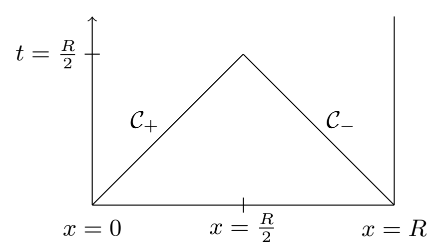

Figure 3: Lightcone factorisation of the monodromy matrix

The contour $\mathcal{C}_+$ is defined by $x = t$ and hence $x_- = 0$ while x_+ runs from 0 to R. The contour $\mathcal{C}_-$ is defined by $x = R - t$, and so by periodicity we have $\phi(R-t, t) \equiv \phi(-t, t)$, hence this is equivalent to $x = -t$ so that $x_+ = 0$ and x_- runs from 0 to R.

The monodromy matrix $\mathcal{M}(\lambda)$ then factors as

$$\mathcal{M}(\lambda) = \mathcal{M}_-(\lambda)\mathcal{M}_+(\lambda) \tag{5.38}$$

where

$$\mathcal{M}_\pm(\lambda) = \mathcal{P}\exp\left(\int_{\mathcal{C}_\pm} dx_\pm\, U_\pm(x_\pm,\lambda)\right) \tag{5.39}$$

Now let us form the two operators $L_\pm = \partial_\pm - U_\pm$ and use the gauge transformation

$$g = \begin{pmatrix} e^{-\phi} & 0 \\ 0 & e^{\phi} \end{pmatrix} \tag{5.40}$$

to obtain

$$L_+ \to \partial_+ - \lambda\sigma_3 - 2\partial_+\phi\,\sigma_1 \tag{5.41}$$

$$L_- \to \partial_- - \lambda^{-1}m^2 \begin{pmatrix} 0 & e^{4\phi} \\ e^{-4\phi} & 0 \end{pmatrix} \tag{5.42}$$

having also rescaled $\lambda \equiv \lambda m$.

We now focus just on L_+; the gauge transformation

$$g = \begin{pmatrix} \lambda & 2\partial_+\phi \\ 0 & \lambda \end{pmatrix} \tag{5.43}$$

takes us to

$$L_+ \to \partial_+ - \lambda \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda^{-1} \begin{pmatrix} 0 & 4(\partial_+\phi)^2 - 2\partial_+^2\phi \\ 0 & 0 \end{pmatrix} \tag{5.44}$$

Letting $\mathcal{U} = 4(\partial_+\phi)^2 - 2\partial_+^2\phi$ and diagonalising we find

$$L_+ \to \partial_+ - \lambda\sigma_3 - \lambda^{-1}\frac{1}{2}\begin{pmatrix} \mathcal{U} & -\mathcal{U} \\ \mathcal{U} & -\mathcal{U} \end{pmatrix} \tag{5.45}$$

We see that L_+ is now in the form (5.13) but with $A = 0$. The gauge transformation $\omega = I + \sum_{k=1}^\infty \lambda^{-k}\omega_k$ can then be used to completely diagonalise L_+ into the final form $\partial_+ - \lambda\sigma_3 - \sum_{k=0}^\infty \lambda^{-k}\tilde{I}_k$: the recursion relations are as in equations (5.18)-(5.21) with $A = 0$:

$$O(\lambda^0)\,:\,\tilde{I}_0 + [\sigma_3,\omega_1] = 0 \tag{5.46}$$

$$O(\lambda^{-1})\,:\,\tilde{I}_1 + [\sigma_3,\omega_2] = \omega_1' - B - \omega_1\tilde{I}_0 \tag{5.47}$$

$$O(\lambda^{-2})\,:\,\tilde{I}_2 + [\sigma_3,\omega_3] = \omega_2' - B\omega_1 - \omega_1\tilde{I}_1 - \omega_2\tilde{I}_0 \tag{5.48}$$

and in general

$$\tilde{I}_k + [\sigma_3,\omega_{k+1}] = \omega_k' - B\omega_{k-1} - \sum_{r=1}^{k}\omega_r\tilde{I}_{k-r} \tag{5.49}$$

We find immediately that $\tilde{I}_0 = \omega_1 = 0$, and assuming as before that $\tilde{I}_k$ is diagonal and ω_k is off-diagonal we then have

$$\tilde{I}_1 = -\frac{\mathcal{U}}{2}\sigma_3 \qquad \omega_2 = \frac{\mathcal{U}}{4}\sigma_1 \tag{5.50}$$

$$\tilde{I}_2 = 0 \qquad \omega_3 = \frac{\mathcal{U}'}{8}i\sigma_2 \tag{5.51}$$

$$\tilde{I}_3 = \frac{1}{8}\mathcal{U}^2\sigma_3 \tag{5.52}$$

We see that we have k, $\tilde{I}_k = I_k^+ \sigma_3$ for k odd and I_k^+ some polynomial in the field and its derivatives, and $\tilde{I}_k = 0$ for k even.

Returning to L_- we apply

$$g = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{5.53}$$

such that

$$L_- \to \partial_- - \lambda^{-1} m^2 \begin{pmatrix} \cosh 4\phi & -\sinh 4\phi \\ \sinh 4\phi & -\cosh 4\phi \end{pmatrix} \tag{5.54}$$

We now turn to the construction of the monodromy matrix. It is clear that

$$\mathcal{M}_+(\lambda) = \begin{pmatrix} \exp\left(\lambda R - \sum_{k=0}^\infty \lambda^{-k} h_k^+\right) & 0 \\ 0 & \exp\left(-\lambda R + \sum_{k=0}^\infty \lambda^{-k} h_k^+\right) \end{pmatrix} \tag{5.55}$$

with $h_k^+ = \int_{\mathcal{C}_+} dx_+ \, I_k^+$. To calculate $\mathcal{M}_-(\lambda)$ we expand

$$\mathcal{M}_-(\lambda) = \mathcal{P} \exp\left(\int_{\mathcal{C}_-} dx_- \, U_-\right) \approx I + \int_{\mathcal{C}_-} dx_- \, U_-$$

which is allowable as we have U_- proportional to λ^{-1}, thus

$$\mathcal{M}_-(\lambda) \approx I + \lambda^{-1} m^2 \int_{\mathcal{C}_-} dx_- \begin{pmatrix} \cosh 4\phi & -\sinh 4\phi \\ \sinh 4\phi & -\cosh 4\phi \end{pmatrix} + O(\lambda^{-2}) \tag{5.56}$$

from which

$$\mathcal{M}(\lambda) = \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix} + \lambda^{-1} m^2 \begin{pmatrix} \left(\int dx_- \cosh 4\phi\right) e^+ & -\left(\int dx_- \sinh 4\phi\right) e^- \\ \left(\int dx_- \sinh 4\phi\right) e^+ & -\left(\int dx_- \cosh 4\phi\right) e^- \end{pmatrix} \tag{5.57}$$

where $e^\pm$ stand for the entries of $\mathcal{M}_+(\lambda)$. Taking the trace we have

$$\mathcal{T}(\lambda) = \operatorname{tr} \mathcal{M}(\lambda) = \exp\left(\lambda R - \sum_{k=0}^\infty \lambda^{-k} h_k^+\right)\left(1 + \lambda^{-1} m^2 \int dx_- \cosh 4\phi + O(e^{-\lambda R})\right) \tag{5.58}$$

so that taking the logarithm for $\lambda \to \infty$ we find

$$\log \mathcal{T}(\lambda) = \lambda R - \lambda^{-1} H_1^+ + O(\lambda^{-2}) \tag{5.59}$$

where

$$H_1^+ = -\left(\int_{\mathcal{C}_+} dx_+ \, (2(\partial_+\phi)^2 - \partial_+^2\phi) + m^2 \int_{\mathcal{C}_-} dx_- \, \cosh 4\phi\right) \tag{5.60}$$

Notice that the term $\partial_+^2\phi$ is a total derivative and so normally would vanish at the boundary. However we do not have periodicity at the end points of the lightcone contours $\mathcal{C}_\pm$. Note also that $2(\partial_+\phi)^2 = \frac{1}{2}(\dot{\phi} + \phi')^2$ and that if we had expanded for $\lambda \to 0$ we would obtain

$$H_1^- = -\left(\int_{\mathcal{C}_-} dx_- \, (2(\partial_-\phi)^2 - \partial_-^2\phi) + m^2 \int_{\mathcal{C}_+} dx_+ \, \cosh 4\phi\right) \tag{5.61}$$

displaying a certain similarity between these and the quantities $H_1^\pm$ found in normal coordinates.

By keeping terms of higher order in the above expansions we could generate the remaining conserved quantities H_3^+, H_5^+. Interestingly these would involve multiple integrals in the exponential series for $\mathcal{M}_-$, seemingly implying that these conserved quantities are not of the local type.

Finally let us note that we can introduce a lightcone Hamiltonian formalism by defining the Poisson brackets

$$
\begin{aligned}
\{\phi(x_+), \phi(x'_+)\} &= \frac{1}{4}\varepsilon(x_+ - x'_+) \\
\{\phi(x_-), \phi(x'_-)\} &= \frac{1}{4}\varepsilon(x_- - x'_-) \\
\{\phi(x_+), \phi(x'_-)\} &= 0
\end{aligned}
\tag{5.62}
$$

where $\phi(x_\pm)$ is understood to mean ϕ restricted to $\mathcal{C}_\pm$, and $\varepsilon(x-y)$ is the step function,

$$
\varepsilon(x-y) = \begin{cases} 1 & x > y \\ 0 & x < y \end{cases} \qquad \partial_x \varepsilon(x-y) = \delta(x-y)
\tag{5.63}
$$

With these Poisson brackets we can recover the sinh-Gordon equation on the lightcone as

$$
\begin{aligned}
\partial_+\partial_-\phi = \{H_1^+, \partial_-\phi\} &= -m^2 \int dx'_- \{\cosh 4\phi(x'_-), \partial_+\phi\} \\
&= -4m^2 \int dx_-\ \sinh 4\phi\, \partial_+\{\phi(x'_-), \phi(x_-)\} \\
&= m^2 \int dx_-\ \sinh 4\phi\, \delta(x'_- - x_-)
\end{aligned}
$$

so that we obtain

$$
\partial_+\partial_-\phi = m^2 \sinh 4\phi
\tag{5.64}
$$

Note that $\partial_+\partial_-\phi = \frac{1}{4}(\partial_x^2 - \partial_t^2)\phi$ so this agrees with (5.7).

5.3 The r-matrix

In order to give a complete picture of the $\mathfrak{sl}(2)$ conserved quantities, we sketch here the derivation of the r-matrix for the connection $U(x,t,\lambda)$. Recall from section 3 that we seek a 4×4 matrix $r(\lambda,\mu)$ such that

$$
\{U(x,\lambda) \widetilde{\otimes} U(y,\mu)\} = [r(\lambda,\mu), U(x,\lambda)\otimes I + I \otimes U(x,\mu)]\delta(x-y)
\tag{5.65}
$$

It is easiest to write our connection (5.8) in terms of the Pauli matrices as

$$
U(x,t,\lambda) = \pi\,\sigma_3 + m(\lambda+\lambda^{-1})\cosh 2\phi\,\sigma_1 + m(\lambda^{-1}-\lambda)\cosh 2\phi\, i\sigma 2
\tag{5.66}
$$

where $\pi = \dot{\phi}$. We further define $k_0(\lambda) = m(\lambda+\lambda^{-1})$ and $k_1(\lambda) = m(\lambda^{-1}-\lambda)$ so that

$$
\begin{aligned}
\frac{1}{2}\{U(x,\lambda)\widetilde{\otimes} U(x,\mu)\} = \Big[&k_0(\mu)\sinh 2\phi\,\sigma_3\otimes\sigma_1 + k_1(\mu)\cosh 2\phi\,\sigma_3\otimes i\sigma_2 - k_0(\lambda)\sinh 2\phi\,\sigma_1\otimes\sigma_3 \\
&- k_1(\lambda)\cosh 2\phi\, i\sigma_2\otimes\sigma_3\Big]\delta(x-y)
\end{aligned}
\tag{5.67}
$$

using the Poisson brackets (4.7). Next we write

$$
\begin{aligned}
[r(\lambda,\mu), U(\lambda)\otimes I + I\otimes U(\mu)]\delta(x-y) = [r(\lambda,\mu), \pi(\sigma_3\otimes I + I\otimes\sigma_3) + \cosh 2\phi(k_0(\lambda)\sigma_1\otimes I + k_0(\mu) I\otimes\sigma_1) \\
+ \sinh 2\phi(k_1(\lambda) i\sigma_2\otimes I + k_1(\mu) I\otimes i\sigma_2)]\delta(x-y)
\end{aligned}
$$

which we want to equal (5.67). The first thing we note is that no terms containing a π factor appear in (5.67), so seek our $r(\lambda,\mu)$ so that it commutes with $\sigma_3\otimes I + I\otimes\sigma_3$. Noting that $(a\otimes b)(c\otimes d) = ac\otimes bd$ we clearly have $[I\otimes I, \sigma_3\otimes I + I\otimes\sigma_3] = [\sigma_3\otimes\sigma_3, \sigma_3\otimes_3 I + I\otimes\sigma_3] = 0$, while as $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for distinct

i, j we see that $\sigma_1 \otimes \sigma_1$ and $\sigma_2 \otimes \sigma_2$ also commute. Taking a small amount of inspiration from [1] we try the following ansatz for $r(\lambda, \mu)$:

$$r(\lambda, \mu) = f(\lambda, \mu)(I \otimes I - \sigma_3 \otimes \sigma_3) + g(\lambda, \mu)(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) \tag{5.68}$$

Our commutator then works out as

$$\begin{aligned} -2f\Big[&\cosh 2\phi(k_0(\lambda)i\sigma_2 \otimes \sigma_3 + k_0(\mu)\sigma_3 \otimes i\sigma_2) + \sinh 2\phi(k_1(\lambda)\sigma_1 \otimes \sigma_3 + k_1(\mu)\sigma_3 \otimes \sigma_1\Big]\delta(x-y) \\ - 2g\Big[&\cosh 2\phi(k_0(\lambda)\sigma_3 \otimes i\sigma_2 + k_0(\mu)i\sigma_2 \otimes \sigma_3 + \sinh 2\phi(k_1(\lambda)\sigma_3 \otimes \sigma_1 + k_1(\mu)\sigma_{\otimes}\sigma_3)\Big]\delta(x-y) \end{aligned} \tag{5.69}$$

We now equate the coefficients of the matrix tensor products in (5.69) and (5.67), finding the four equations

$$\sigma_3 \otimes \sigma_1 \Rightarrow \quad k_0(\mu) = -fk_1(\mu) - gk_1(\lambda) \tag{5.70}$$
$$\sigma_1 \otimes \sigma_3 \Rightarrow \quad k_0(\lambda) = fk_1(\lambda) + gk_1(\mu) \tag{5.71}$$
$$\sigma_3 \otimes i\sigma_2 \Rightarrow \quad k_1(\mu) = -fk_0(\mu) - gk_0(\lambda) \tag{5.72}$$
$$i\sigma_2 \otimes \sigma_3 \Rightarrow \quad k_1(\lambda) = fk_0(\lambda) + gk_0(\mu) \tag{5.73}$$

Subtracting the third equation from the first we have

$$\mu f + \lambda g = \mu \tag{5.74}$$

while subtracting the fourth from the second gives

$$\lambda f + \mu g = -\lambda \tag{5.75}$$

We combine these into the one matrix equation

$$\begin{pmatrix} \mu & \lambda \\ \lambda & \mu \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \mu \\ -\lambda \end{pmatrix} \tag{5.76}$$

which is easily solved by inversion, giving

$$f = \frac{\mu^2 + \lambda^2}{\mu^2 - \lambda^2} \qquad g = -\frac{2\lambda\mu}{\mu^2 - \lambda^2} \tag{5.77}$$

Explicitly, we have found the r-matrix to be

$$r(\lambda, \mu) = \frac{2}{\mu^2 - \lambda^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu^2 + \lambda^2 & -2\lambda\mu & 0 \\ 0 & -2\lambda\mu & \mu^2 + \lambda^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.78}$$

and thus shown the validity of equation (5.65) for $\mathfrak{sl}(2)$ affine Toda field theory, which in turn implies the relation

$$\{T(x, y, \lambda) \widetilde{\otimes} T(x, y, \mu)\} = [r(\lambda, \mu), T(x, y, \lambda) \otimes T(x, y, \mu)] \tag{5.79}$$

for $T(x, y, \lambda)$ the transfer matrix. As explained in section 3, this guarantees that the conserved quantities $H_k^{\pm}$ are mutually Poisson commuting, and so ensures integrability.

6 Example: $\mathfrak{sl}(3)$

A Cartan-Weyl basis for the Lie algebra $\mathfrak{sl}(3)$ consists of the following Cartan generators

$$H^1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad H^2 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{6.1}$$

together with the following step operators

$$E^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.2}$$

$$E^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad E^{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{6.3}$$

$$E^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E^{-3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.4}$$

where the roots are

$$\pm\,\alpha_1 = (\pm 1, 0) \qquad \pm\,\alpha_2 = \left(\mp\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \qquad \pm\,\alpha_3 = \left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) \tag{6.5}$$

We take α_1 and α_2 to be our simple roots, so that $\alpha_3 = \alpha_1 + \alpha_2$ is the highest root. The affine root is then $\alpha_0 = -\alpha_3$ with corresponding step operators $E^0 = E^{-3}$ and $E^{-0} = E^3$.

From (4.6) the Hamiltonian is

$$H = \int_0^R dx\, \left(\frac{1}{2}\pi_1^2 + \frac{1}{2}\pi_2^2 + \frac{1}{2}(\partial_x\phi_1)^2 + \frac{1}{2}(\partial_x\phi_2)^2 + 2m^2\left[e^{2\phi_1} + e^{-\phi_1+\sqrt{3}\phi_2} + e^{-\phi_1-\sqrt{3}\phi_2}\right]\right) \tag{6.6}$$

with the Poisson brackets (4.7) giving $\pi_1 = \dot{\phi}_1, \pi_2 = \dot{\phi}_2$ and so equations of motion

$$\ddot{\phi}_1 - \phi_1'' = -2m^2\left(2e^{2\phi_1} - e^{\phi_1+\sqrt{3}\phi_2} - e^{-\phi_1-\sqrt{3}\phi_2}\right) \tag{6.7}$$

$$\ddot{\phi}_2 - \phi_2'' = -2\sqrt{3}m^2\left(e^{\phi_1+\sqrt{3}\phi_2} - e^{-\phi_1-\sqrt{3}\phi_2}\right) \tag{6.8}$$

6.1 The connection

From equations (4.3), (4.4), (4.5) we have the following connection:

$$U(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} \dot{\phi}_1 + \frac{1}{\sqrt{3}}\dot{\phi}_2 & 0 & 0 \\ 0 & -\dot{\phi}_1 + \frac{1}{\sqrt{3}}\dot{\phi}_2 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}\dot{\phi}_2 \end{pmatrix} + m\begin{pmatrix} 0 & \lambda^{-1}e^{\phi_1} & \lambda e^{-\frac{1}{2}\phi_1-\frac{\sqrt{3}}{2}\phi_2} \\ \lambda e^{\phi_1} & 0 & \lambda^{-1}e^{-\frac{1}{2}\phi_1+\frac{\sqrt{3}}{2}\phi_2} \\ \lambda^{-1}e^{-\frac{1}{2}\phi_1-\frac{\sqrt{3}}{2}\phi_2} & \lambda e^{-\frac{1}{2}\phi_1+\frac{\sqrt{3}}{2}\phi_2} & 0 \end{pmatrix} \tag{6.9}$$

$$V(x,t,\lambda) = \frac{1}{2}\begin{pmatrix} \phi_1' + \frac{1}{\sqrt{3}}\phi_2' & 0 & 0 \\ 0 & -\phi_1' + \frac{1}{\sqrt{3}}\phi_2' & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}}\phi_2' \end{pmatrix} + m\begin{pmatrix} 0 & -\lambda^{-1}e^{\phi_1} & \lambda e^{-\frac{1}{2}\phi_1-\frac{\sqrt{3}}{2}\phi_2} \\ \lambda e^{\phi_1} & 0 & -\lambda^{-1}e^{-\frac{1}{2}\phi_1+\frac{\sqrt{3}}{2}\phi_2} \\ -\lambda^{-1}e^{-\frac{1}{2}\phi_1-\frac{\sqrt{3}}{2}\phi_2} & \lambda e^{-\frac{1}{2}\phi_1+\frac{\sqrt{3}}{2}\phi_2} & 0 \end{pmatrix} \tag{6.10}$$

Applying the zero curvature condition to this connection leads to the equations of motion (6.7), (6.8), as can be directly verified.

We next consider the operator $L = \partial_x - U$. For slight notational convenience we rescale the fields as

$$\varphi_1 \equiv \frac{1}{2}\phi_1 \qquad \varphi_2 \equiv \frac{1}{2\sqrt{3}}\phi_2 \tag{6.11}$$

Then the gauge transformation

$$g_1 = \begin{pmatrix} e^{-\varphi_1-\varphi_2} & 0 & 0 \\ 0 & e^{\varphi_1-\varphi_2} & 0 \\ 0 & 0 & e^{2\varphi_2} \end{pmatrix} \tag{6.12}$$

sends L to the form

$$\partial_x - \lambda m \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \dot\varphi_1 + \varphi_1' + \dot\varphi_2 + \varphi_2' & 0 & 0 \\ 0 & \dot\varphi_2 + \varphi_2' - \dot\varphi_1 - \varphi_1' & 0 \\ 0 & 0 & -2\left(\dot\varphi_2 + \varphi_2'\right) \end{pmatrix} - \lambda^{-1} m \begin{pmatrix} 0 & e^{4\varphi_1} & 0 \\ 0 & 0 & e^{-2\varphi_1+6\varphi_2} \\ e^{-2\varphi_1-6\varphi_2} & 0 & 0 \end{pmatrix}$$

We now also rescale $\lambda \equiv m\lambda$ and we define $f_i = \dot\varphi_i + \varphi_i'$, $i = 1, 2$ as well as $x = e^{4\varphi_1}$, $y = e^{-2\varphi_1+6\varphi_2}$, $z = e^{-2\varphi_1-6\varphi_2}$ before applying the gauge transformation

$$g_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \qquad \omega = e^{2\pi i/3} \tag{6.13}$$

to find that

$$L \to \partial_x - \lambda E - A - \lambda^{-1} B \tag{6.14}$$

where

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{6.15}$$

$$A = \frac{1}{3} \begin{pmatrix} 0 & (1-\omega^2) f_1 - 3\omega f_2 & (1-\omega) f_1 - 3\omega^2 f_2 \\ (1-\omega) f_1 - 3\omega^2 f_2 & 0 & (1-\omega^2) f_1 - 3\omega f_2 \\ (1-\omega^2) f_1 - 3\omega f_2 & (1-\omega) f_1 - 3\omega^2 f_2 & 0 \end{pmatrix} \tag{6.16}$$

$$B = \frac{m^2}{3} \begin{pmatrix} x+y+z & \omega^2 x + \omega y + z & \omega x + \omega^2 y + z \\ x + \omega y + \omega^2 z & \omega^2 (x+y+z) & \omega x + y + \omega^2 z \\ x + \omega^2 y + \omega z & \omega^2 x + \omega y + z & \omega (x+y+z) \end{pmatrix} \tag{6.17}$$

having noted the relation $1 + \omega + \omega^2 = 0$.

Our connection L is now in the same form (5.13) as we obtained for the $\mathfrak{sl}(2)$ connection in the previous section, so that we can apply the gauge transformation $\sum_{k=1}^{\infty} \lambda^{-k} \omega_k$ that sends the connection to the desired form

$$\partial_x - \lambda E + \sum_{k=1}^{\infty} \lambda^{-k} \tilde I_k \tag{6.18}$$

with the same recursion relations as before:

$$\tilde I_0 + [E, \omega_1] = -A \tag{6.19}$$

$$\tilde I_1 + [E, \omega_2] = \omega_1' - A\,\omega_1 - B - \omega_1 \tilde I_0 \tag{6.20}$$

and

$$\tilde I_k + [E, \omega_{k+1}] = \omega_k' - A\omega_k - B\omega_{k-1} - \sum_{r=1}^{k} \omega_r \tilde I_{k-r} \tag{6.21}$$

To solve these we again assume ω_k is off-diagonal and $\tilde I_k$ is diagonal. Note that if

$$\omega_k = \begin{pmatrix} 0 & p & q \\ r & 0 & s \\ t & u & 0 \end{pmatrix}$$

then

$$[E,\omega_k] = \begin{pmatrix} 0 & p(1-\omega) & q(1-\omega^2) \\ r(\omega-1) & 0 & s(\omega-\omega^2) \\ t(\omega^2-1) & u(\omega^2-\omega) & 0 \end{pmatrix}$$

It is then simple to obtain

$$\tilde{I}_0 = 0 \qquad \omega_1 = \frac{1}{3}\begin{pmatrix} 0 & \omega^2 f_1 + (\omega-1)f_2 & \omega f_1 + (\omega^2-1)f_2 \\ f_1 + (\omega-\omega^2)f_2 & 0 & \omega f_1 - (\omega^2-1)f_2 \\ f_1 - (\omega-\omega^2)f_2 & \omega^2 f_1 - (\omega-1)f_2 & 0 \end{pmatrix} \tag{6.22}$$

We find that $\tilde{I}_1 = -(A\omega_1)|_{diag} - B|_{diag}$ is given by

$$\tilde{I}_1 = -\frac{1}{3}\left(f_1^2 + 3f_2^2 + m^2(x+y+z)\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \tag{6.23}$$

so that our first conserved quantity is

$$H_1^+ = -\frac{1}{3}\int_0^R dx\ \left((\dot{\varphi}_1 + \varphi_1')^2 + 3(\dot{\varphi}_2 + \varphi_2')^2 + m^2\left[e^{4\varphi_1} + e^{-2\varphi_1+6\varphi_2} + e^{-2\varphi_1-6\varphi_2}\right]\right) \tag{6.24}$$

or, changing back to the original field variables ϕ_i,

$$H_1^+ = -\frac{1}{3}\int_0^R dx\ \left(\frac{1}{4}(\dot{\phi}_1 + \phi_1')^2 + \frac{1}{4}(\dot{\phi}_2 + \phi_2')^2 + m^2\left[e^{2\phi_1} + e^{-\phi_1+\sqrt{3}\phi_2} + e^{-\phi_1-\sqrt{3}\phi_2}\right]\right) \tag{6.25}$$

By choosing the first gauge transformation (6.12) to be $g_1 = \mathrm{diag}\left(e^{\varphi_1+\varphi_2}, e^{-\varphi_1+\varphi_2}, e^{-2\varphi_2}\right)$ we can carry out the above procedure with the leading term being proportional to λ^{-1}, we then obtain

$$H_1^- = -\frac{1}{3}\int_0^R dx\ \left(\frac{1}{4}(\dot{\phi}_1 - \phi_1')^2 + \frac{1}{4}(\dot{\phi}_2 - \phi_2')^2 + m^2\left[e^{2\phi_1} + e^{-\phi_1+\sqrt{3}\phi_2} + e^{-\phi_1-\sqrt{3}\phi_2}\right]\right) \tag{6.26}$$

so that the Hamiltonian (6.6) is given by

$$H = -3\left(H_1^+ + H_1^-\right) \tag{6.27}$$

while the momentum is

$$P = -3\left(H_1^+ - H_1^-\right) = \int_0^R dx\ \left(\dot{\phi}_1\phi_1' + \dot{\phi}_2\phi_2'\right) \tag{6.28}$$

It is of course also possible to investigate the $\mathfrak{sl}(3)$ in lightcone coordinates and to obtain its r-matrix: indeed a general formula for the r-matrices of affine Toda field theory is well-known, see for instance [5].

7 Deformations of $\mathfrak{sl}(2)$ solutions and KdV flows

7.1 Higher Hamiltonian flows

In the previous sections we have shown how to obtain an infinite number of Poisson-commuting conserved quantities, $H_k^\pm$. We can take each of these quantities as a Hamiltonian, and introduce auxiliary time parameters $t_k^\pm$ so that we can evolve our fields ϕ_i by

$$\frac{\partial \phi_i}{\partial t_k^\pm} = \{H_k^\pm, \phi_i\} \tag{7.1}$$

We thus obtain fields

$$\phi_i = \phi_i(x, t, t_1^{\pm}, t_2^{\pm}, \dots) \tag{7.2}$$

which are solutions not only to the original affine Toda field equations, but also to the infinity of higher Hamiltonian flows defined by (7.1) - this compatibility is ensured by the mutual commutativity of all the $H_k^{\pm}$ amongst themselves and with the original Hamiltonian, which also shows that each of these equations are themselves integrable.

7.2 Obtaining the MKdV equation from lightcone $\mathfrak{sl}(2)$

As an example we consider affine $\mathfrak{sl}(2)$ Toda theory on the lightcone. We focus on the positive lightcone contour $\mathcal{C}_+$ where we had the conserved quantity (5.52) given by

$$h_3^+ = \frac{1}{8}\int_{\mathcal{C}_+} dx_+ \,\left(4(\partial_+\phi)^2 - 2\partial_+^2\phi\right)^2 \tag{7.3}$$

We calculate $\partial_{t_3^+}\partial_+ = \{h_3^+, \partial_+\}$ using the lightcone Poisson brackets (5.62):

$$\{\phi(x_+), \phi(x'_+)\} = \frac{1}{4}\varepsilon(x_+ - x'_+)$$

Dropping the subscript $+$ for convenience, we have

$$\begin{aligned}
\{h_3^+, \partial\phi\} &= \int dx\,\left(2(\partial\phi)^2 - \partial^2\phi\right)\{2(\partial\phi)^2 - \partial^2\phi, \partial'\phi(x')\} \\
&= \int dx\,\left(2(\partial\phi)^2 - \partial^2\phi\right)\left(4\partial\phi\{\partial\phi, \partial'\phi(x')\} - \partial^2\partial'\{\phi(x), \phi(x')\}\right) \\
&= -\frac{1}{4}\int dx\,\left(2(\partial\phi)^2 - \partial^2\phi\right)\left(4\partial\phi\partial\delta(x-x') - \partial^2\delta(x-x')\right) \\
&= \partial\Big(\left(2(\partial\phi)^2 - \partial^2\phi\right)\partial\phi\Big) + \frac{1}{4}\partial^2\Big(2(\partial\phi)^2 - \partial^2\phi\Big)
\end{aligned}$$

having integrated by parts. Evaluating the derivatives, we find

$$\partial_{t_3^+}\partial_+\phi = -\frac{1}{4}\partial_+^4\phi + 6(\partial_+\phi)^2\partial_+^2\phi \tag{7.4}$$

and rescaling $u = 2\partial_+\phi$, $\tau = \frac{1}{4}t_3^+$, we find the modified KdV equation in the form

$$u_\tau + u_{+++} - 6u^2 u_+ = 0 \tag{7.5}$$

with subscripts denoting derivatives.

8 Conclusion

We have thus shown how to exploit the zero curvature condition

$$[\partial_t - V, \partial_x - U] = 0 \tag{8.1}$$

to construct an infinite number of conserved quantities. The fact that this is possible is a remarkable feature of integrable systems.

Although we used the examples of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ in this review, the general methodology should apply to any system which admits a zero curvature representation. Hence the ideas we have discussed provide powerful tools for many problems.

Asides from the zero curvature condition, the other central theme we have discussed is the Hamiltonian formalism. The use of Poisson brackets and the introduction of r-matrices provided important insights into the integrability of affine Toda field theory, and again apply extensively throughout the field of integrable systems.

We mention here one subtlety that arose in the writing of this report. In the theory of affine $\mathfrak{sl}(2)$ in lightcone coordinates we used the unusual gauge transformation (5.43). This was to ensure the appearance of the total derivative term $\partial_+^2\phi$ in the our answer for H_1^+, in agreement with for instance [9]. Had we carried on without this gauge transformation we would not have obtained this term, which is slightly mysterious. This issue seems to involve the non-periodicity of the lightcone fields, which should require an additional gauge transformation of the monodromy matrices $\mathcal{M}_\pm(\lambda)$ - however when we implemented this correction we failed to find the necessary terms. It would seem there is some subtle point here that we did not have time to cover in this review.

A Lie algebras and affine Lie algebras

We review some background material on Lie algebras, drawing on the presentations in [7], [8].

A.1 Lie algebras

We consider first an ordinary finite-dimensional compact connected Lie group G with associated Lie algebra $\mathfrak{g}$. Recall that a Lie group is a group with the structure of a smooth manifold, and its Lie algebra consists of the tangent space to the group at the identity, the idea being that in the vicinity of the identity we can expand a general group element g as

$$g \approx 1 - iT^a x_a$$

where x_a are some infinitesimal parameters and the T^a are elements of the Lie algebra, called the generators of the group.

Mathematically the Lie algebra has a vector space structure, with the T_a forming a basis. We also have a bilinear skew-symmetric form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie product, which satisfies Jacobi's identity: $[X,[Y,Z]]+$ cyclic permutations $= 0$ for any $X, Y, Z \in \mathfrak{g}$. In the case of a matrix Lie group (or a matrix representation of a Lie group) the Lie product is simply the matrix commutator. We will implicitly assume we are dealing with this case in what follows.

The starting point in classifying Lie algebras is to find the largest possible set of (Hermitian) generators which commute with each other under the Lie product. This set is called the Cartan subalgebra, and we denote its elements (the Cartan generators) by H^i, $i = 1,\ldots,r$. We then take a basis for $\mathfrak{g}$ in which the remaining generators E^α satisfy the eigenvalue equations

$$[H^i, E^\alpha] = \alpha^i E^\alpha \qquad i = 1,\ldots r$$

The r-component vector $\alpha = (\alpha^i)$ is called a root vector. By conjugating the above equation we find that $E^{\alpha\dagger} \equiv E^{-\alpha}$ has root vector $-\alpha$. In fact $\pm\alpha$ are the only possible scalar multiples of a root α which are also roots. Note that for any of the Cartan generators H^i the root vector is the zero vector.

The set $\{H^i, E^\alpha : i = 1,\ldots,r, \alpha \text{ a root}\}$ then forms a basis for the Lie algebra known as the Cartan-Weyl basis. The matrices $E^{\pm\alpha}$ can be thought of as raising and lowering operators, as if we suppose v_μ is a common eigenvector of the H^i such that $H^i v_\mu = \mu^i v_\mu$, then $H^i E^{\pm\alpha} v_\mu = [H^i, E^{\pm\alpha}]v_\mu + E^{\pm\alpha} H^i v_\mu = (\pm\alpha^i + \mu^i)E^{\pm\alpha} v_\mu$.

The Jacobi identity with $X = H^i$, $Y = E^\alpha$ and $Z = E^\beta$ shows that

$$[H^i, [E^\alpha, E^\beta]] = (\alpha+\beta)^i [E^\alpha, E^\beta]$$

and so for $\alpha \neq -\beta$, $[E^\alpha, E^\beta] = \varepsilon(\alpha,\beta)E^{\alpha+\beta}$ if $\alpha+\beta$ is also a root, for some number $\varepsilon(\alpha,\beta)$. If $\alpha+\beta$ is not a root, then $[E^\alpha, E^\beta] = 0$. For the final case, if $\alpha = -\beta$ then $[E^\alpha, E^\beta]$ commutes with all H^i and so must be expressible as a linear combination of the H^i. In fact we can choose our basis such that

$$[E^\alpha, E^{-\alpha}] = \frac{2\alpha \cdot H}{\alpha^2} \qquad \alpha \cdot H \equiv \alpha^1 H^1 + \cdots + \alpha^r H^r$$

The benefit of this is that for all roots α then $E^{\pm\alpha}$ and $\frac{2\alpha \cdot H}{\alpha^2}$ forms an algebra isomorphic to that of $\mathfrak{su}(2)$, which is well known to physicists as corresponding to the quantum mechanical description of angular momentum. This then allows us to define representations of our Lie algebra $\mathfrak{g}$.

However for our purposes, the only point we want to note from the theory of $\mathfrak{su}(2)$ is that $\frac{2\alpha \cdot H}{\alpha^2}$ has integral eigenvalues. In particular if we consider the adjoint representation defined by the action of the Lie algebra on itself

$$\text{ad X}(Y) = [X, Y]$$

then clearly $\alpha \cdot H$ has eigenvalues $\alpha \cdot \beta$ for β a root, and we conclude that for all roots α, β

$$\frac{2\alpha \cdot \beta}{\alpha^2} \in \mathbb{Z}$$

Let us now consider the space of roots α. We can choose an r-dimensional basis $\{\alpha_{(i)} : i = 1, \ldots, r\}$ such that any root is expressed as a linear combination

$$\alpha = \sum_{i=1}^{r} n_i \alpha_{(i)}$$

where the n_i are integers, which are either all positive or zero (in which case we say the root α is positive) or all negative or zero (in which case we say the root α is negative). The sum of all the n_i for a root α is called the height of the root, while the basis elements $\alpha_{(i)}$ are called simple roots.

We end our discussion by noting the remarkable fact that any Lie algebra is completely determined by its simple roots, as it is possible to reconstruct the entire root system by taking appropriate linear combinations of simple roots. There is even a way to encode all the information about the simple roots in a matrix called the Cartan matrix, with entries defined by

$$A_{ij} = 2\frac{\alpha_{(i)} \cdot \alpha_{(j)}}{\alpha_{(j)}^2} \qquad i, j = 1, \ldots, r$$

For any Cartan matrix A all the entries are integers, with 2s on the diagonal and the off-diagonal entires zero or negative.

It is also possible to represent this graphically, using Dynkin diagrams, where each simple root is represented by a dot, with 0,1,2 or 3 lines joining the dots depending on whether the angle between the roots is 0, 120, 135 or 150 degrees. Longer roots are represented by shaded dots (for a simple Lie algebra there are at most two roots lengths).

A.2 Affine Lie algebras

An affine Lie algebra is a certain type of infinite-dimensional Lie algebra. The starting point in their construction is to consider $\mathcal{G}$, the set of smooth maps from the circle S^1 into a finite-dimensional compact connected Lie group G. This forms an infinite-dimensional Lie group known as the loop group of G.

Let $\gamma : z \to \gamma(z)$ be an element of $\mathcal{G}$. We can write $\gamma(z)$ as

$$\gamma(z) = \exp(-T^a \theta_a(z)) \approx 1 - iT^a \theta_a(z)$$

where the T_a are the generators of the Lie group (i.e. the basis elements of the Lie algebra), and $\theta_a(z)$ are certain functions defined on the circle. On the right-hand side we have expanded $\gamma(z)$ about the identity element. If we now also expand $\theta_a(z) = \sum_{-\infty}^{\infty} \theta_a^{-n} z^n$ as a Laurent polynomial in z we find

$$\gamma(z) \approx 1 - i \sum_{n,a} T^a_{-n} \theta^n(z)$$

which implies we should take

$$T_n^a \equiv T^a z^n$$

as generators for the loop algebra. The defining commutation relations are

$$[T_n^a, T_m^n] = i f_{abc} T_{n+m}^c$$

Next we append a single extra element k to the Lie algebra, with the defining property that it commutes with all elements of the Lie algebra. For a simple compact Lie group the element k, which is known as a central extension of the Lie algebra, enters into the commutation relations as

$$[T_m^a, T_n^b] = i f_{abc} T_{m+n}^c + km\delta^{ab}\delta_{m+n,0}$$

In all cases that we will discuss, however, we may set $k = 0$. Now, k also lies in the Cartan subalgebra, the elements of which are now written as H_0^i. The equations for the root vectors are

$$[H_0^i, E_n^\alpha] = \alpha^i E_n^\alpha \qquad [k, E_n^\alpha] = 0$$

This would imply we take as roots $(\alpha, 0)$ - however these are infinitely degenerate with respect to k. To remove this degeneracy we add one last element to our algebra, in the form of a derivation d which satisfies

$$[d, T_n^a] = nT_n^a \qquad [d, k] = 0 \qquad d^\dagger = d$$

Concretely, we can write $d = z\frac{d}{dz}$.

We now have all the components of our affine Lie algebra in place. The Cartan subalgebra consists of H_0^i, k and d, with roots

$$(\alpha, 0, n)$$

corresponding to the step operators E_α^n, and

$$n\delta = (0, 0, n)$$

corresponding to the generators H_n^i.

Our new simple roots are given by

$$a_i = (\alpha_{(i)}, 0, 0) \quad i = 1, \ldots, r \qquad a_0 = (-\psi, 0, 1)$$

where the $\alpha_{(i)}$ are the simple roots of the original Lie algebra $\mathfrak{g}$ and ψ is the highest root of that algebra.

The reasoning here is that for a given value of n we can form any root $(\alpha, 0, n)$ by adding simple roots to $na_0 = (-n\psi, 0, n)$. Thus the roots a_0, a_i form a basis for the root system of the affine Lie algebra.

Finally, we define a scalar product by

$$(\alpha, k, n) \cdot (\beta, l, m) = \alpha \cdot \beta + km + ln$$

(This is a result of seeking a symmetric scalar product invariant under the action of the group G on the Lie algebra.) Hence in practice for affine roots $(\alpha, 0, n)$, $(\beta, 0, m)$ the scalar product agrees with that for non-affine roots: $(\alpha, 0, n) \cdot (\beta, 0, m) = \alpha \cdot \beta$.

In practice this means when constructing an affine Lie algebra out of a normal finite-dimensional one $\mathfrak{g}$, we start with the root system of $\mathfrak{g}$ and find the highest root ψ. Then we define the affine simple root $\alpha_{(0)} = -\psi$ and so essentially have the complete root system. We may also extend our definition of the Cartan matrix so that i, j run from 0 to r rather than from 1 to r, and append an extra node to the corresponding Dynkin diagram.

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