QCD and Renormalization: Dependence of Observables on the Renormalization Scale in Higher Orders of the Strong Coupling α_s

Elina Fuchs
DESY Summerstudent Program 2009

Supervisor: Dr. Markus Diehl
Theory Group

September 10, 2009

Abstract

The dependence of the hadronic in Higgs decay on the renormalization scale μ is analyzed in different orders up to NNNLO. According to the requirement of Fastest Apparent Convergence, the Principle of Minimal Sensitivity and the procedure by Brodsky, Lepage and Mackenzie, the points with the least dependence are determined. These optimized scales differ from the simple choice $Q = \mu$. The solutions are not unique beyond NLO, but the different principles lead to comparable results.

Contents

1 Renormalization of the strong coupling

Despite its name, the effective QCD coupling $\alpha_s = \frac{g_s^2}{4\pi}$ is not a constant. Due to the renormalization of strong vertices $\alpha_s(\mu)$ runs with the energy scale μ. After renormalization at μ_0, the values of $\alpha_s(\mu)$ can be calculated for any μ with the help of the renormalization group equation (3) [1].

1.1 Quantum Chromodynamics

QCD behaves quite differently from QED. While QED is an Abelian gauge theory, the non-Abelian nature of QCD allows for the self-interaction of the gauge bosons (gg-coupling). Virtual gluons lead to the anti-shielding effect (opposite to QED) which determines the sign of the beta function (3): the strong coupling decreases.

1.2 Perturbation theory

Physically relevant variables are expanded in powers of the strong coupling α_s. This expansion is truncated after a certain order and therefore only exact up to higher terms. For example, the strong coupling at the momentum transfer Q and any observable C can be expressed in the following way.

$$
\begin{aligned}
\alpha_s(Q) &= \alpha_s(\mu) \sum_{i=0}^{\infty} a_i \left(\frac{Q}{\mu}\right) \alpha_s^i(\mu) \\
&= \alpha_s(\mu) \cdot \left(1 + a_1 \left(\frac{Q}{\mu}\right) \alpha_s(\mu) + a_2 \left(\frac{Q}{\mu}\right) \alpha_s^2(\mu) + a_3 \left(\frac{Q}{\mu}\right) \alpha_s^3(\mu)\right) + \mathcal{O}(\alpha_s^5) \qquad (1) \\
C &= \alpha_s^m(\mu) \cdot \left[C_0 + \alpha_s(\mu) C_1 \left(\frac{Q}{\mu}\right) + \alpha_s^2(\mu) C_2 \left(\frac{Q}{\mu}\right) + \alpha_s^3(\mu) C_3 \left(\frac{Q}{\mu}\right)\right] + \mathcal{O}(\alpha_s^{m+4}) \qquad (2)
\end{aligned}
$$

1.3 Renormalization group equation

In the renormalization group equation (RGE) (3) the β-function describes the change of α_s with μ.

$$
\begin{aligned}
\frac{d}{d\ln(\mu^2)} \alpha_s(\mu) &= \beta(\alpha_s(\mu)) \qquad (3) \\
&= -\alpha_s^2(\mu) \cdot [\beta_0 + \beta_1 \alpha_s(\mu) + \beta_2 \alpha_s^2(\mu) + \beta_3 \alpha_s^3(\mu)] + \mathcal{O}(\alpha_s^5(\mu))
\end{aligned}
$$

$\beta(\alpha_s(\mu))$ is a solution of the RGE [2] and its Taylor expansion of $\beta(\alpha_s)$ is determined by the coefficients β_i[1]. The first and second one are scheme independent, but for β_i, $i \geq 2$ the renormalization scheme must be specified. Generally they depend on the number of

[1] Comparing different articles, the β_i often differ by factors of $\frac{1}{4\pi}$.

fermion flavours n involved at a certain energy. In $\overline{MS}$ they read [3]:

$$\begin{aligned}
\beta_0 &= \frac{1}{4\pi}\left[11-\frac{2}{3}n\right]\\
\beta_1 &= \frac{1}{(4\pi)^2}\left[102-\frac{38}{3}n\right]\\
\beta_2 &= \frac{1}{(4\pi)^3}\left[\frac{2857}{2}-\frac{5033}{18}n+\frac{325}{54}n^2\right]\\
\beta_3 &= \frac{1}{(4\pi)^4}\left[\frac{149753}{6}+3564\zeta(3)-\left(\frac{1078361}{162}+\frac{6508}{27}\zeta(3)\right)n+\left(\frac{50065}{162}+\frac{6472}{81}\zeta(3)\right)n^2+\frac{1093}{729}n^3\right]
\end{aligned}$$

The ζ-function has the values $\zeta(2)=\frac{\pi^2}{6}$ and $\zeta(3)=1.202$.

These coefficients are needed for the running coupling in 4-loop approximation [4]. The short notation $L \equiv \ln(\mu^2/\Lambda_{\overline{MS}})$ is used[2].

$$\begin{aligned}
\alpha_s(\mu^2) = &\frac{1}{\beta_0 L}-\frac{1}{\beta_0^3 L^2}\beta_1\ln L+\frac{1}{\beta_0^3 L^3}\left(\frac{\beta_1^2}{\beta_0^2}\left(\ln^2 L-\ln L-1\right)+\frac{\beta_2}{\beta_0}\right)\\
&+\frac{1}{\beta_0^4 L^4}\left(\frac{\beta_1^3}{\beta_0^3}\left(-\ln^3 L+\frac{5}{2}\ln^2 L+2\ln L-\frac{1}{2}\right)-3\frac{\beta_1\beta_2}{\beta_0^2}\ln L+\frac{\beta_3}{2\beta_0}\right) \qquad (4)
\end{aligned}$$

This function is verified by comparing its derivative to the β-function. They differ only by terms of $\mathcal{O}(L^{-6})$. Thus (4) obeys the RGE to the fourth order.
In order to plot $\alpha_s(\mu)$ the parameter $\Lambda_{\overline{MS}}$ must be calculated. Therefore the experimental results of e.g. $\alpha_s(M_Z)$ and $\alpha_s(M_\tau)$ are used as reference values for the number of flavours of interest. Due to $M_\tau < M_b < M_Z$ here it is $n = 4$ or 5, respectively. These reference values are helpful because they have been measured experimentally with a high precision. The corresponding Λ's are used to compute the strong coupling also at other scales. The Λ is determined which fulfils $\alpha_s(\Lambda_{\overline{MS}}) = \alpha_s(M_X)$. In case of τ the fitted value $\alpha_s(M_\tau) = 0.32$ was taken instead of the measured 0.34 since it deviates strongly from the fit average [6] which would influence the $\Lambda_{\overline{MS}}$ significantly. As a cross check α_s is evaluated at the bottom threshold with n =4 or 5. The values in tab.(1) agree up to 0.4%.

Table 1: Determintation of $\Lambda_{\overline{MS}}$

M_X [GeV] [7]	$\alpha_s(M_X)$	n	$\Lambda_{\overline{MS}}$	$\alpha_s^{(n)}(M_b)$
$M_Z = 91.1876$	0.1176 [7]	5	0.2037	0.223
$M_\tau = 1.77684$	0.32 [6]	4	0.2862	0.224

[2]Be careful, in both of Bethke's papers [5], [6] the logarithm is forgotten in the definition of L!

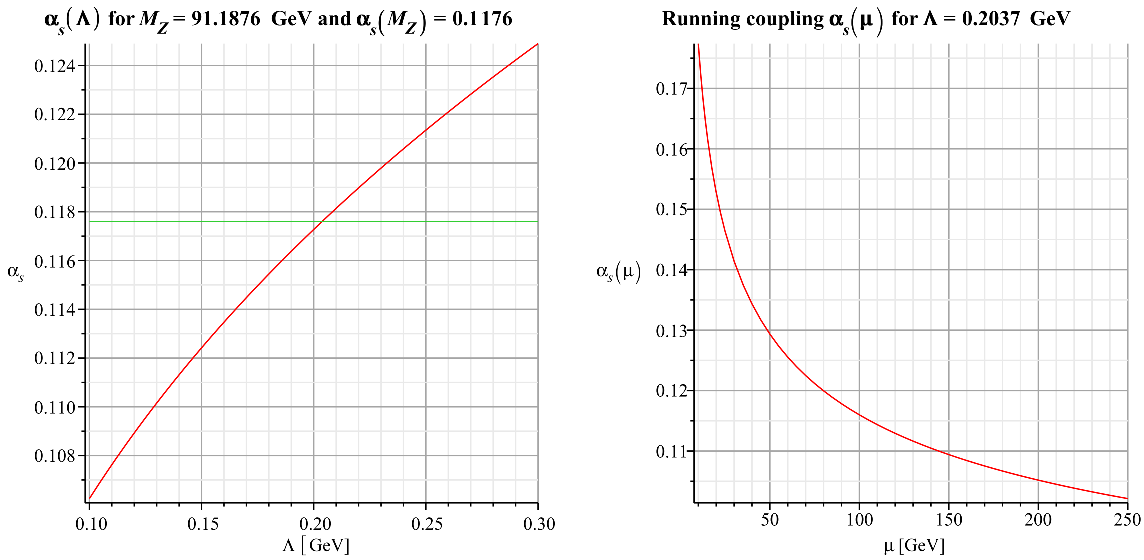

Figure 1: Determination of Λ (left) and running coupling (right) with $n = 5$ and $\Lambda = 0.2037$

2 Scale dependence

2.1 Arbitrary choice of the renormalization scale

Having specified a renormalization scheme, the choice of the renormalization scale μ is still arbitrary. A priori there is no unique principle to resolve the scale ambiguity. Nevertheless, physical observables must not depend on this arbitrary scale since they are well-defined, measurable quantities. This is provided as long as the infinite perturbation series is taken into account. The coefficients themselves can be scale dependent, but the depencies of different order cancel each other. However, if the series is truncated at the nth order, a scale dependence remains. In order to minimize the theoretical uncertainties it is crucial to reduce the μ-dependence by an optimal fixing of the scale. Furthermore, it would be interesting to estimate from the nth order also the contribution of the (n+1)th order.

2.1.1 Calculation of coefficients

A physical observable does not depend on μ, but the the coefficients $C_i\left(\frac{Q}{\mu}\right)$ depend logarithmically on the ratio of the physical scale Q and the renormalization scale μ [8]. In the following the coefficients in eq.(2) are calculated up to the third order by expanding $\alpha_s(Q)$ and C in terms of $\alpha_s(\mu)$ and by using the renormalization group equation (3). In each order one expects the structure

$$C_i\left(\frac{Q}{\mu}\right) = C_i^{(0)} + C_i^{(1)}\mathrm{ln}\left(\frac{\mathrm{Q}^2}{\mu^2}\right) + \mathrm{C}_\mathrm{i}^{(2)}\mathrm{ln}^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) + ... + \mathrm{C}_\mathrm{i}^{(\mathrm{i})}\mathrm{ln}^\mathrm{i}\left(\frac{\mathrm{Q}^2}{\mu^2}\right)$$

.

Differentiating both the left and right hand side of eq.(1) allows for a comparison of coefficients.

$$\begin{aligned}
\frac{d}{d\mathrm{lnQ}^2}(l.h.s.) &= \frac{d}{d\mathrm{lnQ}^2}\alpha_s(Q) = \beta(\alpha_s(Q)) \\
&= -\alpha_s^2(Q)\cdot(\beta_0 + \beta_1\alpha_s(Q) + \beta_2\alpha_s^2(Q)) + O(\alpha_s^5) \qquad (5)
\end{aligned}$$

Then the square of (1) is inserted into eq.(5), where the arguments are left out for brevity $\left(a_i \equiv a_i\left(\frac{Q}{\mu}\right)\right)$. Therefore the binomial theorem

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

is applied to calculate $\alpha_s^m(Q)$ with an arbitrary power m to order $\mathcal{O}(\alpha_s^{m+3}(\mu))$.

$$\begin{aligned} \alpha_s^m(Q) = &\ \alpha_s^m(\mu) \cdot \left[1 + a_1\alpha_s(\mu) + a_2\alpha_s^2(\mu) + a_3\alpha_s^3(\mu)\right]^m \\ = &\ \alpha_s^m(\mu) \cdot [1^m + m a_1 \alpha_s(\mu) + (m a_2 + \frac{m(m-1)}{2} a_1^2)\alpha_s^2(\mu) \\ &+ (m a_3 + m(m-1)a_1 a_2 + \frac{m(m-1)(m-2)}{6} a_1^3)\alpha_s^3(\mu)] + \mathcal{O}(\alpha_s^{m+4}(\mu)) \end{aligned} \tag{6}$$

$$\begin{aligned} \Rightarrow \alpha_s^2(Q) = &\ \alpha_s^2(\mu) \cdot [1 + 2a_1\alpha_s(\mu) + (2a_2 + a_1^2)\alpha_s^2(\mu)] + \mathcal{O}(\alpha_s^5(\mu)) \\ \Rightarrow \beta(\alpha_s(Q)) = &\ -\alpha_s^2(\mu) \cdot [\beta_0 + (\beta_1 + 2\beta_0 a_1)\alpha_s(\mu) \\ &+ (\beta_0 a_1^2 + \beta_1 a_1 + \beta_2 + 2(\beta_1 a_1 + \beta_0 a_2))\alpha_s^2(\mu)] + \mathcal{O}(\alpha_s^5(\mu)) \end{aligned} \tag{7}$$

The right hand side of (1) yields the derivatives of the a_i.

$$\frac{d}{d\,\mathrm{lnQ}^2}\alpha_s(Q) = \frac{d\,a_1}{d\,\mathrm{lnQ}^2}\alpha_s^2(\mu) + \frac{d\,a_2}{d\,\mathrm{lnQ}^2}\alpha_s^3(\mu) + \frac{d\,a_3}{d\,\mathrm{lnQ}^2}\alpha_s^4(\mu) \tag{8}$$

Comparing the coefficients of eq. (7) and (8) leads to:

$$\frac{d\,a_1}{d\,\mathrm{lnQ}^2} = -\beta_0 \tag{9}$$

$$\frac{d\,a_2}{d\,\mathrm{lnQ}^2} = -\beta_1 - 2\beta_0\ a_1 \tag{10}$$

$$\frac{d\,a_3}{d\,\mathrm{lnQ}^2} = -\beta_2 - \beta_0 a_1^2 - 3\beta_1 a_1 - 2\beta_0 a_2 \tag{11}$$

This system of equations is solved by integrating (9) and inserting it in (10), then these two are used for eq. (11).

$$a_1\left(\frac{Q}{\mu}\right) = -\beta_0 \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \tag{12}$$

$$a_2\left(\frac{Q}{\mu}\right) = +\beta_0^2 \ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) - \beta_1 \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \tag{13}$$

$$a_3\left(\frac{Q}{\mu}\right) = -\beta_0^3 \ln^3\left(\frac{\mathrm{Q}^2}{\mu^2}\right) + \frac{5}{2}\beta_0\beta_1 \ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) - \beta_2 \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \tag{14}$$

The expansion of C must also hold for the special case $Q = \mu$.

$$\begin{aligned} C &= \alpha_s^m(\mu) \cdot \left[C_0 + C_1\left(\frac{Q}{\mu}\right)\alpha_s(\mu) + C_2\left(\frac{Q}{\mu}\right)\alpha_s^2(\mu) + C_3\left(\frac{Q}{\mu}\right)\alpha_s^3(\mu)\right] \\ &= \alpha_s^m(Q) \cdot \left[C_0 + C_1(1)\alpha_s(Q) + C_2(1)\alpha_s^2(Q) + C_3(1)\alpha_s^3(Q)\right] \end{aligned}$$

Taking $\alpha_s^3(Q) = \alpha_s^3(\mu) \cdot (1 + 3a_1\alpha_s(\mu))$ from (6) with $m = 3$, one gets the μ-dependent coefficients C_i by sorting the terms of C by powers of $\alpha_s(\mu)$ and using the α-coefficients a_i known from (12-14).

$$
\begin{aligned}
C = &\alpha_s^m(\mu) \cdot \Big[\quad C_0 + \alpha_s(\mu) \cdot \underbrace{[C_1(1) + mC_0a_1]}_{C_1(Q/\mu)} \\
&\alpha_s^2(\mu) \cdot \underbrace{[C_2(1) + (m+1)a_1C_1(1) + ma_2C_0 + \frac{m(m-1)}{2}a_1^2C_0]}_{C_2(Q/\mu)} \\
&\alpha_s^3(\mu) \cdot [C_3(1) + (m+2)C_2(1)a_1 + (m+1)C_1(1)a_2 + \frac{m(m+1)}{2}C_1(1)a_1^2 \\
&+ m(m-1)C_0a_1a_2 + \frac{m(m-1)(m-2)}{6}C_0a_1^3 + mC_0a_3] \quad \Big]
\end{aligned}
$$

Now the coefficients can be read off:

$$
C_1\left(\frac{Q}{\mu}\right) = C_1(1) - \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \mathrm{m}\beta_0\mathrm{C}_0 \tag{15}
$$

$$
C_2\left(\frac{Q}{\mu}\right) = C_2(1) - \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot [(\mathrm{m}+1)\beta_0\mathrm{C}_1(1) + \mathrm{m}\beta_1\mathrm{C}_0] + \ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \frac{\mathrm{m}(\mathrm{m}+1)}{2}\beta_0^2\mathrm{C}_0 \tag{16}
$$

$$
\begin{aligned}
C_3\left(\frac{Q}{\mu}\right) = &C_3(1) + (m+2)C_2(1)a_1 + (m+1)C_1(1)a_2 + \frac{m(m+1)}{2}C_1(1)a_1^2 \\
&+ m(m-1)C_0a_1a_2 + \frac{m(m-1)(m-2)}{6}C_0a_1^3 + mC_0a_3 \\
= &C_3(1) - \ln\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot [\mathrm{m}\beta_2\mathrm{C}_0 + (\mathrm{m}+1)\beta_1\mathrm{C}_1(1) + (\mathrm{m}+2)\beta_0\mathrm{C}_2(1)] \\
&+ \ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot [(\mathrm{m}+1)\left(\frac{\mathrm{m}}{2}+1\right)\beta_0^2\mathrm{C}_1(1) + \mathrm{m}\left(\mathrm{m}+\frac{3}{2}\right)\beta_0\beta_1\mathrm{C}_0] \\
&- \ln^3\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \frac{\mathrm{m}(\mathrm{m}+1)(\mathrm{m}+2)}{6}\beta_0^3\mathrm{C}_0
\end{aligned} \tag{17}
$$

The last step in C_3 required the auxiliary calculations for the coefficients of $\ln^2$ and $\ln^3$:

$$
\begin{aligned}
&\ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \left[\beta_0^2\mathrm{C}_1(1) \cdot \left\{1 + 2\mathrm{m} + \frac{\mathrm{m}(\mathrm{m}-1)}{2}\right\} + \beta_0\beta_1\mathrm{C}_0 \cdot \left\{\frac{5}{2}\mathrm{m} + \mathrm{m}(\mathrm{m}-1)\right\}\right] \\
= &\ln^2\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \left[\beta_0^2\mathrm{C}_1(1) \cdot (\mathrm{m}+1)\left(\frac{\mathrm{m}}{2}+1\right) + \beta_0\beta_1\mathrm{C}_0 \cdot \mathrm{m}\left(\mathrm{m}+\frac{3}{2}\right)\right] \\
&\ln^3\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \beta_0^3\mathrm{C}_0\left[\mathrm{m}(\mathrm{m}-1) + \frac{\mathrm{m}(\mathrm{m}-1)(\mathrm{m}-2)}{6} + \mathrm{m}\right] = \ln^3\left(\frac{\mathrm{Q}^2}{\mu^2}\right) \cdot \beta_0^3\mathrm{C}_0\frac{\mathrm{m}(\mathrm{m}+1)(\mathrm{m}+2)}{6}
\end{aligned}
$$

2.2 Best choices of the scale

There exist some semi-heuristic-deductive approaches to resolve the scale ambiguity to some extent. Nevertheless, the procedures described in the following do not always provide unique solutions and cannot (so far) be applied to arbitrarily high orders. But at least the requirement of Fastest Apparent Convergence, the Principle of Miminal

SENSITIVITY and the scheme of BRODSKY-LEPAGE-MACKENZIE enable us to figure out where physical observables depend least on the scale, on higher orders or on the number of flavours. This is nicely reviewed in [9].

2.2.1 Fastest Apparent Convergence (FAC)

Grunberg presented the FAC procedure [10]. This allows to truncate the expansion of an observable R at a low order by choosing the renormalization scheme (RS) such that the higher coefficients r_i vanish. Here the r_i represent the RS so that R can be described by

$$R = \alpha_s(\mu) \cdot (1 + r_1\alpha_s(\mu) + r_2\alpha_s^2(\mu) + ...).$$

Let $\bar{\alpha}$ be the coupling in the chosen scheme having the coefficients v_i. Then R reduces to (with $\alpha \equiv \alpha_s(\mu)$)

$$R = \bar{\alpha} = \alpha \cdot (1 + v_1\alpha + v_2\alpha^2 + ...).$$

In second order v_1 follows from $R = \alpha \cdot (1 + r_1\alpha) = \alpha \cdot (1 + v_1\alpha) = \bar{a} \Rightarrow v_1 = r_1$. Changing the RS with a fixed scale μ is equivalent to redefining the scale in a certain scheme [9] according to $\mu \to \mu^* = \mu \cdot \exp(-\mathrm{r}_1/\mathrm{b})$ with $b = \frac{\beta_0}{8\pi}$ or by adjusting $\Lambda \to \Lambda^* = \Lambda \cdot \exp(+\mathrm{r}_1/\mathrm{b})$. In higher orders ($\geq$NLO) also the expansion coefficients $r_2, r_3, ...$ or $v_2, v_3, ...$ must be changed with logarithms $\ln(\Lambda^{*2}/\Lambda^2)$ [10]. But this corresponds to a change not only of the optimized scale, but also of the renormalization scheme. Alternatively, the $\overline{MS}$ scheme is maintained and the coefficients are chosen as follows. In NNLO the requirement of Fastest Apparent Convergence allows different constraints on the expansion parameters. Either the NLO or NNLO contribution vanishes, e.g.

$$R_{LO} = R_{NLO} \qquad \Leftrightarrow r_1 = 0 \tag{18}$$

$$R_{NLO} = R_{NNLO} \qquad \Leftrightarrow r_2 = 0 \tag{19}$$

Another possibility is to constrain r_1 and r_2 simultaneously: $R_{LO} = R_{NNLO}$.

2.2.2 Principle of Minimal Sensitivity (PMS)

In Stevenson's Principle of Minimal Sensitivity [11] the optimum RS is characterized by the condition

$$\frac{\partial R}{\partial RS}|_{RS=RS_{best}} = 0$$

As the RS is represented by Λ, this can be written more formally as

$$\frac{\partial R^{(2)}}{\partial \tau} = \frac{\partial}{\partial \tau}\left[\alpha^m(\tau) + \alpha^{m+1}(\tau) \cdot r_1(\tau)\right],$$

where $\tau := b\ln(\mu/\tilde{\Lambda})$ and $R^{(2)}$ is the second order approximation. The idea behind it is to find the τ or corresponding scale μ, respectively, at which the truncated observable changes least when μ is varied. Using the RGE (3) with $c = \beta_1/\beta_0$ and $m = 1$ here the optimized coefficient $\bar{r}_1$ is obtained from a comparison of coefficients in the derivative which is postulated to equal zero in order to achieve a minimal sensitivity.

$$\frac{\partial R^{(2)}}{\partial \tau} = \alpha^2 \cdot \left(\underbrace{\frac{\partial r_1}{\partial \tau} - 1}_{=0\,(i)} - \alpha \cdot \underbrace{(2r_1(1 + \alpha c) + c)}_{=0\,(ii)} \right) = 0 \tag{20}$$

With $\bar{\tau} \equiv \tau_{opt}$ and $\bar{\alpha} \equiv \alpha(\bar{\tau})$, $r_1(\tau) - \tau$ must be constant w.r.t. τ (i) and the coefficient $\bar{r_1} = -\frac{c}{2(1+\bar{\alpha}c)}$ (ii) leads to the optimized second order approximation

$$R^{(2)}_{opt} \quad = \quad R^{(2)}(\bar{\tau}) = \bar{\alpha} \cdot [1 + \bar{r}_1 \bar{\alpha}] = \bar{\alpha} \cdot \left[\frac{1 + \frac{1}{2}\bar{\alpha}c}{1 + \bar{\alpha}} \right] \tag{21}$$

2.2.3 Brodsky-Lepage-Mackenzie (BLM)

The ansatz by Brodsky, Lepage and Mackenzie [12] absorbs all contributions from vacuum polarization into corrections of $\alpha_s(\mu)$. The number of possible fermion loops depends directly on the number of flavours n. Therefore the BLM-criterion rescales $\mu \to \mu^*$ such that the second order coefficient becomes independent of n. Writing $a := \frac{\alpha_s}{\pi}$ a physical observable has the following coefficients in second order [9]:

$$R = a(\mu) \cdot (1 + \underbrace{(An + B)}_{r_1} a(\mu))$$

Changing the scale introduces logarithms.

$$\begin{aligned} a(\mu) \to a(\mu^*) \quad &= \quad a(\mu) \cdot (1 - ba(\mu) \ln(\mu^*/\mu)) \\ \Rightarrow R \quad &= \quad a(\mu^*) \cdot (1 + \underbrace{(An + B + b \ln(\mu^*/\mu))}_{r_1^*} a(\mu)) \end{aligned} \tag{22}$$

Inserting $b = -\frac{1}{3}n + \frac{11}{2}$, r_1^* can be made independent of n by choosing $\mu^* = \lambda\mu$.

$$\begin{aligned} r_1^* \quad &= \quad n \cdot \underbrace{(A - \frac{1}{3} \ln(\lambda))}_{=0} + B + \frac{11}{2} \ln(\lambda) \\ \Rightarrow \ln(\lambda) \quad &= \quad 3A \quad \Rightarrow \quad \mu^* = \mu e^{3A} \end{aligned} \tag{23}$$

3 Corrections for the Higgs decay in higher orders

3.1 Hadronic Higgs decay

The Higgs boson can decay in first order over a top quark triangle into two gluons. This includes one $t\bar{t}H$ vertex and two α_s vertices. Accordingly, the decay width starts with α_s^2 in LO. The Born level reads [13]:

$$\Gamma_{Born}(H \to gg) = \frac{G_F M_H^3}{36\pi\sqrt{2}} \left(\frac{\alpha_s^{(n)}(\mu)}{\pi} \right)^2$$

But the LO is not sufficient since the diagrams with more quarks and gluons in the final states in $(O)(\alpha_s^2)$ (see [13]) contribute reasonably. So the general decay width

$$\Gamma(H \to gg) = \frac{\sqrt{2} G_F}{M_H} C_1^2 \cdot \Im(V_H)$$

with the renormalized M_t-dependent coefficient function C_1 and the vacuum polarization of the Higgs field V_H is expressed as the Born width times the correction factor [14] (here in $\mathcal{O}(\alpha_s^3)$) [15]

$$K = 1 + 17.9167a + (156.81 - 5.7083L_{tH})a^2 + (467.68 - 122.44L_{tH} + 10.94L_{tH}^2)a^3 \quad (24)$$

Here is $L_{tH} := \ln\left(\frac{M_t^2}{M_H^2}\right)$ and $a := \frac{\alpha_s}{\pi}$. In [13] and [15] $Q = \mu = M_H$ has already been inserted. Even though the Higgs mass is not known, $M_H \leq 114\,\mathrm{GeV}$ was excluded by LEP and a light Higgs is favoured by the electroweak χ^2 fit. For this reason $M_H = 120\,\mathrm{GeV}$ is assumed here[3]. CDF and DØ[16] published the combined result $M_t = 173.1\,\mathrm{GeV}$.

3.2 Reconstruction of the μ-dependence

For analyzing the μ-dependence of the K-factor (similar to [17]) it is necessary to regain the terms in μ. There are two possibilities: one can multiply $C_1^2 \cdot V_H$ using the decoupling relation between $\alpha_s^{(5)}$ and $\alpha_s^{(6)}$ [18] and the running top mass [19]. This procedure yields a μ-dependent expression which reproduces the result in [15] when the Higgs mass is inserted.
A more efficient way is the application of the RGE (3) on (24) which recoveres the logarithms $\ln\left(\frac{M_H^2}{\mu^2}\right)$. The correction factor consists of the contributions from LO (1), NLO ($K_1(\mu) \propto \alpha$) and NNLO ($K_2(\mu) \propto \alpha^2$). The coefficients are obtained with $m = 2$ from (15)-(17) where the constants C_0, $C_1(1)$, $C_2(1)$, $C_3(1)$ emerge from comparison to [13] because they have set $\mu = M_H$ or to [15] in case of $C_3(1)$.

$$\begin{aligned}
K(\mu) = \underbrace{1}_{C_0} \quad &+ \alpha \cdot \left[\underbrace{\frac{1}{\pi}\left(\frac{95}{4} - \frac{7}{6}n\right)}_{C_1(1)} - 2\,\beta_0 \ln\left(\frac{M_H^2}{\mu^2}\right)\right] \\
+ \alpha^2 \cdot \quad & [\frac{1}{\pi^2}(\frac{149533}{288} - \frac{363}{8}\zeta(2) - \frac{495}{8}\zeta(3) - \frac{19}{8}\ln\left(\frac{M_H^2}{\mu^2}\right) \\
&+ n \cdot \left(-\frac{4157}{72} + \frac{11}{2}\zeta(2) + \frac{5}{4}\zeta(3) - \frac{2}{3}\ln\left(\frac{M_H^2}{\mu^2}\right)\right) + n^2 \cdot \left(\frac{127}{108} - \frac{1}{6}\zeta(2)\right)) \\
&- \left(\frac{3\,\beta_0}{\pi}\left(\frac{95}{4} - \frac{7}{6}n\right) + 2\,\beta_1\right) \ln\left(\frac{M_H^2}{\mu^2}\right) + 3\,\beta_0^2 \ln^2\left(\frac{M_H^2}{\mu^2}\right)] \\
+ \alpha^3 \cdot \quad & [\frac{C_3(1)}{\pi^3} - (2\beta_2 C_0 + 3\beta_1 C_1(1) + 4\beta_0 C_2(1)) \ln\left(\frac{M_H^2}{\mu^2}\right) \\
&+ \left(6\beta_0^2 C_1(1) + 7\beta_0\beta_1 C_0\right) \ln^2\left(\frac{M_H^2}{\mu^2}\right) - 4\beta_0^3 C_0 \cdot \ln^3\left(\frac{M_H^2}{\mu^2}\right)]
\end{aligned} \quad (25)$$

Since the decay width $\Gamma(H \to gg) \propto K \cdot \alpha^2$ as the observable of interest is proportional to α^2 and the Higgs mass is the physical reference value for the scale with $\alpha_H \equiv \alpha_s(M_H) =$

[3]unlike [13] where they use a LEP excluded Higgs mass

0.116, we define

$$C_{LO}(\mu) = \frac{\alpha^2(\mu)}{\alpha_H^2}$$
$$C_{NLO}(\mu) = (1 + K_1(\mu)) \cdot \frac{\alpha^2(\mu)}{\alpha_H^2}$$
$$C_{NNLO}(\mu) = (1 + K_1(\mu) + K_2(\mu)) \cdot \frac{\alpha^2(\mu)}{\alpha_H^2}$$

and analyze the scale dependence of the different orders in fig. (2).

3.3 Application of the scale finding procedures

In this section the principles of chapter (2.2) are applied to the Higgs decay width. The results of the determination of the best scales are summarized in tab. (2).

FAC The scales for vanishing higher order coefficients can be found numerically. According to (18) a vanishing NLO coefficient requires an intersection between the curves $C_{LO}(\mu)$ and $C_{NLO}(\mu)$ which happens at $\boxed{\mu_{FAC} = 11.5946\,\mathrm{GeV}}$. The NLO does not contribute here. Requiring a zero NNLO ($C_{NLO}(\mu) = C_{NNLO}(\mu)$) leads to two intersections. Thus, the solution is not unique! However, the solution $\mu_{FAC} = 2.1219$ can be excluded from a physical point of view as this scale is two orders of magnitude smaller than M_H. So $\boxed{\mu_{FAC} = 50.8799}$ is regarded as the proper scale. NLO and NNLO cancel each other at $\boxed{\mu_{FAC} = 20.5534\,\mathrm{GeV}}$.

PMS A minimal sensitivity is reached at a stationary point. For this reason the derivatives are calculated. The LO decreases monotonously so that there is no maximum or minimum. But the higher orders have maxima.

$$\alpha_H^2 \cdot \mu^2 \frac{\mathrm{dC}(\mu)}{\mathrm{d}\mu^2} = \mu^2 \frac{\mathrm{d}\alpha^2(\mu)}{\mathrm{d}\mu^2} \cdot K(\mu) + \mu^2 \alpha^2(\mu) \frac{\mathrm{dK}(\mu)}{\mathrm{d}\mu^2} \tag{26}$$

With the knowledge of the β-function we can use:

$$\mu^2 \frac{\mathrm{d}\alpha^{\mathrm{n}}(\mu)}{\mathrm{d}\mu^2} = n\,\alpha^{n-1} \cdot \underbrace{\mu^2 \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu^2}}_{\beta} = -n\alpha^{n+1}(\beta_0 + \alpha\beta_1 + \alpha^2\beta_2) + \mathcal{O}(\alpha^{n+3}) \tag{27}$$

Writing $L_{\mu H} \equiv \ln\left(\frac{\mu^2}{M_H^2}\right)$ the C-factor becomes

$$C_{NNLO}(\mu) = \alpha^2 \cdot [C_0 + \alpha\underbrace{(C_1 + C_1' L_{\mu H})}_{D_1} + \alpha^2 \underbrace{(C_2 + C_2' L_{\mu H} + C_2'' L_{\mu H}^2)}_{D_2}]$$

so that the derivative in (26) can be expressed as

$$\begin{aligned}\frac{\mathrm{dC_{NNLO}}(\mu)}{\mathrm{d}\mu^2} &= \alpha^7[-4\beta_2 D_2] + \alpha^6[-4\beta_1 D_2 - 3\beta_2 D_1] + \alpha^5[-2\beta_2 C_0 - 3\beta_1 D_1 - 2\beta_0 D_2] \\ &+ \alpha^4[-2\beta_2 C_0 - 3\beta_0 D_1 + C_2' + C_2'' L_{\mu H}] + \alpha^3[-2\beta_0 C_0 + C_1']\end{aligned}$$

For the NLO (27) is applied only up to β_1 and (3.3) up to D_1 resulting in the following total derivative:

$$\frac{\mathrm{dC_{NLO}}(\mu)}{\mathrm{d}\mu^2} = \alpha^5\left[-3\beta_1 D_1\right] + \alpha^4\left[-2\beta_1 C_0 - 3\beta_0 D_1\right] + \alpha^3\left[-2\beta_0 C_0 + C_1'\right]$$

The NLO has one stationary point at $\boxed{\mu_{PMS} = 10.4658\,\mathrm{GeV}}$ whereas the NNLO solution is again not unique: $\boxed{\mu_{PMS} = 43.95\,\mathrm{GeV}\text{ or }1.822\,\mathrm{GeV}}$. The second value is close to the unphysical FAC solution and is neglected for the same reason.

BLM According to (23) with $A = -\frac{7}{6\pi}$ the BLM-scale is calculated as

$$\boxed{\mu_{BLM} = M_H \exp\left(-\tfrac{7}{2\pi}\right) = 39.38\,\mathrm{GeV}}$$

3.4 Comparison of the results

The correction factors as functions of the μ and the optimized scales are visualized in fig. (2). The intersections and stationary points are close to each other. This shows that FAC and PMS yield similar, but not equal results. The values for the best scales are larger for NNLO than for NLO. The NLO BLM result, however, is close to the NNLO FAC and PMS results.

C is evaluated at the optimized scales as listed in tab. (2). The values are listed for the different principles and orders. For instance NLO means a zero next-to-leading order, NNNLO+NNLO+NLO denotes that these three orders vanish simultaneously, i.e. $C_{L0} = C_{NNNLO}$.
A common method to estimate the error is to insert the half and the double scale. Sometimes it works to predict the subsequent order if

$$C_{N^{n+1}LO}(\mu) \in [C_{N^nLO}(\mu/2), C_{N^nLO}(2\mu)]$$

But in this case the estimation only works for the simple choice $\mu = Q = M_H$. On the other hand it becomes evident that this simple arbitrary choice can only be regarded as a first guess since these C-values differ reasonably from the optimized ones. However, in NNNLO the best FAC-scale $\mu = 118\,\mathrm{GeV} \approx \mathrm{M_H}$ is quite close to the inserted Higgs mass $M_H = 120\,\mathrm{GeV}$. The NNNLO results for the scale from FAC and PMS are much higher than the lower orders.

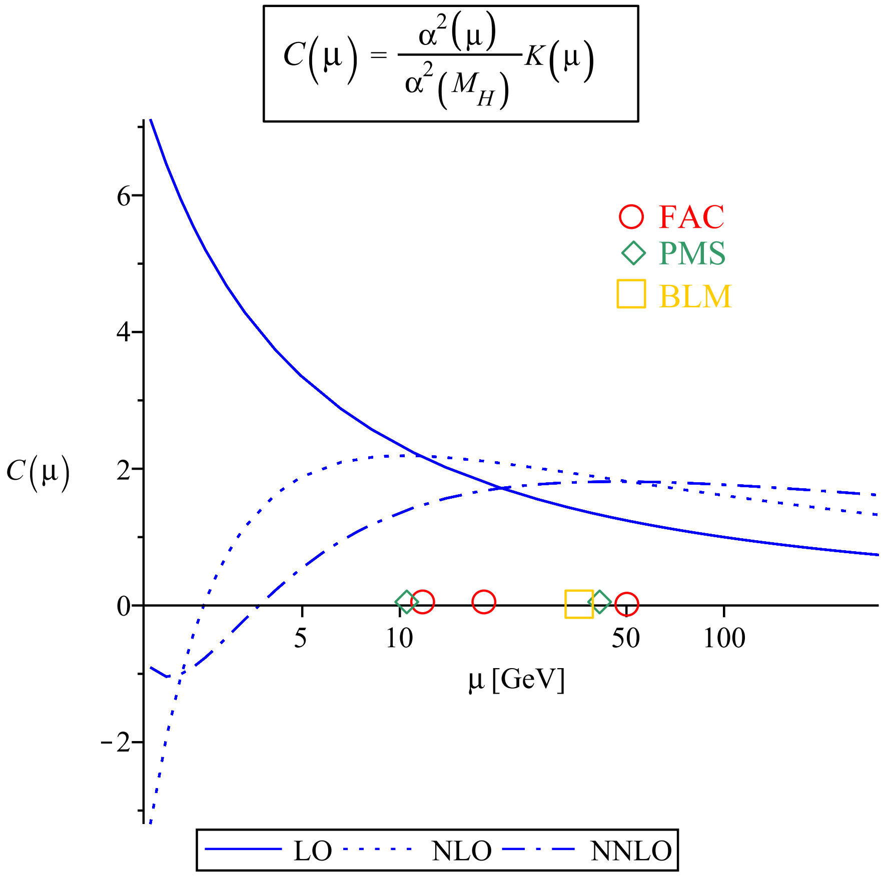

Figure 2: μ-dependence of C in several orders

Table 2: Best scales

Principle	Order	μ [GeV]	$C(\mu)$	$C(\mu/2)$	$C(2\,\mu)$
FAC	NLO	11.5946	2.1853	2.0184	2.0483
FAC	NNLO	50.8799	1.8114	1.7637	1.7640
FAC	NNLO	(2.1219)	(-1.0006)	(6.3876)	(0.2709)
FAC	NLO+NNLO	20.5534	1.7163	2.3094	1.3287
FAC	NNNLO	118	1.7463	0.7379	2.1757
FAC	NNNLO+NNLO	103	1.6075	0.4273	2.1204
FAC	NNNLO+NNLO+NLO	71.7	1.1072	-0.6765	1.9089
PMS	NLO	10.4658	2.1895	1.9334	2.0759
PMS	NNLO	43.95	1.8117	1.7333	1.7788
PMS	NNLO	(1.822)	(-1.0161)	(16.0554)	(-0.0198)
PMS	NNNLO	645	2.3296	2.2646	2.2883
BLM	NLO	39.38	1.8906	2.0916	1.6789
$\mu = M_H$	LO	M_H	0.9477	1.1710	0.7834
$\mu = M_H$	NLO	M_H	1.5581	1.7608	1.3782
$\mu = M_H$	NNLO	M_H	1.7450	1.8062	1.6488

4 Further observables in higher orders

This project also comprised searching in literature for more observables which are already available in higher orders and could be studied with the methods presented above. Some of them are listed in the following.

τ-Decays

- $\mathcal{O}(\alpha_s^3)$ accuracy [20]
- towards $\mathcal{O}(\alpha_s^4)$ accuracy [21]
- hadronic Z- and τ- decays in $\mathcal{O}(\alpha_s^4)$ [22]

Moments and Sum Rules

- Higher moments of DIS structure functions at NNLO [23]
- Towards NNLO in the sum rule for the kaon distribution amplitude [24]
- $\mathcal{O}(\alpha_s^3)$ corrections to the Bjorken sum rule for polarized electroproduction and to the GLLS sum rule

Others

- Higgs decay into b-quarks at $\mathcal{O}(\alpha_s^4)$ [25]
- Vector boson production at hadron colliders in NNLO [26]
- NNLO corrections to four-fermion production [27]

5 Conclusion

5.1 Summary

The depenence of the hadronic decay width of the Higgs boson on the renormalization scale was analyzed by regaining the scale dependent coefficients from the renormalization group equation. The arbitrary choice of the scale was optimized by the application of the requirement of Fastest Apparent Convergence (FAC), the Principle of Minimal Sensitivity (PMS) and the flavour-independent procedure by Brodsky, Lepage and Mackenzie (BLM). The optimized scales differ notably from the simple choice $\mu = Q = M_H$. Compared to each other, PMS and FAC result in similar, but not exactly equal values. However, the solution is not unique in NNLO! One still has to judge from a physical point of view which of the two solutions is regarded as the best value. The BLM procedure is so far only possible in NLO.

5.2 Outlook

The analyses could be extended to as high orders as possible, in case of FAC and PMS if numerical methods are used to determine intersections and vanishing derivatives. Furthermore, it would be interesting to investigate the scale dependence of the observables listed in chapter (4).
The concepts still need to be extended to allow for analytical calculations of coefficients beyond the NLO. Although some authors have claimed to have resolved the scale ambiguities in NLO, the solutions are not unique in NNLO. Therefore the procedures should be improved.

Acknowledgements

I would like to thank Dr. Markus Diehl for supervising this project. His explanations and advises were always very helpful.
I am also thankful to Prof. Dr. Ahmed Ali for answering some questions.
Special thanks to Aleksandra for interesting discussions and mutual encouragement!
Further thanks go to Prof. Dr. Joachim Meyer and Andrea Schrader for organizing this great summer students program 2009 at DESY.

References

[1] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory.* Perseus Books, 1995.

[2] S. Pokorski. *Gauge Field Theories.* Cambridge University Press, 1987.

[3] G. M. Prosperi, M. Raciti, and C. Simolo. On the running coupling constant in QCD. *Prog. Part. Nucl. Phys.*, 58:387–438, 2007.

[4] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser. Strong coupling constant with flavor thresholds at four loops in the modified minimal-subtraction scheme. *Phys. Rev. Lett.*, 79(12):2184–2187, Sep 1997.

[5] Siegfried Bethke. Experimental tests of asymptotic freedom. *Prog. Part. Nucl. Phys.*, 58:351–386, 2007.

[6] Siegfried Bethke. The 2009 World Average of $\alpha_s(M_Z)$. 2009.

[7] C. Amsler et al. Particle Data Group. *Physics Letters B667*, 1, 2008.

[8] M. Diehl. Theoretical uncertainties: selected issues. DESY seminar, 2009.

[9] D. W. Duke and R. G. Roberts. Determinations of the QCD Strong Coupling α_s and the Scale Lambda (QCD). *Phys. Rept.*, 120:275, 1985.

[10] G. Grunberg. Renormalization Group Improved Perturbative QCD. *Phys. Lett.*, B95:70, 1980.

[11] Paul M. Stevenson. Optimized Perturbation Theory. *Phys. Rev.*, D23:2916, 1981.

[12] Stanley J. Brodsky, G. Peter Lepage, and Paul B. Mackenzie. On the Elimination of Scale Ambiguities in Perturbative Quantum Chromodynamics. *Phys. Rev.*, D28:228, 1983.

[13] K. G. Chetyrkin, Bernd A. Kniehl, and M. Steinhauser. Hadronic Higgs decay to order α_s^4. *Phys. Rev. Lett.*, 79:353–356, 1997.

[14] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas. Higgs boson production at the LHC. *Nucl. Phys.*, B453:17–82, 1995.

[15] P. A. Baikov and K. G. Chetyrkin. Higgs decay into hadrons to order α_s^5. *Phys. Rev. Lett.*, 97:061803, 2006.

[16] Tevatron Electroweak Working Group for the CDF and D0 Collaborations. Combination of CDF and D0 Results on the Mass of the Top Quark. *arXiv:0903.2503v1 [hep-ex]*, 2009.

[17] G. Kramer and B. Lampe. Jet production rates at LEP and the scale of α_s. *Z. Phys.*, A339:189–193, 1991.

[18] S. A. Larin, T. van Ritbergen, and J. A. M. Vermaseren. The Large quark mass expansion of Gamma (Z0 $\rightarrow$ hadrons) and Gamma (tau- $\rightarrow$ tau-neutrino + hadrons) in the order α_s^3. *Nucl. Phys.*, B438:278–306, 1995.

[19] K. G. Chetyrkin, Bernd A. Kniehl, and M. Steinhauser. Three-loop $\mathcal{O}(\alpha_s^2 G_F M_t^2)$ corrections to hadronic Higgs decays. *Nucl. Phys.*, B490:19–39, 1997.

[20] P. A. Baikov, K. G. Chetyrkin, and Johann H. Kuhn. Strange quark mass from tau lepton decays with $\mathcal{O}(\alpha_s^3)$ accuracy. *Phys. Rev. Lett.*, 95:012003, 2005.

[21] P. A. Baikov, Konstantin G. Chetyrkin, and Johann H. Kuhn. Towards order α_s^4 accuracy in tau decays. *Phys. Rev.*, D67:074026, 2003.

[22] P. A. Baikov, K. G. Chetyrkin, and Johann H. Kuhn. Order α_s^4 QCD Corrections to Z and τ Decays. *Phys. Rev. Lett.*, 101:012002, 2008.

[23] A. Retey and J. A. M. Vermaseren. Some higher moments of deep inelastic structure functions at next-to-next-to leading order of perturbative QCD. *Nucl. Phys.*, B604:281–311, 2001.

[24] K. G. Chetyrkin, A. Khodjamirian, and A. A. Pivovarov. Towards NNLO Accuracy in the QCD Sum Rule for the Kaon Distribution Amplitude. *Phys. Lett.*, B661:250–258, 2008.

[25] P. A. Baikov, K. G. Chetyrkin, and Johann H. Kuhn. Scalar correlator at $\mathcal{O}(\alpha_s^4)$, Higgs decay into b- quarks and bounds on the light quark masses. *Phys. Rev. Lett.*, 96:012003, 2006.

[26] Stefano Catani, Leandro Cieri, Giancarlo Ferrera, Daniel de Florian, and Massimiliano Grazzini. Vector boson production at hadron colliders: a fully exclusive QCD calculation at NNLO. *Phys. Rev. Lett.*, 103:082001, 2009.

[27] S. Actis, M. Beneke, P. Falgari, and C. Schwinn. Dominant NNLO corrections to four-fermion production near the W-pair production threshold. *Nucl. Phys.*, B807:1–32, 2009.