Compactification of extra dimensions and the hierarchy problem

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Abstract

Here I present my investigation in the summer student program at DESY 2009. In this I studied three mechanisms to hide extra-dimension. I focused on the Kaluza-Klein compactification and on the necessary calculations to arrive since from a five dimensional theory of gravity to a four dimensional theory of gravity and electromagnetism. As a continuation of the way that we can hide extra dimensions I touch on models of brane worlds.

1 Introduction

At present superstring and M theory are probably our best candidates for providing a unified description of all the fundamental forces in nature, but almost all their versions are formulated in space time of more than four dimensions. If these theories are truly fundamental, and we want to describe our four-dimensional world, we need to have a way to extract the four-dimensional physics from these higher dimensional theories, therefore an important issue in multi-dimensional theories is the mechanism by which extra dimensions are hidden. Studying these mechanisms one can find that they propose different ways to study and address the gauge hierarchy problem. In this summer I studied three hiding mechanisms: Kaluza- Klein compactification, unwarped and warped brane worlds.

2 The hierarchy problem

Before starting with the compactification mechanisms, in this section I will present a sketch of the hierarchy problem.

In theoretical physics a hierarchy problem occurs when the fundamental parameters, couplings constants or masses, of some Lagrangian are vastly different from the parameters measured by experiment. This can happen because measured parameters are related to the fundamental parameters by a prescription known as renormalization. Typically the renormalization parameters are closely related to the fundamental parameters, but in some cases, it appears that there has been a delicate cancellation between the fundamental quantity and the quantum corrections to it.

Studying the renormalization in hierarchy problems is difficult, because such quantum corrections are usually power-law divergent which means that the shortest-distance physics are most important. Because we do not know the precise details of the shortest-distance theory of physics (quantum gravity), we cannot even address how this delicate cancellation between two large terms occurs.

In particle physics, the gauge hierarchy problem is the question why the weak force is 10^{32} times stronger than gravity, technically, the question is why the Higgs boson is so much lighter than the Planck mass. Because the Higgs expectation value $v^2 = \mu^2/\lambda$ is measured as $v = 246GeV$ and the validity of the perturbation theory (see [?]) requires $\lambda/16\pi^2 \leq 1$, μ should obey: $\mu \leq 4\pi v = 3TeV$. But when we considered virtual effects involving very heavy new particles having mass M, generically contribute to μ at an amount order:

$$\delta\mu^2 \sim \frac{gM^2}{16\pi^2} \tag{1}$$

where g is a dimensionless measure of the coupling strength between the heavy virtual particle and the Higgs doublet. Strictly speaking explicit calculations give results which diverge quadratically leading to expressions of the form $\delta^2\mu = c_2\Lambda^2 + c_0 M^2 ln\Lambda^2/M^2 + ...$ the contribution to the effective coupling $\delta\mu^2$, is defined to reproduce the difference between the theory with and without the massive particle, after renormalization, and this lead an estimate of the form (1). Thus the question here is why $\mu^2 << \delta\mu^2$

Although it is possible that the bare term μ_0^2, appearing in the Lagrangian precisely cancels these types of contributions for all such massive particles, the required cancellation is fantastically accurate for M extremely large. A cancellation this accurate is generally known as a fine tuning, and it is widely considered unlikely that such cancelation is the explanation. Rather, it is generally believed that some new physics must be involved.

Below we will discuss the possibility that the new physics is due to extra dimensions.

3 Kaluza-Klein compactification

The idea of extra-dimensions actually arose before the development of string theory, as a route to the unification of theories. Indeed in the 1920's T. Kaluza and O. Klein, proposed to consider an extra dimension to unify geometrically the theories of electromagnetism and gravitation introducing the electromagnetic field as a component of the metric of a five-dimensional space-time, where the extra dimension is compact. This work is the basis of the theories of compactification, where the compactness of the extra dimensions ensures that space-time is effectively four-dimensional at distances exceeding the compactification scale. Hence the size of extra dimensions must be microscopic. For a nice review [?] and [2]

To understand this mechanism, let us assume we are starting from the Einstein- Hilbert gravity in (D+1) dimensions, described by the Einstein- Hilbert Lagrangian (with convention $(+----)$):

$$L = -\sqrt{\hat{g}}\hat{R} \tag{2}$$

where $\hat{R}$ is the Ricci scalar, $\hat{g}$ is the determinant of the metric and the hat means that they are in $(D+1)$ dimensions. Now suppose that we wish to reduce the theory to D dimensions by compactificaing one of the coordinates on a circle S^1 of radius L, let this coordinate be called z, where $\theta = z/L$ and $0 < \theta < 2\pi$. In principle we could simply now expand all the components of the (D+1) dimensions metric tensor as Fourier series of the form:

$$\hat{g}_{MN}(x,z) = \sum_n g^n_{MN}(x)e^{inz/L} \tag{3}$$

where x denotes collectively the D coordinates of lower-dimensional space-time. If we do this, we get an infinitive number of fields in D-dimensions, labelled by the Fourier mode number n. We will see that modes with $n \neq 0$ are associated with massive fields, and modes with $n = 0$ are massless.

In this summer I considered the example of a massless scalar field ϕ in a (D+1)-dimensional space, that satisfies the equation of motion:

$$\hat{\Box}\hat{\phi} = 0 \tag{4}$$

where $\hat{\Box} = \partial^M \partial_M$. If we Fourier expand $\hat{\phi}$ after compactifing the coordinate z, so that:

$$\hat{\phi}(x,z) = \phi_n(x)e^{inz/L} \tag{5}$$

then we inmediatly see that the lower-dimensional fields $\phi_n(x)$ will satisfy:

$$\Box\phi_n - \frac{n^2}{L^2}\phi_n = 0 \tag{6}$$

this is the wave equation for Kaluza-Klein modes ϕ_n of mass n/L. In the Kaluza-Klein theory we assume that the radius L of the compactifing circle is very small, and we require enormous energies to access the heavy modes, so at low energies $< 1/L$ we can truncate the non zero modes. Thus, a Kaluza- Klein reduction, implies a compactification together with a truncation to the massless sector. In our case this means that the fields $\hat{g}_{MN}(x,z)$ are independent of z.

Now we can denote the components of the metric $\hat{g}_{MN}$ by $\hat{g}_{\mu\nu}$, $\hat{g}_{z\mu}$ and $\hat{g}_{zz}$, from a D-dimensional viewpoint these look like a symmetric tensor (the metric), a 1-form (a maxwell potential) and a scalar field respectively.

Doing a parametrization as follows, we write the (D+1) dimensional metric in terms of D-dimensional fields: $g_{\mu\nu}$, A_μ and ϕ as follows:

$$d\hat{s}^2 = ds^2 - \phi(dz + A)^2 \tag{7}$$

Where $A = A_\mu dx^\mu$. All the fields on the right-hand side are independent of z. Note that this ansatz means that the components of hte higher dimensional metric $\hat{g}_{MN}$ are given in terms of the lower-diemnsional fields by:

- $\hat{g}_{\mu\nu} = g_{\mu\nu} - \phi A_\mu A_\nu$
- $\hat{g}_{\mu z} = -\phi A_\mu$
- $\hat{g}_{zz} = -\phi$

With the goal of expressing the (D+1)-dimensional quantities in terms of the D-dimensional ones, and to be able to express the (D+1)-dimensional Hilbert-Einstein Lagrangian in terms of the canonical D-dimensional Lagrangian, we have to ensure that the dimensionally-reduced Lagrangian is the Hilbert-Einstein form $L = \sqrt{-g}R + ...$ and ensure that the scalar field ϕ acquires a kinetic term with the canonical normalization, meaning a term of the form: $-\frac{1}{2}\sqrt{-g}(\partial\phi)^2$. This requires a so called Weyl rescaling of the four-dimensional metric, which we will not discuss further.

Considering a space with (4+1)-dimensions lead us to a metric of the form:

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & -\phi A_\mu \\ -\phi A_\mu & -\phi \end{pmatrix}$$

Having in mind that the Christoffel symbols are given by:

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{8}$$

and that the Ricci tensors are given by:

$$R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\alpha_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu} \tag{9}$$

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{10}$$

I could calculate the necessary Christoffels:

$$\hat{\Gamma}^\sigma_{\lambda\mu} = \Gamma^\sigma_{\lambda\mu} + \frac{1}{2}\phi A_\mu F^\sigma_\lambda + A_\lambda F^\sigma_\mu \tag{11}$$

$$\hat{\Gamma}^\lambda_{5\sigma} = \frac{1}{2}\phi F^\lambda_\sigma \tag{12}$$

$$\hat{\Gamma}^5_{\lambda\sigma} = -\frac{1}{2}\phi A^\rho A_\lambda F_{\rho\mu} - \frac{1}{2}\phi A^\rho A_\mu F_{\rho\lambda} + \frac{1}{2}(A_{\lambda,\mu} + A_{\mu,\lambda} - \Gamma^\gamma A_\gamma - \Gamma^\gamma_{\mu\lambda} A_\gamma) \tag{13}$$

$$\hat{\Gamma}^5_{5\mu} = -\frac{1}{2} A^\nu \phi F_{\lambda\nu} \tag{14}$$

$$\hat{\Gamma}^5_{55} = 0 \tag{15}$$

$$\hat{\Gamma}^\lambda_{55} = 0 \tag{16}$$

to achieve the components of the Ricci tensor:

$$\hat{R}_{5\mu} = \frac{1}{2} F^\sigma_{\mu;\sigma} + \frac{1}{4}\phi^2 A_\mu F^{\sigma\rho} F_{\sigma\rho} \tag{17}$$

$$\hat{R}_{55} = \frac{1}{4}\phi^2 F^{\sigma\rho} F_{\sigma\rho} \tag{18}$$

$$\hat{R}_{\nu\mu} = R_{\nu\mu} - \frac{1}{2}\phi F^\rho_\beta F_{\rho\delta} + \frac{1}{2}\phi A_\beta F^\rho_{\delta;\rho} + \frac{1}{2}\phi A_\delta F^\rho_{\beta;\rho} + \frac{1}{4}\phi^2 A_\beta A_\delta F^{\sigma\rho} F_{\sigma\rho} \tag{19}$$

with which I calculated the Ricci scalar $\hat{R}$:

$$\hat{R} = R + \frac{1}{4}\phi F^{\rho\nu}F_{\rho\nu} \tag{20}$$

Computing the determinant of the metric $\mid G \mid = \phi g$, we can see that the dimensional reduction of higher -dimensional Hilbert-Einstein Lagrangian gives:

$$L = \phi g[R + \frac{1}{4}\phi F^{\rho\nu}F_{\rho\nu}] \tag{21}$$

This is as we wanted the five-dimensional Hilbert-Einstein Lagrangian in terms of a four-dimensions theory of a massless spin-2 particle, $g_{\mu\nu}$ (the four-dimensional graviton), a massless spin-1, A_μ (the Kaluza-Klein gauge boson), and if we had kept ϕ we would have had a massless spin-0, ϕ (the dilaton). In modern language the massless scalar field is called a modulus field, and the presence of moduli are one of the big problems in string compactifications because we do not observe them.

Recalling the mass of the Kaluza-Klein modes (see (6)) the energy scale necessary to observe the extra-dimensions depends on L. As we have not seen yet the extra-dimensions in the particles accelerators experiments, which have reached energies of a few hundred of GeV, we know that $L \leq 10^{-17}cm$.

4 Unwarped brane world

Another mechanism to hide extra-dimensions is based on the assumption that the matter and gauge fields (in the standard model) are trapped to a three-dimensional submanifold, the brane, embedded in a fundamental multi-dimensional space. In this picture , the standard model is thus four-dimenisonal (explaning why we have not observed extra-dimensions in particle accelerators), but gravity is higher dimensional. Since we have only tested gravity to around $10\mu m$, in this scenario, extra dimensions may be large; this implies that they may have experimentally observable effects, for instance at the LHC. This fact and the fact that branes are inherent in string and M theory, make interesting to study this mechanism.

The ADD [6], [7] mechanism consists in neglecting the brane tension, σ, which is the energy density per unit three-volume of the brane, and considering a brane in large compact extra dimensions. In this way the Kaluza- Klein picture is reintroduced, but in this approach the size of the extra-dimensions R do not need to be microscopic, because only gravity is in the bulk. This approach may address in a new way the hierarchy problem, since in multidimensional theories the four-dimensional Planck scale is not a fundamental parameter. Rather the, fundamental mass of multidimensional gravity M obeys:

$$G_{(D)} = \frac{1}{M^{D-2}} = \frac{1}{M^{d+2}} \tag{22}$$

where G_D is the D-dimensional Newton‘s constant, and $D - 4 = d$ is the number of extra-dimensions. As in the Kaluza-Klein compactification, if we consider the metric independent of the extra-dimensions, we see that the four dimensional Planck mass is:

$$M_{Pl} = M(MR)^{d/2} \tag{23}$$

where R is the size of the extra-dimensions. Then we can suppose that the fundamental scale is of the order of the electroweak scale and thus so is the renormalized Higgs mass. So assuming that $M \sim 1TeV$ one can calculate from (23) the value of $R = 10^{32/d}10^{-17}cm$, thus if there were for example two extra-dimensions R should be $R = 1mm$ then we should be able to measure effects of them. In this picture the hierarchy problem then transforms to the large size of the extra-dimensions.

5 Warped extra dimensions

The last mechanism ignores the energy density of the brane itself, i.e., the gravitational field that the brane produces, but when we consider distance scales much larger than the brane thickness, the brane can be seen as source of the gravitational field which induces an interesting geometry in the multi-dimensional space. The mechanism studied in this section considers only one extra-dimension compactified by the introduction of two branes at $z = 0$ with positive tension and $z = z_c$ with negative tension. For this configuration the next metric is a solution of the complete set of Einstein equations (see [4] and [5])

$$ds^2 = a^2(z)\eta_{\mu\nu}dx^\mu dx^\nu - dz^2 \tag{24}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $a(x) = e^{-k|z|}$ is the warp factor, $k = \frac{4\pi}{3}G\hat{(5)}\sigma$. Unlike the metric of the Kaluza-Klein compactification, this metric is non factorizable, i.e., is not the product between four dimensional Minkowski space and the manifold of the compact extra-dimension. Now if we consider small fluctuations as in the Kaluza-Klein compactification, we can get an equation with a mass spectrum m_n^2, of Kaluza-Klein gravitons. The solution for the zero mode $m_0^2 = 0$ is:

$$h_0(z) = e^{-kz} \tag{25}$$

this mode describes the usual four dimensional gravity. Unlike the Kaluza Klein compactification with factorizable geometry, the zero mode function depends on z non trivially, and decrease towards $z = z_c$. This suggests that the gravitational coupling between particles depends on which brane they are residing. Considering that the matter resides in the negative tension brane one can arrive to the next expression for the four-dimensional gravitational constant:

$$G_{(4)} = G_{(5)}k\frac{1}{e^{2kz_c} - 1} \tag{26}$$

From the last equation using (22)we can write:

$$\frac{1}{M_{Pl}^2} = k\frac{1}{M^3}\frac{1}{e^{2kz_c} - 1} \tag{27}$$

Taking the fundamental mass like M_{ew} and also $k \sim M$, with $kz_c >> 1$ we can get:

$$M_{Pl} = M_{ew}e^{2kz_c} \tag{28}$$

This gives a novel possibility to address the hierarchy problem thanks to the exponential warp factor.

And for matter residing in the positive tension brane one can arrive to the next expression for the four-dimensional gravitational constant:

$$G_{(4)} = G_{(5)} k \frac{1}{1 - e^{-2kz_c}} \tag{29}$$

Notice that taking $z_c \to \infty$ we get an infinite large extra-dimension and yet recover a finite four-dimesnion Planck mass, which indicates that four-dimensional physics can be recovered.

6 Conclusions

The search for a theory which includes all interactions and in particular a quantum theory of gravity has driven us to multi-dimensional theories, and with this the necessity to find mechanisms to hide extra-dimensions. Moreover the study of these mechanisms opens the possibility to solve in new ways the gauge hierarchy problem. In the ADD mechanism compactification the hierarchy problem depends on the large of the size of the extra-dimensions R, leading to the possibility of detecting them. While in the warped extra-dimension the large difference between the M_{pl} and the M_{ew} depends on the the warp factor. This gives us even more reasons to continue to study these theories, as so far they just take the problem to different variables.

References

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