Journal Club - 2401.10981

Smoothed asymptotics

We all know that

$$\sum_{n=0}^{N} n = \frac{1}{2} N (N + 1)$$

Turn this into a weighted sum

$$\sum_{n=0}^{\infty} n\, \Theta[n / N] = \frac{1}{2} N (N + 1)$$

Where

In[∘]:=
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Θ → Plot[HeavisideTheta[x], {x, -1, 1}, ImageSize → Medium]
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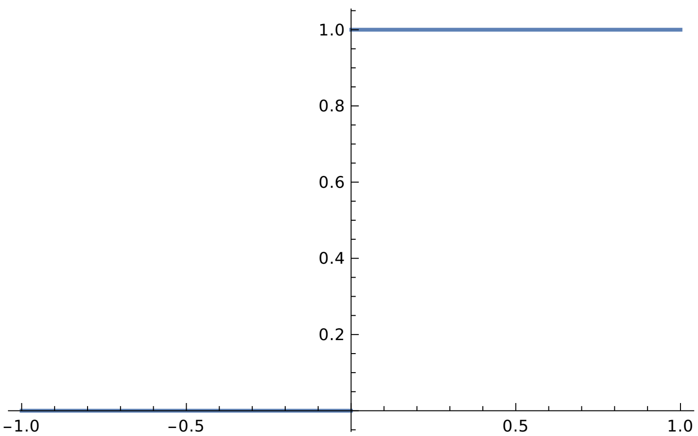
Out[∘]= Θ →



Replace the regulator by a smooth function:

- $\eta[0] = 1$
- $\eta[\infty] \to 0$
- Schwartz function

Then

$$\sum_{n=0}^{\infty} n\, \eta[n / N] = C_1[\eta]\, N^2 - \frac{1}{12} + O\left(\frac{1}{N}\right)$$

Where

$$C_1[\eta] = \int_0^{\infty} dx\, x\, \eta[x]$$

Example: $\eta[x] = e^{-x}$

In[∘]:=
```
η[x_] := Exp[-x];
Sum[n η[n/N], {n, 0, ∞}] // Series[#, {N, ∞, 0}] & // Normal
C1 * N^2 - 1/12 /. C1 → Integrate[η[x] x, {x, 0, ∞}]
```

Out[∘]= $-\frac{1}{12}+N^2$

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Example: $\eta[x]=e^{-x}\,\text{Cos}[x]$

In[∘]:=
```
η[x_] := Exp[-x] Cos[x];
Sum[n η[n/N], {n, 0, ∞}] // Series[#, {N, ∞, 0}] & // Normal
C1 * N^2 - 1/12 /. C1 → Integrate[η[x] x, {x, 0, ∞}]
```

Out[∘]= $-\frac{1}{12}$

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Features:

1) There is no linear divergence.
2) Quadratic divergence depends on the regulator (can be made zero -- Enhanced regulators).
3) There is a universal behaviour (regulator independent): -1/12.

Similar feature to QFT:

1) Power law divergences are meaningless (regulator dependent).
2) Log-divergences are universal (regulator independent).
3) What are enhanced regulators for QFTs? Are they physical in some sense?

(Number Theory $\sum_{n=0}^{\infty}\# \to \sum_{n=0}^{\infty}\eta\left(\frac{n}{N}\right)\#.$) → (QFT $\int d^4k\# \to \int d^4k\ \eta\left(\frac{|k|}{\Lambda}\right)\#,$ η-regularization)

What they did:

1) Start with one fold irreducible integrals (ILS).
2) Implement the smoothed regularization to the integrals.
3) Get that power law divergences are reg. dependent and log-divergences are log-independent.
4) Gauge invariant regulators → enhanced regulators

Details

1) Start with one fold irreducible integrals (ILS):

To develop η regularisation in more detail, it is convenient to introduce the concept of irreducible loop integrals (ILIs) [25, 26, 29]. In general, n-fold ILIs are defined as the n-loop integrals for which there are no longer the overlapping factors $(k_i - k_j + p)$ in the denominator of the integrand and no factors of the scalar momentum k^2 in the numerator [29]. In this work we focus on regularising ultra-violet divergences at one-loop, postponing a detailed discussion of higher loops to future work [55]. It was shown in [25] that upon use of the Feynman parameter method, all one-loop perturbative Feynman integrals of the one-particle irreducible graphs can be evaluated as the following one-fold ILIs in Minkowski spacetime:

$$I_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha}}, \tag{3.9}$$

$$I^{\mu\nu}_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 + \mathcal{M}^2)^{3+\alpha}}, \tag{3.10}$$

$$I^{\mu\nu\rho\sigma}_{-2\alpha}(\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\sigma}{(k^2 + \mathcal{M}^2)^{4+\alpha}}, \tag{3.11}$$

where the subscript (-2α) labels the power counting dimension (of energy-momentum) with $\alpha = -1$ and $\alpha = 0$ corresponding to quadratic and logarithmically divergent integrals. The mass term $\mathcal{M}^2 = \mathcal{M}^2(m_1^2, p_1^2, \dots)$ is a function of Feynman parameters, external momenta, p_i and corresponding mass scales, m_i. Note that $k^2 = g_{\mu\nu} k^\mu k^\nu$ where the metric $g_{\mu\nu}$ is written with mostly positive signature.

2) Implement the smoothed regularization to the integrals.

$$J_{-2\alpha}[\eta](\mathcal{M}^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + \mathcal{M}^2)^{2+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right),$$

$$J^{\mu\nu}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4} g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{(k^2 + \mathcal{M}^2)^{3+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right),$$

$$J^{\mu\nu\rho\sigma}_{-2\alpha}[\eta](\mathcal{M}^2) = \frac{1}{4!} S^{\mu\nu\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{k^4}{(k^2 + \mathcal{M}^2)^{4+\alpha}} \eta\left(\frac{|k|}{\Lambda}\right),$$

3) Get that power law divergences are reg. dependent and log-divergences are log-independent.

For $\alpha > 0$ the integrals are convergent as $\Lambda \to \infty$ and one readily obtains

$$J_{-2\alpha}[\eta](\mathcal{M}^2) \sim \frac{1}{16\pi^2\alpha(1+\alpha)\mathcal{M}^{2\alpha}}. \tag{3.20}$$

For $\alpha \leq 0$, the integrals diverge as $\Lambda \to \infty$, where they take the following asymptotic form

$$J_0[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta] - \frac{1}{2}\right], \tag{3.21}$$

$$J_2[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\Lambda^2 C_1[\eta] - \mathcal{M}^2\left(\ln(\Lambda/|\mathcal{M}|) + \gamma[\eta]\right)\right], \tag{3.22}$$

and

$$J_{2s+4}[\eta](\mathcal{M}^2) \sim \frac{1}{8\pi^2}\left[\sum_{z=0}^{s}\binom{s}{z} C_{2z+3}[\eta]\mathcal{M}^{2(s-z)}\Lambda^{2z+4}\right] \tag{3.23}$$

for any natural number s. These expressions are valid provided η is a regulator: a smooth

where

$C_z[\eta] = \int x^z\, \eta[x]\, dx$ and $\gamma[\eta] = \int x^z\, \frac{d\eta}{dx}[x]\, \ln[x]\, dx$

4) Gauge invariant regulators:

To investigate how gauge invariance is affected by η regularisation, we follow [25, 26, 29] and consider a general gauge theory where the gauge group has dimension d_G and where N_f Dirac spinors Ψ_n $(n = 1, \ldots, N_f)$ are interacting with the Yang Mills field A_μ^a $(a = 1, \ldots, d_G)$. Such a theory is described by a Lagrangian

$$\mathcal{L} = \bar{\psi}_n(i\gamma^\mu D_\mu - m)\psi_n - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu}, \tag{3.25}$$

where

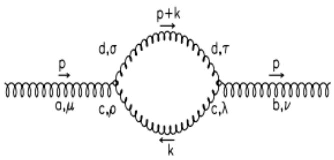
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu, \qquad D_\mu\psi_n = (\partial_\mu + igT^a A^a_\mu)\psi_n, \tag{3.26}$$

and T^a are the generators of the gauge group whose commutator $[T^a, T^b] = if^{abc}T^c$ defines the structure constants f^{abc}. A careful computation of the vacuum polarisation for the gauge field at one-loop (see [25, 26, 29] for details) yields an expression of the form

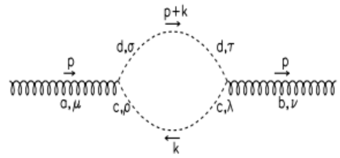
$$\Pi^{ab}_{\mu\nu}(p) = \Pi^{(g)ab}_{\mu\nu}(p) + \Pi^{(f)ab}_{\mu\nu}(p), \tag{3.27}$$

where p^μ is the external momentum. Here $\Pi^{(g)ab}_{\mu\nu}(p)$ are the pure Yang Mills contributions coming from gauge field loops and ghost loops. $\Pi^{(f)ab}_{\mu\nu}(p)$ are the contributions from fermion loops, arising from the interaction of the fermions with the gauge field. Gauge invariance is understood in terms of the Ward identities $p^\mu\Pi^{ab}_{\mu\nu} = \Pi^{ab}_{\mu\nu}p^\nu = 0$. Requiring this to hold for any gauge theory and with any number of fermions means that Ward identities should hold

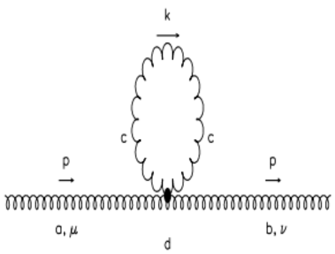
[25] Starts with the diagrams below and write the vacuum polarization in terms of the integrals.



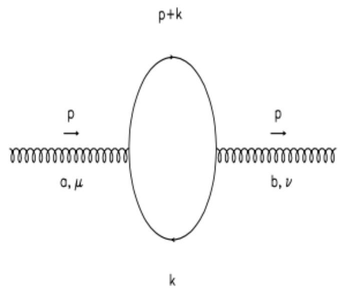
(1)



(3)



(2)



(4)

Requiring gauge invariance puts the following constraint in the integrals

$$I^{\mu\nu}_{-2\alpha}|_{\text{regularised}} \sim \frac{1}{2(\alpha+2)} g^{\mu\nu} I_{-2\alpha}|_{\text{regularised}},$$
$$I^{\mu\nu\rho\sigma}_{-2\alpha}|_{\text{regularised}} \sim \frac{1}{4(\alpha+2)(\alpha+3)} S^{\mu\nu\rho\sigma} I_{-2\alpha}|_{\text{regularised}},$$

This constrains in terms of the regulators implies:

i) We need more than one regulator:

$$I_{-2\alpha}|_{\text{regularised}} = iJ_{-2\alpha}[\eta_{-2\alpha}],$$
$$I^{\mu\nu}_{-2\alpha}|_{\text{regularised}} = iJ^{\mu\nu}_{-2\alpha}[\theta_{-2\alpha}],$$
$$I^{\mu\nu\rho\sigma}_{-2\alpha}|_{\text{regularised}} = iJ^{\mu\nu\rho\sigma}_{-2\alpha}[\kappa_{-2\alpha}],$$

ii) The satisfy the following properties:

$$\eta_{-2\alpha}(x) = \eta_{[1]}(x), \quad \theta_{-2\alpha}(x) = \eta_{[1]}(\lambda x), \quad \kappa_{-2\alpha}(x) = \eta_{[1]}(\mu x),$$

where $\lambda = e^{-1/4}$, $\mu = e^{-5/12}$, and $\eta_{[1]}(x)$ is any enhanced regulator of order one.

4) Gauge invariant regulators → enhanced regulators