Journal Club

Definitions

Background:

1) Inversion formula:

- CFT data has a natural analytic continuation in spins

$$c^t(J,\Delta) = \frac{\kappa_{J+\Delta}}{4} \int_0^1 dz d\bar{z}\, \mu(z,\bar{z})\, G_{\Delta+1-d,J+d-1}(z,\bar{z})\, \mathrm{dDisc}\left[\mathcal{G}(z,\bar{z})\right].$$

2) Light-ray operators

- Operators themselves have a natural analytic continuation in spins

Step 1) Make the local operator non-local -- light-transform.

$$\mathbf{L}[\mathcal{O}](x,z) \equiv \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} \mathcal{O}\left(x - \frac{z}{\alpha}, z\right).$$

Step 2) Analytic continue in spin -- light-ray operators.

This can be made explicit in weakly-couple theories. For example:

In weakly-coupled CFTs, analyticity in spin of double-twist families can be made fully explicit at the level of operators. Consider a free field ϕ. The non-local light-ray operator

$$\mathbb{O}_2(S) = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2 \left(\Psi_{2,S}(\alpha_1,\alpha_2) \equiv \frac{|\alpha_1-\alpha_2|^{-1-S}}{\Gamma(-S)} \right) :\phi(\alpha_1 n^+)\phi(\alpha_2 n^+): \qquad (1)$$

(where n^+ is a null vector in the $+$ direction) is manifestly analytic in S. Moreover, for even integer spin $S = 2k$, it reduces to the null-integral of the local double-twist spin $2k$ operator $[\phi\phi]_{2k,0}$ defined by

$$[\phi\phi]_{2k,0} = :\phi\partial_+^{2k}\phi: + \partial_+(\cdots),$$

with the total derivatives $(\cdots)$ fixed so that $[\phi\phi]_{2k,0}$ is a primary. To see this, note that as a distribution, $\Psi_{2,S}(\alpha_1,\alpha_2) = \delta^{(2k)}(\alpha_1-\alpha_2) + O(S-2k)$. The matrix elements of $\mathbb{O}_2(S)$ therefore provide an analytic continuation in S of the matrix elements of null integrals of $[\phi\phi]_{S,0}$.

3) Understand this construction in $\mathcal{N} = 4$ SYM (pertubatively)

For twist-2 operators we have a nice picture:

In[20]:=
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nice
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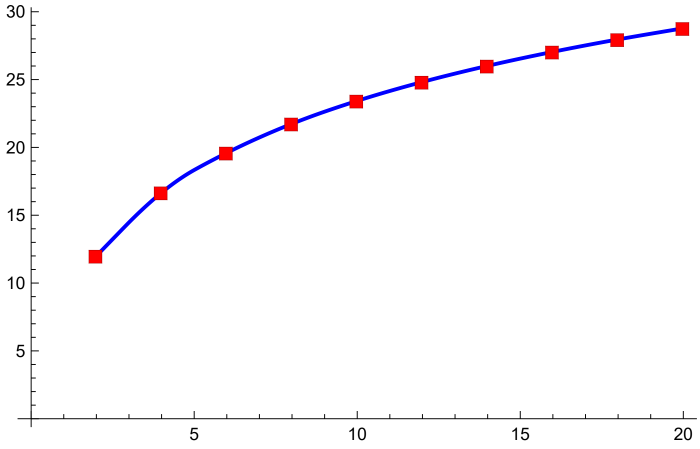
Out[20]=



But for higher twist operators we have a puzzle

In[21]:=
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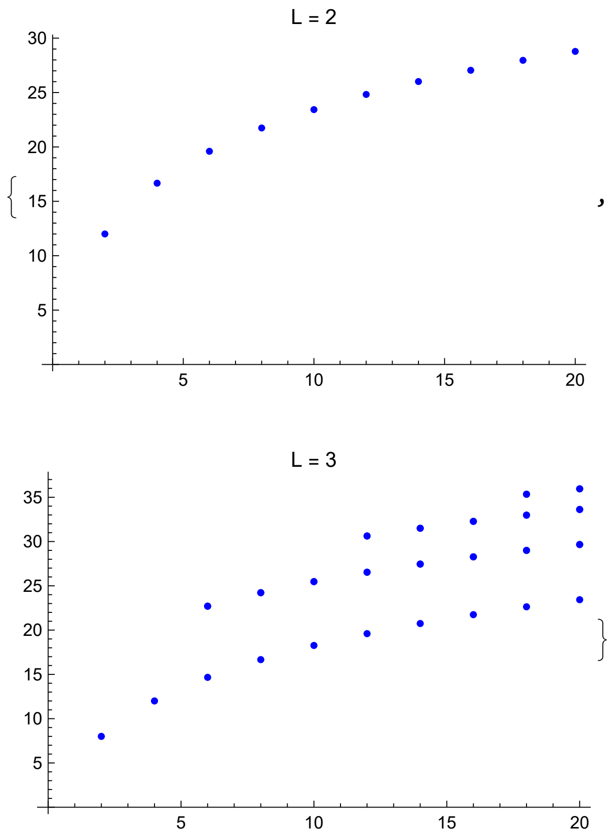
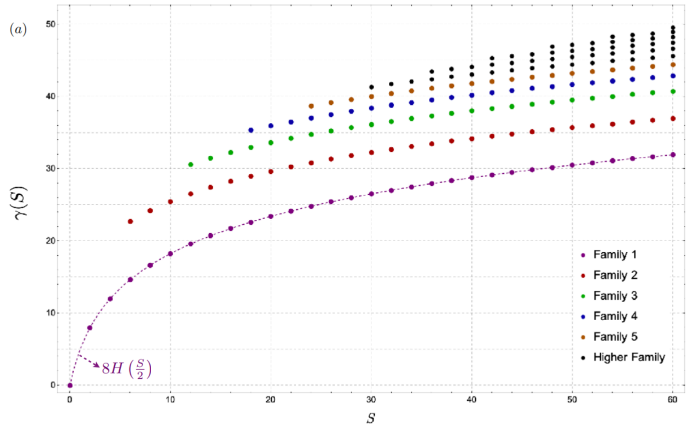
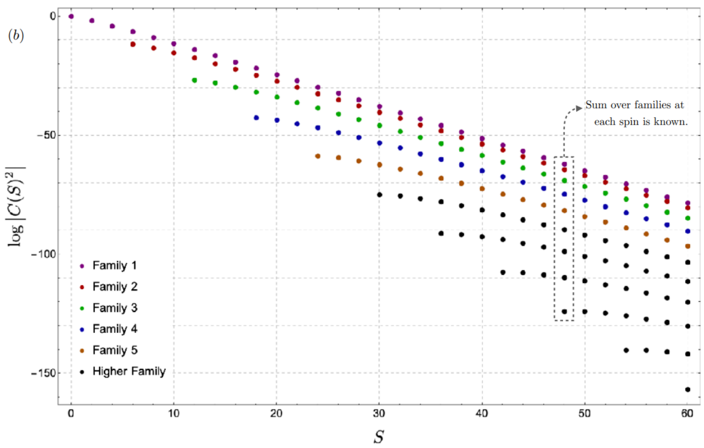


Idea:

Focus on the simplest possible case where this happens: twist-3

In[22]:=





Two puzzles:

1) How many light-ray operators there are?

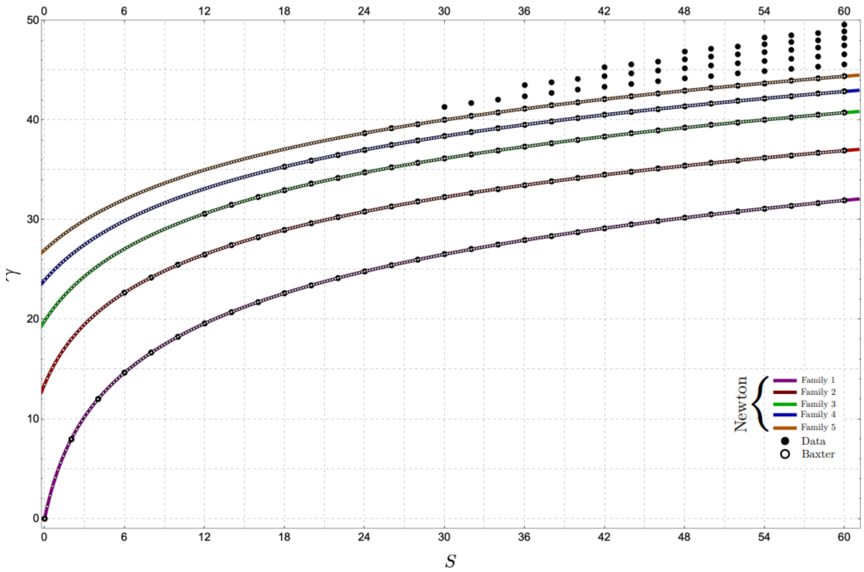
2) How they can give zeros for the three-point functions? (it does not sound very analytic continuous)

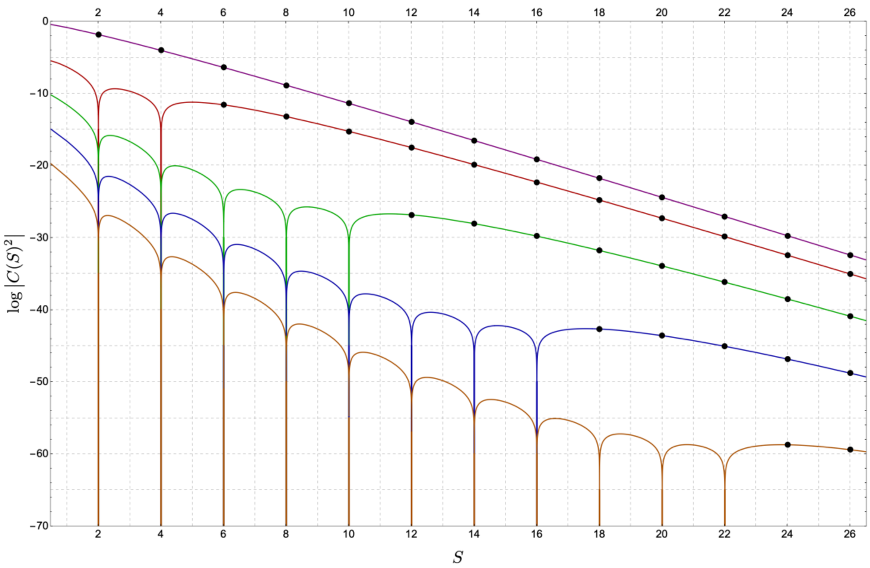
On https://arxiv.org/pdf/2211.13754, they analyzed this numerically.
Feed in the points of the data at physical values of spin and analytic continue using Newton's method:

What is perhaps less well-known is that the unique extension alluded to in Carlson's theorem can be explicitly constructed by a beautiful interpolation series written down by Newton in 1687's *Principia Mathematica*.[16] Newton's series

$$f_N(z) \equiv \sum_{j=0}^{N} \binom{z}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} f(i)\,. \qquad (8)$$

They get this:





They can see that the analytic continuation of the data indeed have the correct zeros!
However, this does not tell anything of the very nature of the light-ray operators.

Now on https://arxiv.org/pdf/2409.02160, they analyzed it analytically.
By explicitly constructing the wave function for twist-3 operators.

$$\mathbb{O}_3(S) = \int_{-\infty}^{\infty} \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3 \Psi_{3,S}(\alpha_1, \alpha_2, \alpha_3)\ {:}\phi(\alpha_1 n^+)\phi(\alpha_2 n^+)\phi(\alpha_3 n^+){:}\ .$$

Now, let’s switch to the paper.