Instantons and the Large $\mathcal{N} = 4$ Algebra

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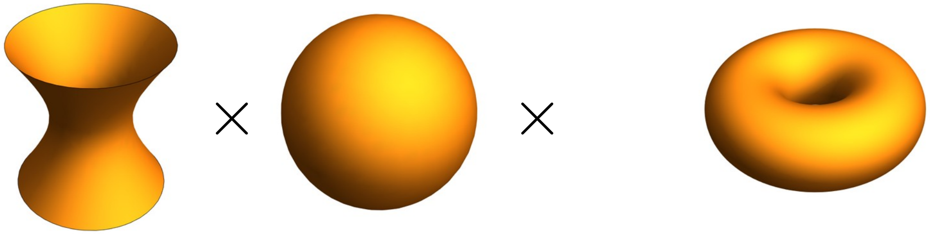
Abstract: We investigate the differential geometry of the moduli space of instantons on $S^3 \times S^1$. Extending previous results, we show that a sigma-model with this target space can be expected to possess a large $\mathcal{N} = 4$ superconformal symmetry, supporting speculations that this sigma-model may be dual to Type IIB superstring theory on $AdS_3 \times S^3 \times S^3 \times S^1$. The sigma-model is parametrized by three integers – the rank of the gauge group, the instanton number, and a "level" (the integer coefficient of a topologically nontrivial B-field, analogous to a WZW level). These integers are expected to correspond to two five-brane charges and a one-brane charge. The sigma-model is weakly coupled when the level, conjecturally corresponding to one of the five-brane changes, becomes very large, keeping the other parameters fixed. The central charges of the large $\mathcal{N} = 4$ algebra agree, at least semiclassically, with expectations from the duality.

$AdS_3\times S^3$ *Holography*

$AdS_3 \times S^3 \times T^4$



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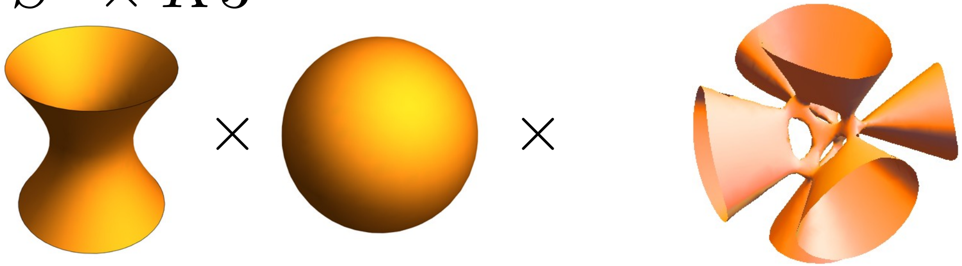
(small $\mathcal{N} = 4$ symmetry)

$AdS_3 \times S^3 \times K3$



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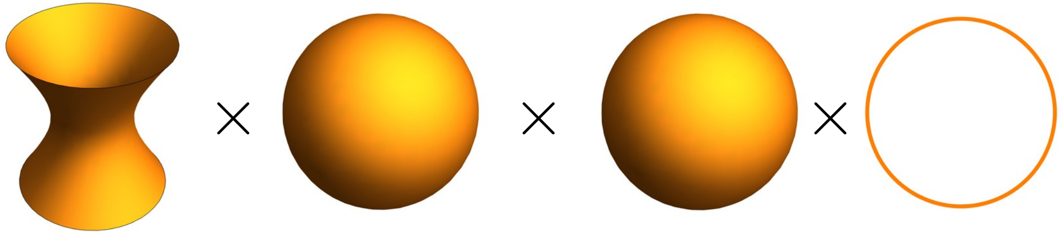
(small $\mathcal{N} = 4$ symmetry)

$AdS_3 \times S^3 \times S^3 \times S^1$



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(large $\mathcal{N} = 4$ symmetry)

Candidate duals

– **Symmetric product orbifolds**

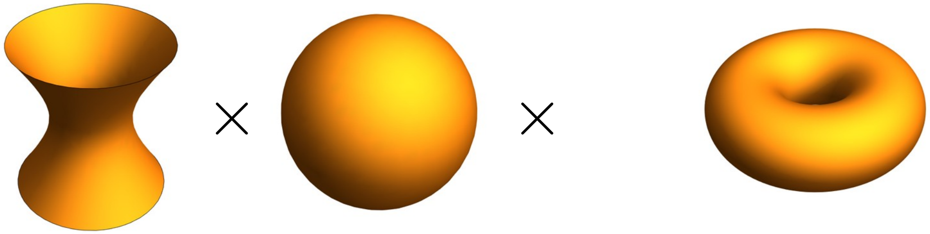
Take a large number of copies of a known SCFT and mod out the symmetric group

– σ**-models**

Find some target space and consider the WS-theory with an appropriate metric and B-field

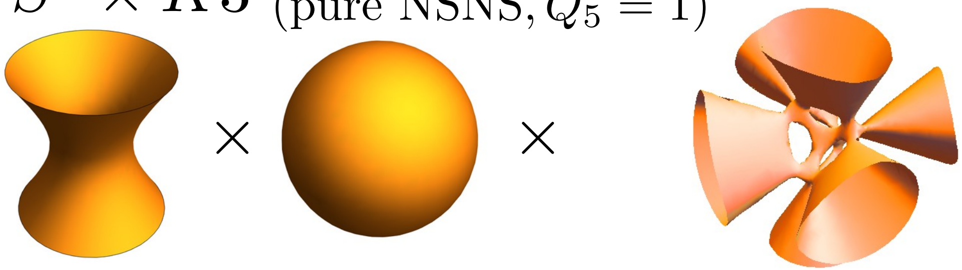
$AdS_3 \times S^3$ *Holography*

$AdS_3 \times S^3 \times T^4$ (pure NSNS, $Q_5 = 1$)



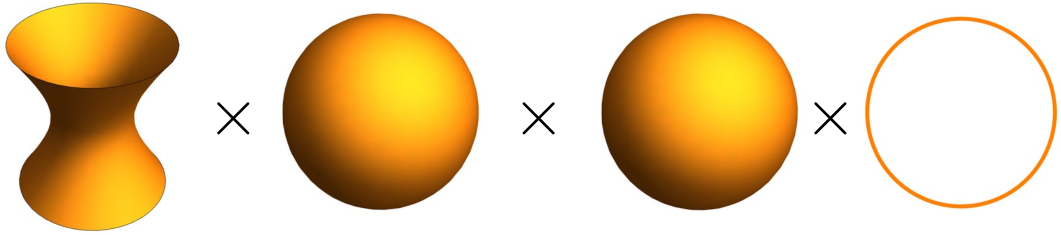
$\longleftrightarrow$ $Sym_\infty(T^4)$

$AdS_3 \times S^3 \times K3$ (pure NSNS, $Q_5 = 1$)



$\longleftrightarrow$ $Sym_\infty(K3)$

$AdS_3 \times S^3 \times S^3 \times S^1$



$\longleftrightarrow$ σ-model of $\mathcal{M}(S^3 \times S^1)$

Brane configuration

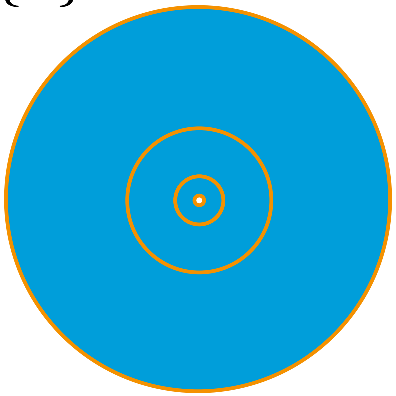
Start with $\mathbb{R}^2 \times S^1 \times T^*S^3 \times \mathbb{R}$

	$\mathbb{R}^2$	S^1	S^3	"T^*"	$\mathbb{R}$
Q_5' units flux			$\times$		
Q_5 D5-branes	$\times$	$\times$	$\times$		
Q_1 D1-branes	$\times$				

→ Q_1 instantons in $SU(Q_5)$ gauge theory with level Q_5'

$S^3 \times S^1$ *geometry*

$\mathbb{C}^2/\{0\}$



$$ds^2 = \frac{d\vec{Y}^2}{\vec{Y}^2}, \quad \vec{Y} \cong e^T \vec{Y}$$

$$ds^2 = d\Omega^2 + d\tau^2, \quad \tau \cong \tau + T,$$

Hermitian, not Kähler

S^3 has $SU(2)_L \times SU(2)_R$ symmetry, *e.g.*

$$g = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}, \quad |z_1|^2 + |z_2|^2 = 1, \quad Z_i = z_i e^\tau.$$

$SU(2)_R$ changes complex structure

→ hypercomplex structure: $\mathcal{I}, \mathcal{J}, \mathcal{K}$

May define $\mathcal{I}', \mathcal{J}', \mathcal{K}'$ for $\mathrm{SU}(2)_L$

σ-model on $S^3 \times S^1$

(4,4) supersymmetry due to $\mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{I}', \mathcal{J}', \mathcal{K}'$

Allows for torsion-full connection generated by $H \sim Q_5' d\Omega_3$

→ WZW model $SU(2)_k$ + 1 free boson + 4 free fermions

$$T = -J^0 J^0 - \frac{\sum_{i=1}^{3} J^i J^i}{\kappa + 2} - \sum_{a=0}^{3} \partial\psi^a \psi^a$$

$$G_a = 2J^0 \psi_a + \frac{4\alpha_{ab}^{+,i} J^i \psi^b}{\sqrt{\kappa+2}} - \frac{2\epsilon_{abcd}\psi^b\psi^c\psi^d}{3\sqrt{\kappa+2}}$$

$$A^{-,i} = \alpha_{ab}^{-,i} \psi^a \psi^b$$

$$A^{+,i} = J^i + \alpha_{ab}^{+,i} \psi^a \psi^b$$
$$U = -\sqrt{\kappa+2} J^0$$
$$Q^a = \sqrt{\kappa+2} \psi^a.$$

Large $\mathcal{N} = 4$ algebra

“Pullback” to moduli space

1. If M satisfies the conditions for $(0,2)$ supersymmetry – it is a complex manifold with a hermitian metric whose torsion is closed in a sense reviewed in section 3.1 – then $\mathcal{M}$ is also a complex manifold[4] [29, 30], with a natural hermitian metric that also has closed torsion [13], so the sigma-model with target $\mathcal{M}$ also has $(0,2)$ supersymmetry,

2. If M is a generalized Kahler manifold (the geometry that leads to $(2,2)$ supersymmetry with a B-field) then so is $\mathcal{M}$ [14, 16].

3. If M is an HKT manifold (the geometry that leads to $(0,4)$ supersymmetry, with a small $\mathcal{N}=4$ algebra), then so is $\mathcal{M}$ [15].

4. If M is generalized hyper-Kahler or bi-HKT (leading to $(4,4)$ supersymmetry with the small $\mathcal{N}=4$ algebra), then so is $\mathcal{M}$. This follows on combining results in [14] and [15]; see section 5.

5. Finally, if M has the properties that lead to invariance under the large $\mathcal{N}=4$ algebra, then so does $\mathcal{M}$. This is shown in section 6.

More things discussed

Conformality of the σ-model

Symmetries, topology of the moduli space

Matching of central charges (functions of Q_1, Q_5, Q'_5)

Orbifolds $S^3/\mathbb{Z}_K \times S^1$

Multiple beautiful discussions, proofs, references...

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