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THE TWO-LOOP AMPLITUHEDRON

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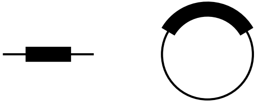
The *loop-Amplituhedron* $\mathcal{A}_n^{(L)}$ is a semialgebraic set in the product of Grassmannians $\mathrm{Gr}_{\mathbb{R}}(2,4)^L$. Recently, many aspects of this geometry for the case of $L = 1$ have been elucidated, such as its algebraic and face stratification, the residual arrangement and the existence and uniqueness of the adjoint. This paper extends this analysis to the simplest higher loop case given by the two-loop four-point Amplituhedron $\mathcal{A}_4^{(2)}$.

Positive Geometries

- **Semi-algebraic sets:** Regions defined by polynomial inequalities.



- **Positive geometries:** Semi-algebraic sets associated with a unique differential forms called the **canonical form**

$$X_{\geq} \longrightarrow \Omega(X_{\geq})$$

- In $\mathcal{N} = 4$ SYM:

Amplituhedron $\Rightarrow$ Superamplitudes integrands

- **Boundaries:** Semi-algebraic sets boundaries have support on **varieties** (solutions of polynomial systems).
- **Residue property:** For a variety C we have:

$$\Omega(\partial_C X_{\geq}) = \text{Res}_C \Omega(X_{\geq})$$

The Grassmannian $Gr(2,4)$

The Grassmannian $Gr(2,4)$ represents lines in $\mathbb{P}^3$. A line through points $A = (a_1, a_2, a_3, a_4)$ and $B = (b_1, b_2, b_3, b_4)$ can be represented by the 2x 4 matrix

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

mod $GL(2)$ The invariants are the minors p_{ij} for example

$$p_{12} = \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

These are related by the Plücker identity

$$p_{12}p_{34} + p_{23}p_{14} - p_{13}p_{24} = 0$$

The Grassmannian is a **variety**.

4-point Amplituhedron:

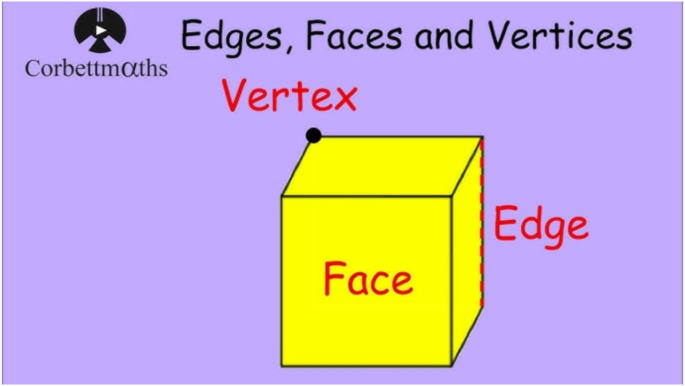
The 4-point loop amplituhedron is a semi-algebraic set

$$\mathcal{A}_{4,L} := \begin{cases} (AB)_l \in Gr_{\geq}(2,4) & \forall\, l = 0, \cdots, L \\ \det \begin{pmatrix} (AB)_l \\ (AB)_m \end{pmatrix} \geq 0 & \forall\, l \neq m \end{cases}$$

Denote the Plückers of $(AB)_1$ as p_{ij} and those of $(AB)_2$ as q_{ij}.

$$\mathcal{A}_{4,2} : \begin{cases} p_{ij}, q_{ij} \geq 0\,, \qquad \forall\, 1 \leq i < j \leq 4 \\ p_{12}q_{34} + p_{34}q_{12} + p_{23}q_{14} + p_{14}q_{23} - p_{13}q_{24} - p_{24}q_{13} \geq 0 \end{cases}$$

Boundary Stratification



+ inclusion relations

Boundary Stratification

The brute-force approach: Maple's `RegularChains` library and **C**ylindric **A**lgebraic **D**ecomposition (CAD): Does not terminate

Algebraic Boundary Stratification:

- is a set of varieties of all codimensions.
- Intersections of strata with the Amplituhedron are either empty or boundaries.
- Use CAD to compute intersections.

Codimension:	1	2	3	4	5	6	7	8
Components:	9	44	144	324	450	370	168	36
Boundaries:	9	44	144	286	356	306	156	34
Residual:	0	0	0	38	94	64	12	2
Regions:	9	52	176	326	416	342	156	34

Residual strata $\Rightarrow$ non-trivial vanishing residue$\Rightarrow$ fix the amplitude

Topology

Theorem 5.1. *The Euclidean interior* $\mathrm{int}(\mathcal{A}_4^{(2)})$ *of* $\mathcal{A}_4^{(2)}$ *is connected and its fundamental group is free of rank one*[3].

We compute boundary-connected components using CAD.

Euler characteristic of a sphere? $\Rightarrow$ triangulate the sphere and count triangles, edges and vertices. You built a **simplicial complex**.

CW complex is built by glueing together balls (so-called cells) of different dimensions in specific ways.

Boundaries are not cells $\rightarrow$ more work needed xP

Conclusions and Outlooks

Conclusions We took the first steps in the understanding of the geometry of loop amplituhedron:

- We computed the boundary structure of $\mathcal{A}_{4,2}$
- We analyzed their topology

Outlooks

- Give the amplituhedron a CW structure.
- Find some nice combinatorics behind the boundary stratification to go beyond $n = 4, L = 2$.