Integrable and critical Haagerup spin chains

Luke Corcoran[*] and Marius de Leeuw[†]
School of Mathematics & Hamilton Mathematics Institute
Trinity College Dublin, Ireland

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We construct the first integrable models based on the Haagerup fusion category H_3. We introduce a Haagerup version of the anyonic spin chain and use the boost operator formalism to identify two integrable Hamiltonians of PXP type on this chain. The first of these is an analogue of the golden chain; it has a topological symmetry based on H_3 and satisfies the Temperley-Lieb algebra with parameter $\delta = (3 + \sqrt{13})/2$. We prove its integrability using a Lax formalism, and construct the corresponding solution to the Yang–Baxter equation. We present numerical evidence that this model is gapless with a dynamical critical exponent $z \neq 1$. The second integrable model we find breaks the topological symmetry. We present numerical evidence that this model reduces to a CFT in the large volume limit with central charge $c \sim 3/2$.

Motivation

- Non-invertible symmetries play an important role for understanding QFTs.
  Ex. the Kramers-Wannier duality defect in the 1+1d critical Ising model.

- The primary example of non-invertible symmetries based on Fusion Categories is "Golden Chain" based on the Fibonacci fusion category.
  Objects, 1 and τ

$$\tau^2 = 1+\tau$$

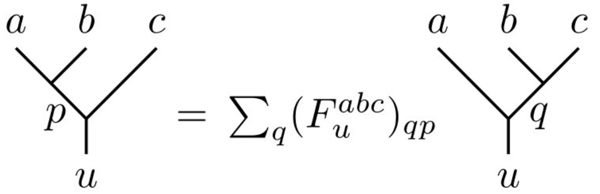
  This is a spin chain which becomes c = 7/10 tricritical Ising CFT in the continuum limit, which realizes the non-invertible Fibonacci symmetry.

- Question: What are the other lattice models that come from fusion categories?

- The simplest fusion category which is not simply related to a affine Lie algebra or a quantum group is Haagerup fusion category.

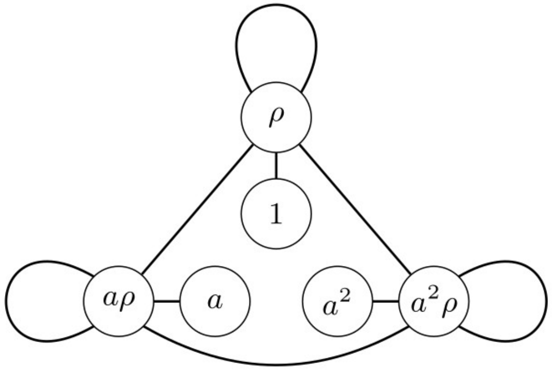
Fusion Category with objects *a,b,c...*

$$a \times b = \bigoplus_{c} N_{ab}^{c}\; c$$

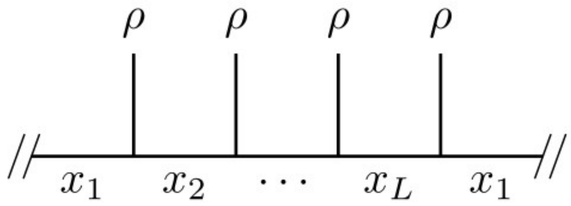


Haagerup Fusion Category and Anyonic Spin Chains

$$\mathrm{Obj}(H_3) = \{1, a, a^2, \rho, a\rho, a^2\rho\}$$



$$|x_1 x_2 \cdots x_L\rangle =$$



$$x_{i+1} \in x_i \times \rho$$

$$\Pi^L = \prod_{i=1}^{L} \Pi_{i,i+1}, \qquad \Pi^L : (\mathbb{C}^6)^{\otimes L} \to V^L,$$

Haagerup Symmetry

$$\langle y_1 y_2 \cdots y_L | \, Y_z \, | x_1 x_2 \cdots x_L \rangle = \prod_{i=1}^{L} (F^{z x_i \rho}_{y_{i+1}})_{x_{i+1} y_i},$$

Integrability with range 3 Hamiltonian, $\mathbb{Q}_2$

$$\Pi^L [\mathbb{Q}_i, \mathbb{Q}_j] \Pi^L = 0, \qquad [\mathbb{Q}_i, \Pi^L] = 0.$$

Boost Operator

$$\mathbb{Q}_3 = \sum_{i=1}^{L} [\mathcal{H}_{i,i+1,i+2}, \mathcal{H}_{i+1,i+2,i+3} + \mathcal{H}_{i+2,i+3,i+4}]$$

Hamiltonian Ansatz

$$\mathcal{H}_{i,i+1,i+2} = D_i \mathcal{O}_{i+1} D_{i+2},$$

Results

Transfer Matrix wrt Lax Operator

$$t(u) := \mathrm{tr}_A[\mathcal{L}_{AL}(u)\mathcal{L}_{A,L-1}(u)\cdots\mathcal{L}_{A1}(u)],$$

$$\Pi^L[\mathbb{Q}_2,\mathbb{Q}_3]\Pi^L = 0, \qquad \Pi^L[\mathbb{Q}_2, Y_\rho]\Pi^L = 0,$$

Model 1. Writing $\mathbb{Q}_2 = -\sum_{i=1}^{L} e_i$ we find that the generators e_i satisfy the Temperley-Lieb algebra

$$e_i^2 = \psi\, e_i, \qquad e_i e_{i\pm1} e_i = e_i,$$
$$[e_i, e_j] = 0, \qquad |i-j| > 1. \tag{17}$$

In such cases the Lax operator can typically be obtained as a linear combination of the identity matrix and $\mathcal{H}$. Indeed, we find that

$$\mathcal{L}_{123}(u) = \mathcal{P}_{13}\mathcal{P}_{23}\left(1 - \frac{1}{\frac{\psi}{2} - \alpha\coth\alpha u}\mathcal{H}_{123}\right)$$
$$:= \mathcal{P}_{13}\mathcal{P}_{23}\check{\mathcal{L}}_{123}(u) \tag{18}$$

$$\Pi^L[\mathbb{Q}_2,\mathbb{Q}_3]\Pi^L = 0,$$

Model 2. The Lax operator for model 2 is given by

$$\check{\mathcal{L}}_{123}(u) = 1 + u\mathcal{H}_{123} + f(u)\mathcal{H}_{123}^2, \tag{21}$$

where $(u + 2\gamma f(u))^2 = f(u)(f(u)+2)$.

$$\check{R}_{1234}(u,v) = \check{\mathcal{L}}_{123}(-v)\check{\mathcal{L}}_{234}(u-v)\check{\mathcal{L}}_{123}(u)$$

$$\Delta E := E_1 - E_0 \propto \frac{1}{L^z}.$$

$$S \sim \frac{c}{3} \log L$$

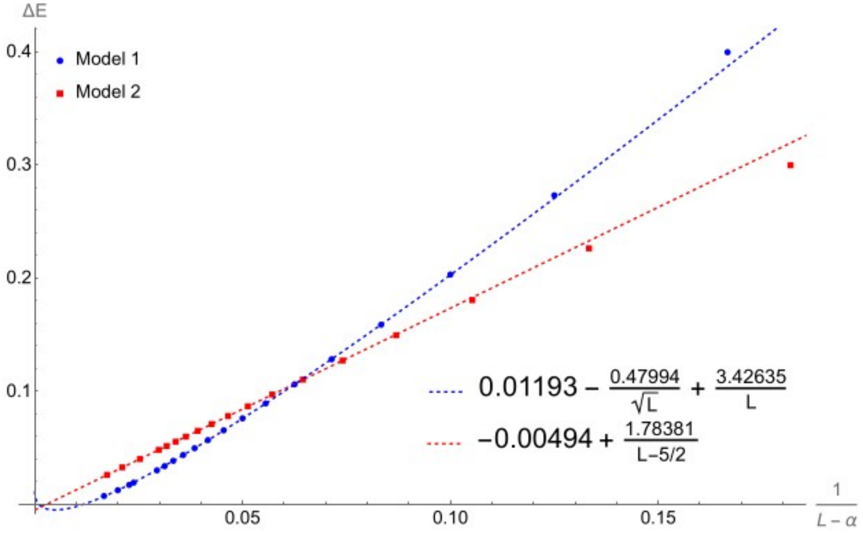


FIG. 2. Energy gap up to $L = 60$ on the periodic chain vs. an inverse shifted length $(L - \alpha)^{-1}$. For model 1 we take $\alpha = 0$ and for model 2 we take $\alpha = 5/2$. For the model 2 fit we used the last 10 points.

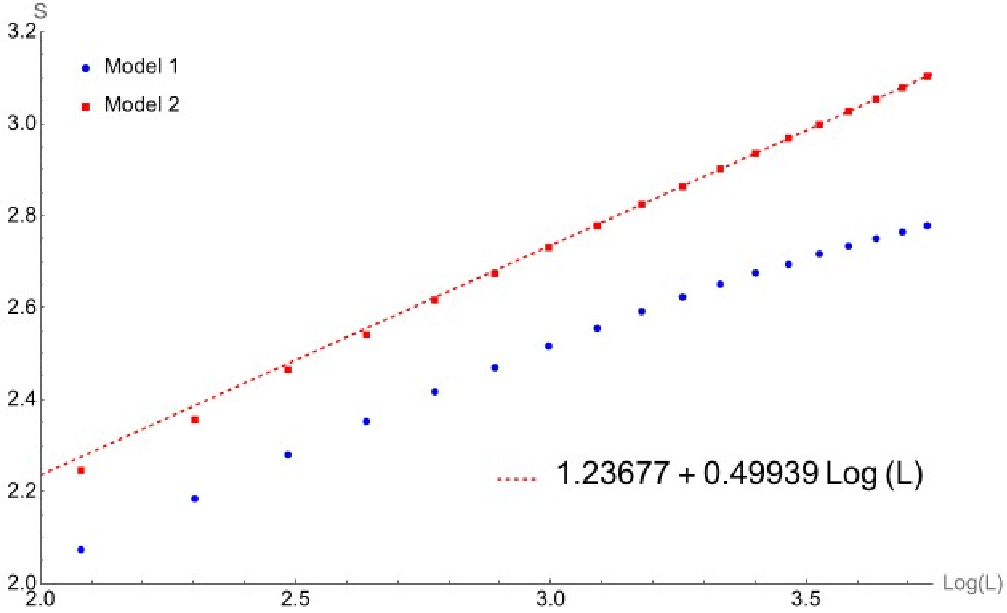


FIG. 3. Half-chain entanglement entropy up to $L = 42$ on the periodic chain. For the fit we used the last 10 points.

Conclusion

- Authors present a new way of generating integrable models with fusion category symmetry.
- They find that only one integrable Hamiltonian is compatible with the full Haagerup symmetry. The other model we present breaks this topological symmetry, but seems to correspond to a CFT in the continuum limit.
- The fact that first model is not critical is surprising. ($\Psi=q+q^{-1}$ is not root of unity.)
- Maybe first model corresponds to a deformation of a CFT.
- They conjecture that the properties that they observed for this fusion category persists to higher rank versions...