Quick overview

- Motivation: By now the Virasoro-Shapiro amplitude for flat space tree-level tachyon scattering has been known for many decades. But determining analogous formulas for tree-level scattering on general curved backgrounds still remains a challenge.
- Concrete Goal: Compute string scattering amplitudes to all orders in α' in a small AdS curvature expansion around flat space.

Scenario 1: World-sheet theory unknown.

- Strategy:

  1. Analytic Bootstrap: Determine the low energy expansion of the amplitude using Regge boundedness of the corresponding correlator in the dual CFT and the CFT data of heavy operators known e.g. from integrability.

  2. Observation: No matter what the World-sheet theory is, it should express the scattering amplitude as some integral over String World-sheets. The integrand is strongly constrained by demanding crossing symmetry and consistency with the low energy expansion.

  3. Assumption: Considering all constraints, it was conjectured that the typical world sheet integrands should be expressible as some weighted sum of Single Valued Multi PolyLogarithms (SVMPL).

  4. Computation: If we can reduce the number of unknowns in the small AdS curvature expansion up to a certain order to finitely many coefficients, we can compute those by matching with the structure imposed by the decomposition of the dual correlator into conformal blocks.
- Some of the papers that have applied/developed this strategy are:

[24] L. F. Alday, T. Hansen, and J. A. Silva, AdS Virasoro-Shapiro from dispersive sum rules, JHEP **10**, 036, arXiv:2204.07542 [hep-th].
[25] L. F. Alday, T. Hansen, and J. A. Silva, AdS Virasoro-Shapiro from single-valued periods, JHEP **12**, 010, arXiv:2209.06223 [hep-th].
[26] L. F. Alday, T. Hansen, and J. A. Silva, Emergent World-sheet for the AdS Virasoro-Shapiro Amplitude, Phys. Rev. Lett. **131**, 161603 (2023), arXiv:2305.03593 [hep-th].
[27] L. F. Alday and T. Hansen, The AdS Virasoro-Shapiro amplitude, JHEP **10**, 023, arXiv:2306.12786 [hep-th].

- Today we look at a paper, where the NLO term in the curvature expansion for the four-point scattering amplitude of Dilatons in AdS_3 xS^3 xM_4 with pure RR flux is computed using the above strategy. (M_4 is T^4 or K3. At tree level the Dilaton is not sensitive to the difference.)

The $\mathrm{AdS}_3\times\mathrm{S}^3$ Virasoro-Shapiro amplitude with RR flux

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We compute the AdS Virasoro-Shapiro amplitude for scattering of dilatons in type IIB theory with pure RR flux on $AdS_3 \times S^3 \times M_4$ for $M_4 = T^4$ or $K3$, to all orders in α' in AdS curvature expansion. This is achieved by comparing the flat space limit of the dual D1D correlator to an ansatz for the amplitude as a worldsheet integral in terms of single valued n polylogarithms. The first curvature correction is fully fixed in this way, and satisfies cons checks in the high energy limit, and by comparison of the energy of massive string operato semiclassical expansion. Our result gives infinite predictions for CFT data in the planar l strong coupling, which can guide future integrability studies.

- Note that a few days ago, there was also another paper on the arxiv, where a similar computation was done for graviton scattering of IIA on AdS4 x CP3. In that case, using extra input from ABJM integrability, allowed for the computation not only of the first, but also of the second curvature correction.

The type IIA Virasoro-Shapiro amplitude in $\mathrm{AdS}_4\times\mathrm{CP}^3$ from ABJM theory

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Abstract: We consider tree level scattering of gravitons in type IIA string theory on $AdS_4 \times \mathbb{CP}^3$ to all orders in α', which is dual to the stress tensor correlator in $U(N)_k \times U(N)_{-k}$ ABJM theory in the planar large N limit and to all orders in large $\lambda \sim N/k$. The small curvature expansion of this correlator, defined via a Borel transform, is given by the flat space Virasoro-Shapiro amplitude plus AdS curvature corrections. We fix curvature corrections by demanding that their resonances are consistent with the superconformal block expansion of the correlator and with a worldsheet ansatz in terms of single-valued multiple polylogarithms. The first correction is fully fixed in this way, and matches independent results from integrability, as well as the R^4 correction at finite AdS curvature that was previously fixed using supersymmetric localization. We are also able to fix the second curvature correction by using a few additional assumptions, and find that it also satisfies various non-trivial consistency checks. We use our results to fix the tree level D^4R^4 correction at finite AdS curvature, and to give many predictions for future integrability studies.

Scenario 2: World-sheet theory known.

- Strategy: Write down the integral and expand it. (Relevance: An important check for the SVMPL Assumption in scenario 1.)

On the AdS_3 Virasoro-Shapiro Amplitude

Luis F. Alday,[a] **Gaston Giribet,**[b] **Tobias Hansen**[c]

Abstract: We consider tree-level scattering amplitudes for four strir
$\mathcal{N}$ with pure NSNS fluxes. We show that in a small curvature expans
the amplitudes take the form of a genus zero integral given by the '
grand with the extra insertion of single valued multiple polylogarith
structure as the one found for the AdS Virasoro-Shapiro amplitude

Some details of the computation

0. Single Valued Multi Polylogarithms

1.1. MPLs

$$L_e(z) = 1, \quad L_{0^p}(z) = \frac{1}{p!}\log^p z, \quad p = 1, 2, \cdots$$

In[∘]:=
```
Clear[L]
```

In[∘]:=
```
L[{}] := 1
L[{0}] := Log[z]
L[{0, 0}] := 1/2 Log[z]^2
L[{0, 0, 0}] := 1/6 Log[z]^3
```

In[∘]:= $$\frac{\partial}{\partial z} L_{\sigma_i w}(z) = \frac{L_w(z)}{z - \sigma_i}.$$

$$L_w(z) \to 0 \text{ as } z \to 0$$

In[∘]:=
```
FullSimplify[f[z] /. DSolve[D[f[z], z] == L[{}]/(z - 1) && f[0] == 0, f, {z}][[1]] // Expand]
```

Out[∘]= $-i\pi + \text{Log}[-1 + z]$

In[∘]:=
```
L[{1}] := Log[1 - z];
```

In[∘]:=
```
L[w_] := L[w] = FullSimplify[
    f[z] /. DSolve[D[f[z], z] == L[w[[2 ;;]]]/(z - w[[1]]) && f[0] == 0, f, {z}][[1]] // Expand, Assumptions → z > 10]
```

In[∘]:=
```
L[{0, 0, 1}]
```

Out[∘]= -PolyLog[3, z]

In[∘]:=
```
L[{0, 1, 1}]
```

Out[∘]= $\frac{1}{2}$ Log[1 - z]^2 Log[z] + Log[1 - z] PolyLog[2, 1 - z] - PolyLog[3, 1 - z] + Zeta[3]

In[∘]:=
```
L[{1, 1, 1}]
```

Out[∘]= $\frac{1}{6}$ Log[1 - z]^3

1.2. SVMPLs

$$\mathcal{L}_e(z) = 1,\ \mathcal{L}_{0^p}(z) = \frac{1}{p!}\log^p |z|^2$$

$$\frac{\partial}{\partial z}\mathcal{L}_{\sigma_i w}(z) = \frac{\mathcal{L}_w(z)}{z-\sigma_i}.$$

Emergent world-sheet for the AdS Virasoro-Shapiro amplitude

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$$\begin{aligned}
\mathcal{L}_{000}(z) &= L_{000}(z) + L_{000}(\bar z) + L_{00}(z)L_0(\bar z) + L_0(z)L_{00}(\bar z),\\
\mathcal{L}_{001}(z) &= L_{001}(z) + L_{100}(\bar z) + L_{00}(z)L_1(\bar z) + L_0(z)L_{10}(\bar z),\\
\mathcal{L}_{010}(z) &= L_{010}(z) + L_{010}(\bar z) + L_{01}(z)L_0(\bar z) + L_0(z)L_{01}(\bar z),\\
\mathcal{L}_{100}(z) &= L_{100}(z) + L_{001}(\bar z) + L_{10}(z)L_0(\bar z) + L_1(z)L_{00}(\bar z),\\
\mathcal{L}_{110}(z) &= L_{110}(z) + L_{011}(\bar z) + L_{11}(z)L_0(\bar z) + L_1(z)L_{01}(\bar z),\\
\mathcal{L}_{101}(z) &= L_{101}(z) + L_{101}(\bar z) + L_{10}(z)L_1(\bar z) + L_1(z)L_{10}(\bar z),\\
\mathcal{L}_{011}(z) &= L_{011}(z) + L_{110}(\bar z) + L_{01}(z)L_1(\bar z) + L_0(z)L_{11}(\bar z),\\
\mathcal{L}_{111}(z) &= L_{111}(z) + L_{111}(\bar z) + L_{11}(z)L_1(\bar z) + L_1(z)L_{11}(\bar z).
\end{aligned}$$

```
In[∘]:= Length3Q[w_] := (Length[w] == 3)
SVL[w_?Length3Q] :=
 L[w] + (L[Reverse[w]] /. z → zb) + L[w[[1 ;; 2]]] (L[{w[[3]]}] /. z → zb) + L[{w[[1]]}] (Reverse[L[w[[2 ;; 3]]]] /. z → zb)
```

```
In[∘]:= SVL[{}] = 1;
SVL[{0}] = Log[z zb];
SVL[{1}] = Log[(1 - z) (1 - zb)];
SVL[{0, 0}] = 1/2 Log[z zb]^2;
```

1. Flat space expansion

- The expansion around flat space is the expansion around $1/\lambda = 0$. Here, λ is defined as

$$\frac{R^2}{\ell_s^2} = gN \equiv \sqrt{\lambda}\,, \qquad (1)$$

In[∘]:= where R is the AdS radius, $\ell_s^2 = \alpha'$ is the string length, $N = \sqrt{Q_1 Q_2}$ for Q_1 D1 and Q_5 D5 branes, and g is the string coupling in six dimensions, which is proportional to both RR and NS-NS flux. Since 2d CFTs are more

- Hence, we are interested in $A^{(1)}$ in the expansion

In[∘]:= $$A(S,T) = A^{(0)}(S,T) + \frac{1}{\sqrt{\lambda}} A^{(1)}(S,T) + \dots$$

- $A^{(0)}$ is just the flat space Virasoro Shapiro amplitude

$$A^{(0)}(S,T) = -\frac{\hat{\sigma}_2 \Gamma(-S)\Gamma(-T)\Gamma(-U)}{4\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}\,, \qquad (2)$$

In[∘]:= where we define the Mandelstam variables $S = -\frac{(p_1+p_2)^2}{4\sqrt{\lambda}}$, $T = -\frac{(p_1+p_3)^2}{4\sqrt{\lambda}}$, $U = -S-T$, and $\hat{\sigma}_2 = S^2+T^2+U^2$. By applying a Borel transform to the

2. World sheet integral Ansatz

- The Ansatz for the world-sheet integral dictated by crossing symmetry and conformal invariance is

In[∘]:= $$A^{(k)}(S,T) = \int \frac{d^2 z |z|^{-2S-2}}{|1-z|^{2T+2}} G^{(k)}(S,T,z) + \text{cross}$$

- For $G^{(0)}$ one gets

In[∘]:= $$G^{(0)}(S,T,z) = \frac{\hat{\sigma}_2}{12U^2}$$

- The $G^{(1)}$ Ansatz dictated by the transcendentality of the coefficients in the low energy expansion is

In[∘]:= $$G^{(1)} = \sum_i p_i^s(S,T)\mathcal{L}_i^s(z) + \sum_j p_j^a(S,T)\mathcal{L}_j^a(z)$$

where, the p^s and p^a are symmetric and anti-symmetric homogenous polynomials of degree 2

In[∘]:=
```
ps[i_] := cs[i, 1](S^2 + T^2) + cs[i, 2] S T;
pa[i_] := ca[i, 1](S^2 - T^2);
```

L^a and L^s are defined as

In[∘]:=
$$\mathcal{L}^s_w(z) = \mathcal{L}_w(z) + \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) + \mathcal{L}_w(1-\bar{z})$$
$$\mathcal{L}^a_w(z) = \mathcal{L}_w(z) - \mathcal{L}_w(1-z) + \mathcal{L}_w(\bar{z}) - \mathcal{L}_w(1-\bar{z})$$

In[∘]:=
```
Ls[w_] := SVL[w] + (SVL[w] /. z → 1 - z) + (SVL[w] /. z → zb) + (SVL[w] /. z → 1 - zb);
La[w_] := SVL[w] - (SVL[w] /. z → 1 - z) + (SVL[w] /. z → zb) - (SVL[w] /. z → 1 - zb);
```

and the specific symmetric and anti-symmetric SVMPL chosen in the Ansatz are

In[∘]:=
$$\mathcal{L}^s = \{\mathcal{L}^s_{000}, \mathcal{L}^s_{001}, \mathcal{L}^s_{010}, \zeta(3)\},$$
$$\mathcal{L}^a = \{\mathcal{L}^a_{000}, \mathcal{L}^a_{001}, \mathcal{L}^a_{010}\},$$

```
In[∘]:= wList = {{0, 0, 0}, {0, 0, 1}, {0, 1, 0}};
        G1 = Sum[Ls[wList[[i]]] ps[i] + La[wList[[i]]] pa[i], {i, 1, 3}] + Zeta[3] ps[4]
```

Out[∘]=

$$
\begin{aligned}
&\left(S^2 - T^2\right) \mathrm{ca}[1,1] \Bigg(-\frac{1}{6}\,\mathrm{Log}[1-z]^3 + \frac{\mathrm{Log}[z]^3}{6} - \frac{1}{6}\,\mathrm{Log}[1-zb]^3 - \frac{1}{2}\,\mathrm{Log}[1-z]^2\,\mathrm{Log}[zb] + \frac{1}{2}\,\mathrm{Log}[z]^2\,\mathrm{Log}[zb] - \\
&\quad \frac{1}{2}\,\mathrm{Log}[1-zb]^2\,\mathrm{Log}[zb] - \frac{1}{2}\,\mathrm{Log}[1-z]\,\mathrm{Log}[zb]^2 + \frac{1}{2}\,\mathrm{Log}[z]\,\mathrm{Log}[zb]^2 - \frac{1}{2}\,\mathrm{Log}[1-zb]\,\mathrm{Log}[zb]^2 + \frac{7\,\mathrm{Log}[zb]^3}{6}\Bigg) + \\
&\left(\left(S^2 + T^2\right) \mathrm{cs}[1,1] + S\,T\,\mathrm{cs}[1,2]\right) \Bigg(\frac{1}{6}\,\mathrm{Log}[1-z]^3 + \frac{\mathrm{Log}[z]^3}{6} + \frac{1}{6}\,\mathrm{Log}[1-zb]^3 + \frac{1}{2}\,\mathrm{Log}[1-z]^2\,\mathrm{Log}[zb] + \\
&\quad \frac{1}{2}\,\mathrm{Log}[z]^2\,\mathrm{Log}[zb] + \frac{1}{2}\,\mathrm{Log}[1-zb]^2\,\mathrm{Log}[zb] + \frac{1}{2}\,\mathrm{Log}[1-z]\,\mathrm{Log}[zb]^2 + \\
&\quad \frac{1}{2}\,\mathrm{Log}[z]\,\mathrm{Log}[zb]^2 + \frac{1}{2}\,\mathrm{Log}[1-zb]\,\mathrm{Log}[zb]^2 + \frac{11\,\mathrm{Log}[zb]^3}{6}\Bigg) + \\
&\left(\left(S^2 + T^2\right) \mathrm{cs}[2,1] + S\,T\,\mathrm{cs}[2,2]\right) \Bigg(\frac{1}{2}\,\mathrm{Log}[1-z]^2\,\mathrm{Log}[1-zb] + \frac{1}{2}\,\mathrm{Log}[z]^2\,\mathrm{Log}[1-zb] + \frac{1}{2}\,\mathrm{Log}[1-zb]^3 + \\
&\quad \frac{5}{2}\,\mathrm{Log}[1-zb]\,\mathrm{Log}[zb]^2 - \mathrm{Log}[1-z]\,\mathrm{PolyLog}[2,zb] - \mathrm{Log}[z]\,\mathrm{PolyLog}[2,zb] - \mathrm{Log}[1-zb]\,\mathrm{PolyLog}[2,zb] + \\
&\quad 3\,\mathrm{Log}[zb]\,\mathrm{PolyLog}[2,zb] - \mathrm{PolyLog}[3,1-z] - \mathrm{PolyLog}[3,z] - \mathrm{PolyLog}[3,1-zb] - 5\,\mathrm{PolyLog}[3,zb]\Bigg) + \\
&\left(S^2 - T^2\right) \mathrm{ca}[2,1] \Bigg(-\frac{1}{2}\,\mathrm{Log}[1-z]^2\,\mathrm{Log}[1-zb] + \frac{1}{2}\,\mathrm{Log}[z]^2\,\mathrm{Log}[1-zb] - \frac{1}{2}\,\mathrm{Log}[1-zb]^3 + \frac{1}{2}\,\mathrm{Log}[1-zb]\,\mathrm{Log}[zb]^2 + \\
&\quad \mathrm{Log}[1-z]\,\mathrm{PolyLog}[2,zb] - \mathrm{Log}[z]\,\mathrm{PolyLog}[2,zb] + \mathrm{Log}[1-zb]\,\mathrm{PolyLog}[2,zb] - \\
&\quad \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,zb] + \mathrm{PolyLog}[3,1-z] - \mathrm{PolyLog}[3,z] + \mathrm{PolyLog}[3,1-zb] - \mathrm{PolyLog}[3,zb]\Bigg) + \\
&\left(S^2 - T^2\right) \mathrm{ca}[3,1] \Bigg(\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2,1-z] + \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,1-z] - \mathrm{Log}[z]\,\mathrm{PolyLog}[2,z] - \\
&\quad \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,z] - \frac{1}{6}\,\mathrm{Log}[1-z]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \frac{1}{6}\,\mathrm{Log}[z]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) - \\
&\quad \frac{1}{6}\,\mathrm{Log}[1-zb]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \frac{1}{6}\,\mathrm{Log}[zb]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \\
&\quad \mathrm{Log}[1-zb]\,\mathrm{PolyLog}[2,1-zb] + \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,1-zb] - 2\,\mathrm{Log}[zb]\,\mathrm{PolyLog}[2,zb] - \\
&\quad 2\,\mathrm{PolyLog}[3,1-z] + 2\,\mathrm{PolyLog}[3,z] - 2\,\mathrm{PolyLog}[3,1-zb] + 2\,\mathrm{PolyLog}[3,zb]\Bigg) + \\
&\left(\left(S^2 + T^2\right) \mathrm{cs}[3,1] + S\,T\,\mathrm{cs}[3,2]\right) \Bigg(-\mathrm{Log}[1-z]\,\mathrm{PolyLog}[2,1-z] - \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,1-z] - \mathrm{Log}[z]\,\mathrm{PolyLog}[2,z] - \\
&\quad \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,z] + \frac{1}{6}\,\mathrm{Log}[1-z]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \frac{1}{6}\,\mathrm{Log}[z]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \\
&\quad \frac{1}{6}\,\mathrm{Log}[1-zb]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) + \frac{1}{6}\,\mathrm{Log}[zb]\left(\pi^2 - 6\,\mathrm{PolyLog}[2,1-zb]\right) - \\
&\quad \mathrm{Log}[1-zb]\,\mathrm{PolyLog}[2,1-zb] - \mathrm{Log}[zb]\,\mathrm{PolyLog}[2,1-zb] - 6\,\mathrm{Log}[zb]\,\mathrm{PolyLog}[2,zb] + 2\,\mathrm{PolyLog}[3,1-z] + \\
&\quad 2\,\mathrm{PolyLog}[3,z] + 2\,\mathrm{PolyLog}[3,1-zb] + 10\,\mathrm{PolyLog}[3,zb]\Bigg) + \left(\left(S^2 + T^2\right) \mathrm{cs}[4,1] + S\,T\,\mathrm{cs}[4,2]\right) \mathrm{Zeta}[3]
\end{aligned}
$$

In total there are 4x2 + 3 = 11 coefficients to fix.

3. Flat space expansion from holography

- The full string scattering amplitude including curvature correction is given by the Borel transform of the corresponding CFT Mellin amplitude.

In[∘]:=
$$A(S,T)=\lambda^{\frac{1}{2}}\int\frac{d\alpha}{2\pi i}\frac{e^{\alpha}}{\alpha^{3}}M\left(\frac{2\sqrt{\lambda}S}{\alpha}+\frac{2}{3},\frac{2\sqrt{\lambda}T}{\alpha}+\frac{2}{3}\right)$$

- The Mellin amplitude decomposes into a sum over Mack polynomials, which are Mellin transforms of conformal blocks, weighted by OPE coefficients (plus a protected part)

In[∘]:=
$$M(s,t)=\text{prot.}+\sum_{\Delta,\ell}C^{2}_{\tau\ell}\sum_{m=0}^{\infty}\frac{\mathcal{Q}_{\tau+2,\ell,m}(t-2)}{s-\tau-2m}$$

- Assuming sufficiently well behaved convergence of the OPE, we swap the Borel transform and the above decomposition of the Mellin Amplitude to get the contributions of individual primaries to the string scattering amplitude.

In[∘]:=
$$A_{\tau,\ell}(S,T)=\lambda^{\frac{1}{2}}\int\frac{d\alpha}{2\pi i}\frac{e^{\alpha}}{\alpha^{3}}\sum_{m=0}^{\infty}\frac{\mathcal{Q}_{\tau+2,\ell,m}\left(\frac{2\sqrt{\lambda}T}{\alpha}-\frac{4}{3}\right)}{\frac{2\sqrt{\lambda}S}{\alpha}+\frac{2}{3}-\tau-2m}\,.$$

- We are interested in the Poles of $A_{\tau,l}$ in S. These have an expansion of the form

In[∘]:=
$$A_{\tau,\ell}(S,T)|_{S-\text{poles}}=\sin\left(\frac{\pi\tau}{2}\right)^{2}\sum_{i=0}^{\infty}\frac{R^{(i)}_{\tau,\ell}(S,T)}{\lambda^{i/4}}\,,$$

where the first few terms are

In[∘]:=
$$\text{RN[0]}[\tau f\_, l\_] := \frac{16\,\text{ChebyshevT}\left[l, -1-\frac{2T}{S}\right]}{\pi^3 S^2 \left(-4 S \tau f + \tau f^3\right)};$$

$$\text{RN[1]}[\tau f\_, l\_] := -\frac{8\left(4\left(3+2l\right)S + \left(1+2l\right)\tau f^2\right)\text{ChebyshevT}\left[l, 1+\frac{2T}{S}\right]}{\pi^3 S^2 \tau f^2 \left(-4S + \tau f^2\right)^2};$$

$$\text{RN[2]}[\tau f\_, l\_] := \Big(-\frac{2}{3} l S (S+2T)(4S-\tau f^2)(48S^2 - 152 S \tau f^2 + 11 \tau f^4)\,\text{ChebyshevT}\left[-1+l, 1+\frac{2T}{S}\right] - \frac{2}{3}\big(384 l^2 S T (S+T) \tau f^2 (4S - \tau f^2) + T(S+T)(192 S^3 + 4080 S^2 \tau f^2 - 668 S \tau f^4 + 5 \tau f^6) - l (4S - \tau f^2)(48 S^4 + 20 T^2 \tau f^4 + 8 S^3 (24 T - 19 \tau f^2) - 20 S T \tau f^2 (64 T - \tau f^2) + S^2 (192 T^2 - 1280 T \tau f^2 + 11 \tau f^4))\big)\,\text{ChebyshevT}\left[l, 1+\frac{2T}{S}\right]\Big) \Big/ \left(\pi^3 S^2 T (S+T) \tau f^3 (-4S + \tau f^2)^4\right);$$

- In Addition to the expansion of the blocks, there is also an expansion of the CFT data.

$$\tau(\delta,\ell) = \tau_0(\delta,\ell)\lambda^{\frac{1}{4}} + \tau_1(\delta,\ell) + \tau_2(\delta,\ell)\lambda^{-\frac{1}{4}} + \dots\,,$$

$$C^2_{\tau,\ell} = \frac{\pi^3}{4^{\tau(\delta,\ell)}\sin^2(\frac{\pi}{2}\tau(\delta,\ell))^2} f(\delta,\ell)\,,$$

$$f(\delta,\ell) = f_0(\delta,\ell) + \frac{f_1(\delta,\ell)}{\lambda^{1/4}} + \frac{f_2(\delta,\ell)}{\lambda^{1/2}} + \dots\,,$$

4. Matching

4.1. The flat space limit

Out[∘]=
$$A^{(0)}(S,T) = -\frac{\hat{\sigma}_2\Gamma(-S)\Gamma(-T)\Gamma(-U)}{4\Gamma(S+1)\Gamma(T+1)\Gamma(U+1)}$$

In[∘]:=
```
σ2 = S^2 + T^2 + U^2;
A0 = -(σ2 Gamma[-S] Gamma[-T] Gamma[-U])/(4 Gamma[S + 1] Gamma[T + 1] Gamma[U + 1]) /. U → -S - T;
FunctionPoles[A0, S]
```

Out[∘]=
$$\{\{-T-2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le -1, \text{Indeterminate}\}, \{1-T-2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le 0, \text{Indeterminate}\}, \{2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\ge 0, 1\}, \{2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le -1, \text{Indeterminate}\}, \{-1+2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le 0, \text{Indeterminate}\}, \{1+2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\ge 0, 1\}, \{-T+2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le 0, 1\}, \{1-T+2c_1 \text{ if } c_1\in\mathbb{Z} \,\&\&\, c_1\le -1, 1\}\}$$

In[∘]:=
```
RN[0][τ, l]
```

Out[∘]=
$$\frac{16\,\text{ChebyshevT}\left[l, -1-\frac{2T}{S}\right]}{\pi^3 S^2\left(-4 S\tau+\tau^3\right)}$$

- We want poles at non negative integer S and thus τ0[δ, l] = $2\sqrt{\delta}$ with δ = 1, 2, ...

In[∘]:=
```
δ = 4;
lhalfmax = 5;
BlockExpansion = π^3 Sum[Residue[RN[0][2 √δ, 2 l] f0[δ, 2 l], {S, δ}] // FullSimplify, {l, 0, lhalfmax}];
VirasoroShapiro = Residue[A0, {S, δ}] // FullSimplify;
Solve[CoefficientList[BlockExpansion - VirasoroShapiro, T] == 0]
```

Out[∘]=
$$\left\{\left\{f0[4,0]\to\frac{13}{6}, f0[4,2]\to\frac{7}{2}, f0[4,4]\to\frac{65}{36}, f0[4,6]\to\frac{1}{2}, f0[4,8]\to\frac{1}{36}, f0[4,10]\to 0\right\}\right\}$$

- We see that coefficients with 2 δ > l vanish. The nonvanishing coefficients match with those given in the paper

In[∘]:=
$$\text{f0list} = \left\{f0[1,0]\to\frac{7}{32}, f0[4,0]\to\frac{13}{6}, f0[9,0]\to\frac{12\,930\,438\,637\,926\,492\,183}{1\,379\,227\,385\,882\,214\,400},\right.$$
$$f0[16,0]\to\frac{30\,540\,149\,740\,456\,016}{1\,123\,136\,891\,296\,875}, f0[1,2]\to\frac{1}{32}, f0[4,2]\to\frac{7}{2},$$
$$f0[9,2]\to\frac{24\,109\,654\,015\,419\,440\,229}{1\,379\,227\,385\,882\,214\,400}, f0[16,2]\to\frac{7\,146\,080\,687\,579\,606\,224}{135\,899\,563\,846\,921\,875},$$
$$f0[2,0]\to\frac{15}{16\sqrt{2}}, f0[3,0]\to\frac{12\,411\sqrt{3}}{16\,384}, f0[5,0]\to\frac{96\,717\,225\sqrt{5}}{67\,108\,864},$$
$$f0[6,0]\to\frac{11\,973\,123\sqrt{\frac{3}{2}}}{3\,276\,800}, f0[7,0]\to\frac{155\,687\,368\,062\,779\sqrt{7}}{69\,578\,470\,195\,200}, f0[8,0]\to\frac{106\,006\sqrt{2}}{19\,845},$$
$$f0[10,0]\to\frac{19\,876\,590\,764\,203\,625\sqrt{\frac{5}{2}}}{2\,761\,569\,482\,047\,488}, f0[11,0]\to\frac{157\,948\,953\,934\,079\,575\,281\,847\,327\sqrt{11}}{38\,609\,539\,749\,432\,357\,027\,840\,000},$$

$$f0[12, 0] \to \frac{319\,212\,278\,817\,\sqrt{3}}{34\,693\,120\,000},\ f0[13, 0] \to \frac{176\,583\,458\,425\,496\,963\,931\,480\,259\,921\,883\,\sqrt{13}}{34\,443\,910\,174\,418\,400\,709\,277\,908\,992\,000},$$

$$f0[14, 0] \to \frac{13\,379\,788\,998\,055\,473\,385\,083\,391\,\sqrt{\frac{7}{2}}}{1\,180\,135\,640\,051\,233\,062\,912\,000},$$

$$f0[15, 0] \to \frac{369\,551\,612\,606\,711\,045\,971\,094\,595\,184\,275\,\sqrt{15}}{59\,355\,899\,128\,804\,224\,336\,553\,389\,850\,624},$$

$$f0[17, 0] \to \frac{318\,016\,920\,095\,921\,418\,691\,025\,353\,151\,599\,616\,027\,421\,056\,759\,\sqrt{17}}{43\,068\,290\,920\,326\,274\,779\,330\,246\,049\,253\,333\,925\,888\,000\,000},$$

$$f0[18, 0] \to \frac{391\,822\,169\,375\,799\,319\,226\,175\,663\,888\,833\,199}{8\,179\,595\,398\,056\,230\,942\,375\,149\,568\,000\,000\,\sqrt{2}},\ f0[4, 4] \to \frac{65}{36},$$

$$f0[9, 4] \to \frac{348\,013\,302\,709\,679\,919}{24\,629\,060\,462\,182\,400},\ f0[16, 4] \to \frac{3\,874\,019\,978\,551\,603\,688}{81\,539\,738\,308\,153\,125},$$

$$f0[4, 6] \to \frac{1}{2},\ f0[9, 6] \to \frac{967\,676\,151\,727\,807\,749}{98\,516\,241\,848\,729\,600},\ f0[16, 6] \to \frac{3\,266\,196\,375\,876\,313\,424}{81\,539\,738\,308\,153\,125},$$

$$f0[4, 8] \to \frac{1}{36},\ f0[9, 8] \to \frac{1\,991\,378\,786\,773\,760\,091}{344\,806\,846\,470\,553\,600},\ f0[16, 8] \to \frac{512\,611\,767\,068\,464\,072}{16\,307\,947\,661\,630\,625},$$

$$f0[9, 10] \to \frac{1\,923\,122\,230\,596\,910\,647}{689\,613\,692\,941\,107\,200},\ f0[16, 10] \to \frac{9\,324\,143\,295\,359\,747\,968}{407\,698\,691\,540\,765\,625},$$

$$f0[2, 2] \to \frac{1}{\sqrt{2}},\ f0[3, 2] \to \frac{35\,703\,\sqrt{3}}{32\,768},\ f0[5, 2] \to \frac{1\,491\,049\,825\,\sqrt{5}}{603\,979\,776},$$

$$f0[6, 2] \to \frac{1\,326\,987\,\sqrt{\frac{3}{2}}}{204\,800},\ f0[7, 2] \to \frac{1\,129\,587\,534\,443\,179\,\sqrt{7}}{278\,313\,880\,780\,800},\ f0[8, 2] \to \frac{325\,594\,\sqrt{2}}{33\,075},$$

$$f0[10, 2] \to \frac{2\,337\,162\,606\,202\,325\,\sqrt{\frac{5}{2}}}{172\,598\,092\,627\,968},\ f0[11, 2] \to \frac{598\,520\,767\,316\,918\,019\,484\,383\,121\,\sqrt{11}}{77\,219\,079\,498\,864\,714\,055\,680\,000},$$

$$f0[12, 2] \to \frac{2\,128\,890\,131\,703\,\sqrt{3}}{121\,425\,920\,000},\ f0[13, 2] \to \frac{48\,293\,798\,417\,918\,747\,099\,524\,735\,117\,241\,\sqrt{13}}{4\,920\,558\,596\,345\,485\,815\,611\,129\,856\,000},$$

$$f0[14, 2] \to \frac{2\,191\,626\,838\,763\,165\,406\,414\,413\,\sqrt{\frac{7}{2}}}{100\,579\,742\,049\,820\,999\,680\,000},$$

$$f0[15, 2] \to \frac{5\,700\,896\,445\,111\,660\,296\,502\,741\,191\,237\,675\,\sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992},$$

$$f0[17, 2] \to \frac{14\,795\,866\,537\,807\,201\,285\,441\,809\,976\,333\,510\,002\,106\,769\,063\,\sqrt{17}}{1\,033\,638\,982\,087\,830\,594\,703\,925\,905\,182\,080\,014\,221\,312\,000},$$

$$f0[18, 2] \to \frac{9\,515\,032\,033\,640\,288\,284\,316\,122\,494\,374\,889}{102\,244\,942\,475\,702\,886\,779\,689\,369\,600\,000\,\sqrt{2}},\ f0[9, 12] \to \frac{25\,875\,765\,257\,907\,897}{24\,629\,060\,462\,182\,400},$$

$$f0[16, 12] \to \frac{6\,255\,810\,680\,750\,301\,952}{407\,698\,691\,540\,765\,625},\ f0[9, 14] \to \frac{775\,703\,736\,696\,989\,259}{2\,758\,454\,771\,764\,428\,800},$$

$$f0[16,14] \to \frac{768\,317\,487\,206\,107\,136}{81\,539\,738\,308\,153\,125},\ f0[9,16] \to \frac{60\,326\,101\,703\,732\,157}{1\,379\,227\,385\,882\,214\,400},$$

$$f0[16,16] \to \frac{142\,509\,000\,791\,324\,672}{27\,179\,912\,769\,384\,375},\ f0[9,18] \to \frac{5\,559\,060\,566\,555\,523}{2\,758\,454\,771\,764\,428\,800},$$

$$f0[16,18] \to \frac{32\,226\,201\,823\,870\,976}{12\,354\,505\,804\,265\,625},\ f0[2,4] \to \frac{1}{16\sqrt{2}},\ f0[3,4] \to \frac{6237\sqrt{3}}{16\,384},$$

$$f0[5,4] \to \frac{231\,490\,625\sqrt{5}}{150\,994\,944},\ f0[6,4] \to \frac{29\,382\,759\sqrt{\frac{3}{2}}}{6\,553\,600},\ f0[7,4] \to \frac{15\,529\,073\,351\,533\sqrt{7}}{5\,153\,960\,755\,200},$$

$$f0[8,4] \to \frac{761\,617\sqrt{2}}{99\,225},\ f0[10,4] \to \frac{31\,070\,056\,931\,389\,825\sqrt{\frac{5}{2}}}{2\,761\,569\,482\,047\,488},$$

$$f0[11,4] \to \frac{28\,230\,741\,698\,348\,579\,718\,113\,687\sqrt{11}}{4\,289\,948\,861\,048\,039\,669\,760\,000},\ f0[12,4] \to \frac{3\,678\,318\,050\,201\sqrt{3}}{242\,851\,840\,000},$$

$$f0[13,4] \to \frac{123\,435\,574\,360\,751\,005\,908\,971\,889\,534\,659\sqrt{13}}{14\,351\,629\,239\,341\,000\,295\,532\,462\,080\,000},$$

$$f0[14,4] \to \frac{1\,368\,263\,817\,079\,908\,428\,184\,945\,881\sqrt{\frac{7}{2}}}{70\,808\,138\,403\,073\,983\,774\,720\,000},$$

$$f0[15,4] \to \frac{2\,553\,376\,537\,541\,052\,412\,443\,624\,077\,990\,075\sqrt{15}}{237\,423\,596\,515\,216\,897\,346\,213\,559\,402\,496},$$

$$f0[17,4] \to \frac{11\,691\,370\,588\,924\,762\,607\,168\,436\,816\,668\,336\,220\,205\,940\,807\sqrt{17}}{897\,256\,060\,840\,130\,724\,569\,380\,126\,026\,111\,123\,456\,000\,000},$$

$$f0[18,4] \to \frac{139\,485\,866\,402\,711\,990\,852\,893\,328\,548\,021\,137}{1\,635\,919\,079\,611\,246\,188\,475\,029\,913\,600\,000\sqrt{2}},$$

$$f0[16,20] \to \frac{154\,845\,122\,714\,927\,104}{135\,899\,563\,846\,921\,875},\ f0[16,22] \to \frac{34\,755\,517\,792\,387\,072}{81\,539\,738\,308\,153\,125},$$

$$f0[16,24] \to \frac{2\,148\,946\,889\,670\,656}{16\,307\,947\,661\,630\,625},\ f0[3,6] \to \frac{729\sqrt{3}}{32\,768},\ f0[5,6] \to \frac{86\,828\,125\sqrt{5}}{134\,217\,728},$$

$$f0[6,6] \to \frac{192\,051\sqrt{\frac{3}{2}}}{81\,920},\ f0[7,6] \to \frac{11\,113\,991\,966\,077\sqrt{7}}{6\,184\,752\,906\,240},\ f0[8,6] \to \frac{1226\sqrt{2}}{245},$$

$$f0[10,6] \to \frac{708\,367\,713\,146\,875\sqrt{\frac{5}{2}}}{86\,299\,046\,313\,984},\ f0[11,6] \to \frac{384\,956\,784\,317\,948\,202\,700\,215\,623\sqrt{11}}{77\,219\,079\,498\,864\,714\,055\,680\,000},$$

$$f0[12,6] \to \frac{18\,647\,356\,293\sqrt{3}}{1\,576\,960\,000},\ f0[13,6] \to \frac{96\,749\,020\,698\,495\,425\,692\,803\,654\,165\,493\sqrt{13}}{14\,058\,738\,846\,701\,388\,044\,603\,228\,160\,000},$$

$$f0[14,6] \to \frac{596\,885\,830\,599\,291\,322\,218\,101\sqrt{\frac{7}{2}}}{37\,824\,860\,258\,052\,341\,760\,000},$$

$f0[15, 6] \to \frac{4242736752798830494095083563895025\sqrt{15}}{474847193030433794692427118804992}$,

$f0[17, 6] \to \frac{71875678749229606207243733593638789936678838463\sqrt{17}}{6460243638048941216899536907388000088883200000}$,

$f0[18, 6] \to \frac{110682306261779297041084543783137}{1503602095230924805583667200000\sqrt{2}}$, $f0[16, 26] \to \frac{2600499618512896}{81539738308153125}$,

$f0[16, 28] \to \frac{18004502904832}{3261589532326125}$, $f0[16, 30] \to \frac{233096465088512}{407698691540765625}$,

$f0[16, 32] \to \frac{8796093022208}{407698691540765625}$, $f0[5, 8] \to \frac{91796875\sqrt{5}}{603979776}$,

$f0[6, 8] \to \frac{2797173\sqrt{\frac{3}{2}}}{3276800}$, $f0[7, 8] \to \frac{6378481360853\sqrt{7}}{7730941132800}$, $f0[8, 8] \to \frac{263857\sqrt{2}}{99225}$,

$f0[10, 8] \to \frac{3588894365265625\sqrt{\frac{5}{2}}}{690392370511872}$, $f0[11, 8] \to \frac{6451364522196238764315779\sqrt{11}}{1930476987471617851392000}$,

$f0[12, 8] \to \frac{402678397529\sqrt{3}}{48570368000}$, $f0[13, 8] \to \frac{1723328663194134099136732207488863\sqrt{13}}{344439101744184007092779089920000}$,

$f0[14, 8] \to \frac{19316716549172712446446103\sqrt{\frac{7}{2}}}{1634033963147861164032000}$,

$f0[15, 8] \to \frac{16628264017037641655039843834375\sqrt{15}}{2422689760359356095369526116352}$,

$f0[17, 8] \to \frac{143613877885693472861214131747346204773046615839\sqrt{17}}{16150609095122353042248842268470000222208000000}$,

$f0[18, 8] \to \frac{3595183951500906295855815192943809}{60144083809236992223346688000000\sqrt{2}}$, $f0[5, 10] \to \frac{9765625\sqrt{5}}{1207959552}$,

$f0[6, 10] \to \frac{72171\sqrt{\frac{3}{2}}}{409600}$, $f0[7, 10] \to \frac{74246641872907\sqrt{7}}{278313880780800}$, $f0[8, 10] \to \frac{36304\sqrt{2}}{33075}$,

$f0[10, 10] \to \frac{241838787109375\sqrt{\frac{5}{2}}}{86299046313984}$, $f0[11, 10] \to \frac{100653689805551704828525013\sqrt{11}}{51479386332576476037120000}$,

$f0[12, 10] \to \frac{2241053883\sqrt{3}}{433664000}$, $f0[13, 10] \to \frac{6778718828237514255615266678137403\sqrt{13}}{2066634610465104042556674539520000}$,

$f0[14, 10] \to \frac{21443140493316402563485643\sqrt{\frac{7}{2}}}{2655305190115274391552000}$,

$f0[15, 10] \to \frac{2303852966664712599912412589359375\sqrt{15}}{474847193030433794692427118804992}$,

$$f0[17, 10] \to \frac{214\,197\,159\,772\,469\,024\,080\,249\,308\,358\,551\,973\,102\,525\,381\,663 \sqrt{17}}{32\,301\,218\,190\,244\,706\,084\,497\,684\,536\,940\,000\,444\,416\,000\,000},$$

$$f0[18, 10] \to \frac{342\,349\,464\,841\,601\,452\,463\,991\,539\,695\,473}{7\,518\,010\,476\,154\,624\,027\,918\,336\,000\,000 \sqrt{2}}, \; f0[6, 12] \to \frac{59\,049 \sqrt{\frac{3}{2}}}{6\,553\,600},$$

$$f0[7, 12] \to \frac{6\,879\,119\,738\,897 \sqrt{7}}{139\,156\,940\,390\,400}, \; f0[8, 12] \to \frac{6368 \sqrt{2}}{19\,845}, \; f0[10, 12] \to \frac{6\,878\,863\,876\,953\,125 \sqrt{\frac{5}{2}}}{5\,523\,138\,964\,094\,976},$$

$$f0[11, 12] \to \frac{75\,454\,010\,670\,950\,667\,100\,975\,531 \sqrt{11}}{77\,219\,079\,498\,864\,714\,055\,680\,000}, \; f0[12, 12] \to \frac{2\,446\,019\,613 \sqrt{3}}{867\,328\,000},$$

$$f0[13, 12] \to \frac{165\,084\,707\,433\,938\,892\,536\,418\,955\,973\,669 \sqrt{13}}{86\,109\,775\,436\,046\,001\,773\,194\,772\,480\,000},$$

$$f0[14, 12] \to \frac{353\,127\,795\,272\,229\,245\,322\,177\,907 \sqrt{\frac{7}{2}}}{70\,808\,138\,403\,073\,983\,774\,720\,000},$$

$$f0[15, 12] \to \frac{744\,240\,989\,867\,707\,110\,096\,318\,603\,515\,625 \sqrt{15}}{237\,423\,596\,515\,216\,897\,346\,213\,559\,402\,496},$$

$$f0[17, 12] \to \frac{37\,095\,518\,035\,658\,699\,053\,793\,556\,416\,135\,991\,485\,979\,296\,653 \sqrt{17}}{8\,075\,304\,547\,561\,176\,521\,124\,421\,134\,235\,000\,111\,104\,000\,000},$$

$$f0[18, 12] \to \frac{7\,800\,350\,313\,156\,067\,988\,348\,244\,434\,045\,379}{240\,576\,335\,236\,947\,968\,893\,386\,752\,000\,000 \sqrt{2}},$$

$$f0[7, 14] \to \frac{678\,223\,072\,849 \sqrt{7}}{278\,313\,880\,780\,800}, \; f0[8, 14] \to \frac{256 \sqrt{2}}{4725}, \; f0[10, 14] \to \frac{149\,412\,841\,796\,875 \sqrt{\frac{5}{2}}}{345\,196\,185\,255\,936},$$

$$f0[11, 14] \to \frac{186\,893\,846\,470\,254\,263\,857\,714\,571 \sqrt{11}}{463\,314\,476\,993\,188\,284\,334\,080\,000}, \; f0[12, 14] \to \frac{79\,960\,644\,207 \sqrt{3}}{60\,712\,960\,000},$$

$$f0[13, 14] \to \frac{1\,015\,473\,408\,770\,489\,713\,153\,602\,346\,750\,403 \sqrt{13}}{1\,033\,317\,305\,232\,552\,021\,278\,337\,269\,760\,000},$$

$$f0[14, 14] \to \frac{18\,256\,662\,843\,157\,223\,541\,350\,231 \sqrt{\frac{7}{2}}}{6\,638\,262\,975\,288\,185\,978\,880\,000},$$

$$f0[15, 14] \to \frac{870\,689\,855\,945\,996\,819\,472\,136\,376\,953\,125 \sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992},$$

$$f0[17, 14] \to \frac{21\,091\,392\,097\,602\,625\,260\,156\,757\,222\,157\,858\,664\,633\,992\,411 \sqrt{17}}{7\,178\,048\,486\,721\,045\,796\,555\,041\,008\,208\,888\,987\,648\,000\,000},$$

$$f0[18, 14] \to \frac{219\,539\,785\,725\,311\,709\,934\,579\,685\,337\,981}{10\,224\,494\,247\,570\,288\,677\,968\,936\,960\,000 \sqrt{2}}, \; f0[8, 16] \to \frac{256 \sqrt{2}}{99\,225},$$

$$f0[10, 16] \to \frac{295\,672\,607\,421\,875 \sqrt{\frac{5}{2}}}{2\,761\,569\,482\,047\,488}, \; f0[11, 16] \to \frac{3\,019\,650\,230\,259\,965\,334\,702\,307 \sqrt{11}}{23\,165\,723\,849\,659\,414\,216\,704\,000},$$

$$f0[12, 16] \to \frac{1\,603\,764\,279 \sqrt{3}}{3\,153\,920\,000}, \; f0[13, 16] \to \frac{222\,908\,354\,400\,351\,119\,593\,920\,700\,143\,491 \sqrt{13}}{516\,658\,652\,616\,276\,010\,639\,168\,634\,880\,000},$$

$$f0[14, 16] \to \frac{70\,698\,138\,997\,738\,459\,514\,202\,917 \sqrt{\frac{7}{2}}}{53\,106\,103\,802\,305\,487\,831\,040\,000},$$

$$f0[15, 16] \to \frac{56\,899\,105\,454\,744\,165\,340\,802\,001\,953\,125 \sqrt{15}}{59\,355\,899\,128\,804\,224\,336\,553\,389\,850\,624},$$

$$f0[17, 16] \to \frac{55\,611\,334\,437\,312\,196\,934\,072\,155\,880\,381\,831\,253\,059\,657\,953 \sqrt{17}}{32\,301\,218\,190\,244\,706\,084\,497\,684\,536\,940\,000\,444\,416\,000\,000},$$

$$f0[18, 16] \to \frac{5\,374\,004\,692\,630\,647\,167\,101\,024\,114\,932\,279}{408\,979\,769\,902\,811\,547\,118\,757\,478\,400\,000 \sqrt{2}}, \; f0[10, 18] \to \frac{762\,939\,453\,125 \sqrt{\frac{5}{2}}}{49\,313\,740\,750\,848},$$

$$f0[11, 18] \to \frac{4\,635\,797\,896\,899\,374\,194\,311\,509 \sqrt{11}}{154\,438\,158\,997\,729\,428\,111\,360\,000}, \; f0[12, 18] \to \frac{1\,697\,127\,309 \sqrt{3}}{11\,038\,720\,000},$$

$$f0[13, 18] \to \frac{53\,952\,758\,401\,426\,084\,988\,476\,460\,332\,249 \sqrt{13}}{344\,439\,101\,744\,184\,007\,092\,779\,089\,920\,000}, \; f0[14, 18] \to$$

$$\frac{10\,440\,753\,141\,129\,127\,900\,097 \sqrt{\frac{7}{2}}}{18\,912\,430\,129\,026\,170\,880\,000}, \; f0[15, 18] \to \frac{208\,949\,570\,745\,513\,627\,562\,591\,552\,734\,375 \sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992},$$

$$f0[17, 18] \to \frac{1\,790\,152\,875\,159\,002\,485\,327\,953\,473\,556\,309\,182\,918\,790\,187 \sqrt{17}}{1\,957\,649\,587\,287\,557\,944\,515\,011\,184\,056\,969\,723\,904\,000\,000},$$

$$f0[18, 18] \to \frac{144\,893\,332\,597\,222\,364\,552\,001\,250\,574\,769}{19\,662\,488\,937\,635\,170\,534\,555\,648\,000\,000 \sqrt{2}}, \; f0[10, 20] \to \frac{3\,814\,697\,265\,625 \sqrt{\frac{5}{2}}}{5\,523\,138\,964\,094\,976},$$

$$f0[11, 20] \to \frac{939\,831\,742\,920\,786\,332\,853\,797 \sqrt{11}}{231\,657\,238\,496\,594\,142\,167\,040\,000}, \; f0[12, 20] \to \frac{8\,051\,862\,591 \sqrt{3}}{242\,851\,840\,000},$$

$$f0[13, 20] \to \frac{11\,505\,464\,793\,675\,213\,585\,069\,467\,609\,819 \sqrt{13}}{258\,329\,326\,308\,138\,005\,319\,584\,317\,440\,000}, \; f0[14, 20] \to$$

$$\frac{618\,664\,332\,623\,548\,270\,943\,723 \sqrt{\frac{7}{2}}}{3\,268\,067\,926\,295\,722\,328\,064\,000}, \; f0[15, 20] \to \frac{3\,160\,155\,524\,181\,509\,639\,739\,990\,234\,375 \sqrt{15}}{18\,263\,353\,578\,093\,607\,488\,170\,273\,800\,192},$$

$$f0[17, 20] \to \frac{28\,981\,670\,082\,523\,264\,323\,087\,933\,995\,336\,372\,805\,976\,463 \sqrt{17}}{66\,738\,054\,112\,075\,839\,017\,557\,199\,456\,487\,604\,224\,000\,000},$$

$$f0[18, 20] \to \frac{15\,322\,533\,711\,635\,132\,564\,857\,554\,582\,975\,303}{4\,089\,797\,699\,028\,115\,471\,187\,574\,784\,000\,000 \sqrt{2}},$$

$f0[11, 22] \to \frac{81\,402\,749\,386\,839\,761\,113\,321\,\sqrt{11}}{463\,314\,476\,993\,188\,284\,334\,080\,000}$, $f0[12, 22] \to \frac{511\,777\,683\,\sqrt{3}}{121\,425\,920\,000}$,
$f0[13, 22] \to \frac{18\,680\,739\,035\,866\,441\,675\,732\,656\,623\,597\,\sqrt{13}}{2\,066\,634\,610\,465\,104\,042\,556\,674\,539\,520\,000}$, $f0[14, 22] \to$
$\frac{10\,383\,287\,821\,041\,952\,078\,129\,\sqrt{\frac{7}{2}}}{204\,254\,245\,393\,482\,645\,504\,000}$, $f0[15, 22] \to \frac{26\,709\,731\,383\,003\,420\,886\,993\,408\,203\,125\,\sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992}$,
$f0[17, 22] \to \frac{1\,949\,185\,124\,895\,233\,946\,714\,032\,968\,465\,927\,775\,999\,582\,939\,\sqrt{17}}{10\,767\,072\,730\,081\,568\,694\,832\,561\,512\,313\,333\,481\,472\,000\,000}$,
$f0[18, 22] \to \frac{12\,801\,606\,924\,018\,101\,926\,917\,980\,001\,087}{7\,518\,010\,476\,154\,624\,027\,918\,336\,000\,000\,\sqrt{2}}$, $f0[12, 24] \to \frac{43\,046\,721\,\sqrt{3}}{242\,851\,840\,000}$,
$f0[13, 24] \to \frac{373\,989\,730\,787\,941\,283\,299\,629\,203\,729\,\sqrt{13}}{344\,439\,101\,744\,184\,007\,092\,779\,089\,920\,000}$, $f0[14, 24] \to$
$\frac{345\,202\,010\,200\,440\,597\,834\,259\,\sqrt{\frac{7}{2}}}{35\,404\,069\,201\,536\,991\,887\,360\,000}$, $f0[15, 24] \to \frac{1\,699\,637\,617\,184\,107\,303\,619\,384\,765\,625\,\sqrt{15}}{118\,711\,798\,257\,608\,448\,673\,106\,779\,701\,248}$,
$f0[17, 24] \to \frac{208\,688\,020\,297\,852\,692\,661\,815\,625\,422\,336\,052\,923\,319\,253\,\sqrt{17}}{3\,230\,121\,819\,024\,470\,608\,449\,768\,453\,694\,000\,044\,441\,600\,000}$,
$f0[18, 24] \to \frac{5\,835\,165\,879\,554\,652\,433\,576\,412\,637\,873}{8\,592\,011\,972\,748\,141\,746\,192\,384\,000\,000\,\sqrt{2}}$,
$f0[13, 26] \to \frac{91\,733\,330\,193\,268\,616\,658\,399\,616\,009\,\sqrt{13}}{2\,066\,634\,610\,465\,104\,042\,556\,674\,539\,520\,000}$, $f0[14, 26] \to$
$\frac{1\,341\,068\,619\,663\,964\,900\,807\,\sqrt{\frac{7}{2}}}{1\,206\,956\,904\,597\,851\,996\,160\,000}$, $f0[15, 26] \to \frac{1\,236\,783\,980\,314\,300\,060\,272\,216\,796\,875\,\sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992}$,
$f0[17, 26] \to \frac{1\,893\,075\,985\,746\,630\,060\,921\,153\,753\,217\,001\,087\,682\,881\,\sqrt{17}}{99\,388\,363\,662\,291\,403\,336\,915\,952\,421\,353\,847\,521\,280\,000}$,
$f0[18, 26] \to \frac{158\,415\,752\,522\,855\,876\,702\,566\,542\,417}{683\,455\,497\,832\,238\,547\,992\,576\,000\,000\,\sqrt{2}}$, $f0[14, 28] \to$
$\frac{9\,387\,480\,337\,647\,754\,305\,649\,\sqrt{\frac{7}{2}}}{212\,424\,415\,209\,221\,951\,324\,160\,000}$, $f0[15, 28] \to \frac{9\,583\,205\,159\,604\,549\,407\,958\,984\,375\,\sqrt{15}}{33\,917\,656\,645\,030\,985\,335\,173\,365\,628\,928}$,
$f0[17, 28] \to \frac{35\,495\,071\,438\,462\,368\,988\,598\,652\,931\,321\,269\,666\,341\,213\,\sqrt{17}}{8\,075\,304\,547\,561\,176\,521\,124\,421\,134\,235\,000\,111\,104\,000\,000}$,
$f0[18, 28] \to \frac{5\,714\,795\,635\,986\,492\,039\,924\,197\,439\,201}{87\,482\,303\,722\,526\,534\,143\,049\,728\,000\,000\,\sqrt{2}}$,
$f0[15, 30] \to \frac{5\,195\,713\,640\,749\,454\,498\,291\,015\,625\,\sqrt{15}}{474\,847\,193\,030\,433\,794\,692\,427\,118\,804\,992}$,

$$f0[17,30] \to \frac{14\,428\,799\,658\,214\,237\,737\,043\,831\,013\,716\,471\,691\,934\,037\,\sqrt{17}}{19\,877\,672\,732\,458\,280\,667\,383\,190\,484\,270\,769\,504\,256\,000\,000},$$
$$f0[18,30] \to \frac{866\,626\,588\,897\,278\,303\,908\,488\,615\,911}{60\,144\,083\,809\,236\,992\,223\,346\,688\,000\,000\,\sqrt{2}},$$
$$f0[17,32] \to \frac{40\,254\,497\,110\,927\,943\,179\,349\,807\,054\,456\,171\,205\,137\,\sqrt{17}}{559\,328\,453\,510\,730\,841\,290\,003\,195\,444\,848\,492\,544\,000\,000},$$
$$f0[18,32] \to \frac{18\,616\,114\,953\,890\,247\,749\,341\,199\,093\,829}{8\,179\,595\,398\,056\,230\,942\,375\,149\,568\,000\,000\,\sqrt{2}},$$
$$f0[17,34] \to \frac{684\,326\,450\,885\,775\,034\,048\,946\,719\,925\,754\,910\,487\,329\,\sqrt{17}}{258\,409\,745\,521\,957\,648\,675\,981\,476\,295\,520\,003\,555\,328\,000\,000},$$
$$f0[18,34] \to \frac{221\,376\,048\,770\,573\,507\,241\,315\,373\,383}{1\,022\,449\,424\,757\,028\,867\,796\,893\,696\,000\,000\,\sqrt{2}},$$
$$f0[18,36] \to \frac{127\,173\,474\,825\,648\,610\,542\,883\,299\,603}{16\,359\,190\,796\,112\,461\,884\,750\,299\,136\,000\,000\,\sqrt{2}}\};$$

4.2. Order $\lambda^{-1/4}$

- At this order, the amplitude should vanish. Imposing this fixes the order $\lambda^{-1/4}$ CFT data.

4.3. Order $\lambda^{-1/2}$

- At this order, we match the conformal block expansion with the world-sheet ansatz, to fix both the order $\lambda^{-1/2}$ CFT data and the free coefficients in the Ansatz.

5. Result

The result is

$$p^s = \left\{\frac{1}{192}\left(9\left(S^2+T^2\right)+52ST\right),\ -\frac{13ST}{48},\ -\frac{1}{96}\left(12S^2+25ST+12T^2\right),\ \left(S^2+T^2\right)\right\},$$
$$p^a = \left(S^2-T^2\right)\{3/32,\ 5/12,\ 5/24\}\ .$$

$$f2list = \left\{f2[1,0] \to \frac{219}{1024},\ f2[1,2] \to \frac{93}{1024},\ f2[2,0] \to -\frac{1221}{1024\,\sqrt{2}},\ f2[2,2] \to \frac{87}{64\,\sqrt{2}},\right.$$

$f2[2,4] \to \frac{213}{1024\sqrt{2}}$, $f2[3,0] \to -\frac{3\,050\,955\sqrt{3}}{524\,288}$, $f2[3,2] \to -\frac{5\,984\,055\sqrt{3}}{1\,048\,576}$,

$f2[3,4] \to -\frac{145\,773\sqrt{3}}{524\,288}$, $f2[3,6] \to \frac{38\,151\sqrt{3}}{1\,048\,576}$, $f2[4,0] \to -\frac{833\,141}{20\,736}$, $f2[4,2] \to -\frac{1\,191\,853}{20\,736}$,

$f2[4,4] \to -\frac{888\,515}{41\,472}$, $f2[4,6] \to -\frac{55\,355}{20\,736}$, $f2[4,8] \to -\frac{299}{4608}$, $f2[5,0] \to -\frac{8\,542\,435\,552\,735\sqrt{5}}{173\,946\,175\,488}$,

$f2[5,2] \to -\frac{13\,833\,719\,779\,295\sqrt{5}}{173\,946\,175\,488}$, $f2[5,4] \to -\frac{1\,853\,060\,419\,375\sqrt{5}}{43\,486\,543\,872}$,

$f2[5,6] \to -\frac{4\,755\,815\,646\,875\sqrt{5}}{347\,892\,350\,976}$, $f2[5,8] \to -\frac{329\,376\,953\,125\sqrt{5}}{173\,946\,175\,488}$, $f2[5,10] \to -\frac{2\,693\,359\,375\sqrt{5}}{38\,654\,705\,664}$,

$f2[6,0] \to -\frac{208\,250\,553\,303\sqrt{\frac{3}{2}}}{1\,048\,576\,000}$, $f2[6,2] \to -\frac{22\,368\,054\,537\sqrt{\frac{3}{2}}}{65\,536\,000}$, $f2[6,4] \to -\frac{2\,277\,062\,867\,799\sqrt{\frac{3}{2}}}{10\,485\,760\,000}$,

$f2[6,6] \to -\frac{12\,865\,141\,413\sqrt{\frac{3}{2}}}{131\,072\,000}$, $f2[6,8] \to -\frac{147\,632\,470\,533\sqrt{\frac{3}{2}}}{5\,242\,880\,000}$, $f2[6,10] \to -\frac{101\,964\,501\sqrt{\frac{3}{2}}}{26\,214\,400}$,

$f2[6,12] \to -\frac{65\,091\,681\sqrt{\frac{3}{2}}}{419\,430\,400}$, $f2[7,0] \to -\frac{3\,560\,268\,200\,161\,592\,143\sqrt{7}}{20\,038\,599\,416\,217\,600}$,

$f2[7,2] \to -\frac{211\,060\,636\,443\,019\,114\,717\sqrt{7}}{667\,953\,313\,873\,920\,000}$, $f2[7,4] \to -\frac{74\,251\,416\,037\,463\,753\,977\sqrt{7}}{333\,976\,656\,936\,960\,000}$,

$f2[7,6] \to -\frac{5\,391\,162\,478\,277\,524\,753\sqrt{7}}{44\,530\,220\,924\,928\,000}$, $f2[7,8] \to -\frac{8\,074\,005\,570\,904\,514\,059\sqrt{7}}{166\,988\,328\,468\,480\,000}$,

$f2[7,10] \to -\frac{8\,445\,040\,150\,227\,707\,453\sqrt{7}}{667\,953\,313\,873\,920\,000}$, $f2[7,12] \to -\frac{67\,705\,998\,948\,549\,997\sqrt{7}}{40\,077\,198\,832\,435\,200}$,

$f2[7,14] \to -\frac{611\,078\,988\,636\,949\sqrt{7}}{8\,906\,044\,184\,985\,600}$, $f2[8,0] \to -\frac{81\,704\,161\,715\,341}{70\,013\,160\,000\sqrt{2}}$, $f2[8,2] \to -\frac{148\,563\,163\,311\,277}{70\,013\,160\,000\sqrt{2}}$,

$f2[8,4] \to -\frac{10\,632\,031\,232\,159}{6\,667\,920\,000\sqrt{2}}$, $f2[8,6] \to -\frac{506\,668\,900\,757}{518\,616\,000\sqrt{2}}$, $f2[8,8] \to -\frac{22\,098\,096\,093\,817}{46\,675\,440\,000\sqrt{2}}$,

$f2[8,10] \to -\frac{1\,497\,642\,991\,421}{8\,751\,645\,000\sqrt{2}}$, $f2[8,12] \to -\frac{3\,686\,245\,601}{89\,302\,500\sqrt{2}}$, $f2[8,14] \to -\frac{2\,358\,383\sqrt{2}}{893\,025}$,

$f2[8,16] \to -\frac{10\,603\sqrt{2}}{99\,225}$, $f2[9,0] \to -\frac{72\,913\,759\,957\,443\,317\,508\,079\,893}{54\,065\,713\,526\,582\,804\,480\,000}$,

$f2[9,2] \to -\frac{134\,643\,287\,123\,897\,830\,242\,254\,991}{54\,065\,713\,526\,582\,804\,480\,000}$, $f2[9,4] \to -\frac{13\,244\,322\,323\,951\,517\,522\,607\,659}{6\,758\,214\,190\,822\,850\,560\,000}$,

$f2[9,6] \to -\frac{1\,408\,386\,559\,520\,149\,760\,160\,081}{1\,081\,314\,270\,531\,656\,089\,600}$, $f2[9,8] \to -\frac{9\,694\,451\,967\,687\,390\,682\,380\,897}{13\,516\,428\,381\,645\,701\,120\,000}$,

$f2[9,10] \to -\frac{8\,544\,307\,048\,431\,761\,577\,970\,053}{27\,032\,856\,763\,291\,402\,240\,000}$, $f2[9,12] \to -\frac{709\,163\,425\,186\,050\,348\,183\,069}{6\,758\,214\,190\,822\,850\,560\,000}$,

$$\text{f2}[9, 14] \to -\frac{51\,966\,809\,641\,299\,653\,389\,809}{2\,206\,763\,817\,411\,543\,040\,000},\ \text{f2}[9, 16] \to -\frac{25\,392\,576\,198\,025\,516\,131}{8\,827\,055\,269\,646\,172\,160},$$

$$\text{f2}[9, 18] \to -\frac{10\,141\,579\,493\,586\,125\,793}{88\,270\,552\,696\,461\,721\,600},\ \text{f2}[10, 0] \to -\frac{102\,806\,681\,191\,612\,804\,465\,615\,\sqrt{\frac{5}{2}}}{77\,942\,537\,061\,308\,301\,312},$$

$$\text{f2}[10, 2] \to -\frac{49\,385\,864\,534\,930\,122\,415\,\sqrt{\frac{5}{2}}}{20\,046\,948\,832\,641\,024},\ \text{f2}[10, 4] \to -\frac{5\,790\,041\,119\,606\,174\,319\,765\,\sqrt{\frac{5}{2}}}{2\,886\,760\,631\,900\,307\,456},$$

$$\text{f2}[10, 6] \to -\frac{1\,148\,482\,965\,262\,962\,780\,625\,\sqrt{\frac{5}{2}}}{811\,901\,427\,721\,961\,472},\ \text{f2}[10, 8] \to -\frac{5\,540\,758\,679\,914\,387\,203\,125\,\sqrt{\frac{5}{2}}}{6\,495\,211\,421\,775\,691\,776},$$

$$\text{f2}[10, 10] \to -\frac{349\,196\,876\,586\,086\,328\,125\,\sqrt{\frac{5}{2}}}{811\,901\,427\,721\,961\,472},\ \text{f2}[10, 12] \to -\frac{27\,236\,845\,995\,886\,357\,421\,875\,\sqrt{\frac{5}{2}}}{155\,885\,074\,122\,616\,602\,624},$$

$$\text{f2}[10, 14] \to -\frac{174\,786\,015\,691\,650\,390\,625\,\sqrt{\frac{5}{2}}}{3\,247\,605\,710\,887\,845\,888},\ \text{f2}[10, 16] \to -\frac{856\,742\,789\,306\,640\,625\,\sqrt{\frac{5}{2}}}{75\,745\,905\,793\,302\,528},$$

$$\text{f2}[10, 18] \to -\frac{87\,406\,158\,447\,265\,625\,\sqrt{\frac{5}{2}}}{66\,277\,667\,569\,139\,712},\ \text{f2}[10, 20] \to -\frac{6\,098\,175\,048\,828\,125\,\sqrt{\frac{5}{2}}}{117\,826\,964\,567\,359\,488},$$

$$\text{f2Zeta3}[1, 0] \to \frac{7}{16},\ \text{f2Zeta3}[1, 2] \to \frac{1}{16},\ \text{f2Zeta3}[2, 0] \to \frac{15}{2\sqrt{2}},\ \text{f2Zeta3}[2, 2] \to 4\sqrt{2},$$

$$\text{f2Zeta3}[2, 4] \to \frac{1}{2\sqrt{2}},\ \text{f2Zeta3}[3, 0] \to \frac{111\,699\,\sqrt{3}}{8192},\ \text{f2Zeta3}[3, 2] \to \frac{321\,327\,\sqrt{3}}{16\,384},$$

$$\text{f2Zeta3}[3, 4] \to \frac{56\,133\,\sqrt{3}}{8192},\ \text{f2Zeta3}[3, 6] \to \frac{6561\,\sqrt{3}}{16\,384},\ \text{f2Zeta3}[4, 0] \to \frac{208}{3},\ \text{f2Zeta3}[4, 2] \to 112,$$

$$\text{f2Zeta3}[4, 4] \to \frac{520}{9},\ \text{f2Zeta3}[4, 6] \to 16,\ \text{f2Zeta3}[4, 8] \to \frac{8}{9},\ \text{f2Zeta3}[5, 0] \to \frac{2\,417\,930\,625\,\sqrt{5}}{33\,554\,432},$$

$$\text{f2Zeta3}[5, 2] \to \frac{37\,276\,245\,625\,\sqrt{5}}{301\,989\,888},\ \text{f2Zeta3}[5, 4] \to \frac{5\,787\,265\,625\,\sqrt{5}}{75\,497\,472},$$

$$\text{f2Zeta3}[5, 6] \to \frac{2\,170\,703\,125\,\sqrt{5}}{67\,108\,864},\ \text{f2Zeta3}[5, 8] \to \frac{2\,294\,921\,875\,\sqrt{5}}{301\,989\,888},$$

$$\text{f2Zeta3}[5, 10] \to \frac{244\,140\,625\,\sqrt{5}}{603\,979\,776},\ \text{f2Zeta3}[6, 0] \to \frac{107\,758\,107\,\sqrt{\frac{3}{2}}}{409\,600},$$

$$\text{f2Zeta3}[6, 2] \to \frac{11\,942\,883\,\sqrt{\frac{3}{2}}}{25\,600},\ \text{f2Zeta3}[6, 4] \to \frac{264\,444\,831\,\sqrt{\frac{3}{2}}}{819\,200},\ \text{f2Zeta3}[6, 6] \to \frac{1\,728\,459\,\sqrt{\frac{3}{2}}}{10\,240},$$

$$\text{f2Zeta3}[6, 8] \to \frac{25\,174\,557\,\sqrt{\frac{3}{2}}}{409\,600},\ \text{f2Zeta3}[6, 10] \to \frac{649\,539\,\sqrt{\frac{3}{2}}}{51\,200},\ \text{f2Zeta3}[6, 12] \to \frac{531\,441\,\sqrt{\frac{3}{2}}}{819\,200},$$

$$\text{f2Zeta3}[7,0]\to\frac{7628681035076171\sqrt{7}}{34789235097600},\ \text{f2Zeta3}[7,2]\to\frac{55349789187715771\sqrt{7}}{139156940390400},$$

$$\text{f2Zeta3}[7,4]\to\frac{760924594225117\sqrt{7}}{2576980377600},\ \text{f2Zeta3}[7,6]\to\frac{544585606337773\sqrt{7}}{3092376453120},$$

$$\text{f2Zeta3}[7,8]\to\frac{312545586681797\sqrt{7}}{3865470566400},\ \text{f2Zeta3}[7,10]\to\frac{3638085451772443\sqrt{7}}{139156940390400},$$

$$\text{f2Zeta3}[7,12]\to\frac{337076867205953\sqrt{7}}{69578470195200},\ \text{f2Zeta3}[7,14]\to\frac{33232930569601\sqrt{7}}{139156940390400},$$

$$\text{f2Zeta3}[8,0]\to\frac{13568768\sqrt{2}}{19845},\ \text{f2Zeta3}[8,2]\to\frac{41676032\sqrt{2}}{33075},\ \text{f2Zeta3}[8,4]\to\frac{97486976\sqrt{2}}{99225},$$

$$\text{f2Zeta3}[8,6]\to\frac{156928\sqrt{2}}{245},\ \text{f2Zeta3}[8,8]\to\frac{33773696\sqrt{2}}{99225},\ \text{f2Zeta3}[8,10]\to\frac{4646912\sqrt{2}}{33075},$$

$$\text{f2Zeta3}[8,12]\to\frac{815104\sqrt{2}}{19845},\ \text{f2Zeta3}[8,14]\to\frac{32768\sqrt{2}}{4725},\ \text{f2Zeta3}[8,16]\to\frac{32768\sqrt{2}}{99225},$$

$$\text{f2Zeta3}[9,0]\to\frac{1047365529672045866823}{689613692941107200},\ \text{f2Zeta3}[9,2]\to\frac{1952881975248974658549}{689613692941107200},$$

$$\text{f2Zeta3}[9,4]\to\frac{28189077519484073439}{12314530231091200},\ \text{f2Zeta3}[9,6]\to\frac{78381768289952427669}{49258120924364800},$$

$$\text{f2Zeta3}[9,8]\to\frac{161301681728674567371}{172403423235276800},\ \text{f2Zeta3}[9,10]\to\frac{155772900678349762407}{344806846470553600},$$

$$\text{f2Zeta3}[9,12]\to\frac{2095936985890539657}{12314530231091200},\ \text{f2Zeta3}[9,14]\to\frac{62832002672456129979}{1379227385882214400},$$

$$\text{f2Zeta3}[9,16]\to\frac{4886414238002304717}{689613692941107200},\ \text{f2Zeta3}[9,18]\to\frac{450283905890997363}{1379227385882214400},$$

$$\text{f2Zeta3}[10,0]\to\frac{496914769105090625\sqrt{\frac{5}{2}}}{345196185255936},\ \text{f2Zeta3}[10,2]\to\frac{58429065155058125\sqrt{\frac{5}{2}}}{21574761578496},$$

$$\text{f2Zeta3}[10,4]\to\frac{776751423284745625\sqrt{\frac{5}{2}}}{345196185255936},\ \text{f2Zeta3}[10,6]\to\frac{17709192828671875\sqrt{\frac{5}{2}}}{10787380789248},$$

$$\text{f2Zeta3}[10,8]\to\frac{89722359131640625\sqrt{\frac{5}{2}}}{86299046313984},\ \text{f2Zeta3}[10,10]\to\frac{6045969677734375\sqrt{\frac{5}{2}}}{10787380789248},$$

$$\text{f2Zeta3}[10,12]\to\frac{171971596923828125\sqrt{\frac{5}{2}}}{690392370511872},\ \text{f2Zeta3}[10,14]\to\frac{3735321044921875\sqrt{\frac{5}{2}}}{43149523156992},$$

$$\text{f2Zeta3}[10,16]\to\frac{7391815185546875\sqrt{\frac{5}{2}}}{345196185255936},\ \text{f2Zeta3}[10,18]\to\frac{19073486328125\sqrt{\frac{5}{2}}}{6164217593856},$$

$$\text{f2Zeta3}[10,20]\to\frac{95\,367\,431\,640\,625\sqrt{\frac{5}{2}}}{690\,392\,370\,511\,872},\ \tau2[1,0]\to\frac{15}{56},\ \tau2[1,2]\to\frac{5}{8},\ \tau2[2,0]\to\frac{89}{40\sqrt{2}},$$

$$\tau2[2,2]\to\frac{15}{8\sqrt{2}},\ \tau2[2,4]\to\frac{21}{8\sqrt{2}},\ \tau2[3,0]\to\frac{20\,677\sqrt{3}}{11\,032},\ \tau2[3,2]\to\frac{168\,623}{31\,736\sqrt{3}},$$

$$\tau2[3,4]\to\frac{3089}{616\sqrt{3}},\ \tau2[3,6]\to\frac{49}{8\sqrt{3}},\ \tau2[4,0]\to\frac{9911}{1872},\ \tau2[4,2]\to\frac{15\,643}{3024},$$

$$\tau2[4,4]\to\frac{15\,527}{3120},\ \tau2[4,6]\to\frac{2093}{432},\ \tau2[4,8]\to\frac{89}{16},\ \tau2[5,0]\to\frac{2\,869\,115\,087\sqrt{5}}{835\,636\,824},$$

$$\tau2[5,2]\to\frac{4\,858\,716\,979\sqrt{5}}{1\,431\,407\,832},\ \tau2[5,4]\to\frac{5\,889\,979\sqrt{5}}{1\,777\,848},\ \tau2[5,6]\to\frac{19\,371\,371}{1\,200\,312\sqrt{5}},$$

$$\tau2[5,8]\to\frac{17\,933}{1128\sqrt{5}},\ \tau2[5,10]\to\frac{141}{8\sqrt{5}},\ \tau2[6,0]\to\frac{90\,004\,261}{3\,547\,592\sqrt{6}},\ \tau2[6,2]\to\frac{619\,372\sqrt{6}}{147\,443},$$

$$\tau2[6,4]\to\frac{154\,262\,003\sqrt{\frac{3}{2}}}{18\,655\,720},\ \tau2[6,6]\to\frac{9610\sqrt{6}}{2371},\ \tau2[6,8]\to\frac{3\,662\,461}{153\,480\sqrt{6}},\ \tau2[6,10]\to\frac{130\sqrt{\frac{2}{3}}}{11},$$

$$\tau2[6,12]\to\frac{205}{8\sqrt{6}},\ \tau2[7,0]\to\frac{2\,684\,330\,302\,694\,221}{76\,255\,037\,418\,504\sqrt{7}},\ \tau2[7,2]\to\frac{32\,324\,089\,806\,855\,151}{922\,112\,273\,014\,840\sqrt{7}},$$

$$\tau2[7,4]\to\frac{11\,532\,669\,047\,539}{332\,627\,262\,840\sqrt{7}},\ \tau2[7,6]\to\frac{55\,317\,926\,557}{1\,619\,440\,680\sqrt{7}},\ \tau2[7,8]\to\frac{4\,462\,704\,511}{132\,774\,360\sqrt{7}},$$

$$\tau2[7,10]\to\frac{49\,745\,497}{1\,501\,960\sqrt{7}},\ \tau2[7,12]\to\frac{56\,065}{1704\sqrt{7}},\ \tau2[7,14]\to\frac{281}{8\sqrt{7}},\ \tau2[8,0]\to\frac{10\,387\,858\,357}{445\,225\,200\sqrt{2}},$$

$$\tau2[8,2]\to\frac{19\,087\,991\,689}{820\,496\,880\sqrt{2}},\ \tau2[8,4]\to\frac{1\,406\,250\,193}{60\,929\,360\sqrt{2}},\ \tau2[8,6]\to\frac{345\,031}{15\,120\sqrt{2}},$$

$$\tau2[8,8]\to\frac{3\,327\,802\,619}{147\,759\,920\sqrt{2}},\ \tau2[8,10]\to\frac{254\,097\,737}{11\,435\,760\sqrt{2}},\ \tau2[8,12]\to\frac{5\,242\,529}{238\,800\sqrt{2}},$$

$$\tau2[8,14]\to\frac{22\,021}{1008\sqrt{2}},\ \tau2[8,16]\to\frac{369}{16\sqrt{2}},\ \tau2[9,0]\to\frac{296\,773\,713\,861\,252\,515\,861}{14\,899\,270\,858\,516\,122\,680},$$

$$\tau2[9,2]\to\frac{1\,656\,534\,145\,985\,722\,154\,201}{83\,342\,013\,880\,462\,262\,520},\ \tau2[9,4]\to\frac{1\,422\,097\,266\,735\,499\,757}{71\,974\,894\,303\,152\,120},$$

$$\tau2[9,6]\to\frac{1\,132\,295\,795\,518\,500\,979}{57\,815\,737\,155\,509\,928},\ \tau2[9,8]\to\frac{6\,776\,798\,475\,429\,569}{349\,732\,181\,180\,760},\ \tau2[9,10]\to\frac{26\,622\,542\,423\,147}{1\,389\,896\,094\,440},$$

$$\tau2[9,12]\to\frac{275\,387\,055\,239}{14\,545\,350\,120},\ \tau2[9,14]\to\frac{686\,271\,371}{36\,620\,520},\ \tau2[9,16]\to\frac{43\,759}{2344},\ \tau2[9,18]\to\frac{469}{24},$$

$$\tau2[10,0]\to\frac{3\,316\,164\,417\,810\,200\,067}{44\,523\,563\,311\,816\,120\sqrt{10}},\ \tau2[10,2]\to\frac{77\,859\,009\,324\,721\,903\sqrt{\frac{5}{2}}}{5\,235\,244\,237\,893\,208},$$

$$\tau 2[10,4] \to \frac{5\,151\,420\,304\,355\,374\,171}{69\,596\,927\,526\,313\,208\,\sqrt{10}},\ \tau 2[10,6] \to \frac{933\,015\,698\,438\,137}{12\,693\,949\,419\,592\,\sqrt{10}},$$

$$\tau 2[10,8] \to \frac{72\,093\,796\,209\,679}{989\,430\,569\,624\,\sqrt{10}},\ \tau 2[10,10] \to \frac{500\,327\,588\,693}{6\,934\,001\,704\,\sqrt{10}},\ \tau 2[10,12] \to \frac{2\,817\,080\,617\,479}{39\,446\,157\,016\,\sqrt{10}},$$

$$\tau 2[10,14] \to \frac{484\,514\,393}{6\,854\,344\,\sqrt{10}},\ \tau 2[10,16] \to \frac{27\,150\,039}{387\,544\,\sqrt{10}},\ \tau 2[10,18] \to \frac{3911}{56\,\sqrt{10}},\ \tau 2[10,20] \to \frac{581}{8\,\sqrt{10}}\};$$