Minimal Models RG flows: non-invertible symmetries & non-perturbative description

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In this letter we continue the investigation of RG flows between minimal models that are protected by non-invertible symmetries. RG flows leaving unbroken a subcategory of non-invertible symmetries are associated with anomaly-matching conditions that we employ systematically to map the space of flows between Virasoro Minimal models beyond the $\mathbb{Z}_2$-symmetric proposed recently in the literature. We introduce a family of non-linear integral equations that appear to encode the exact finite-size, ground-state energies of these flows, including non-integrable cases, such as the recently proposed $\mathcal{M}(kq + I, q) \to \mathcal{M}(kq - I, q)$. Our family of NLIEs encompasses and generalises the integrable flows known in the literature: $\phi_{(1,3)}$, $\phi_{(1,5)}$, $\phi_{(1,2)}$ and $\phi_{(2,1)}$. This work uncovers a new interplay between exact solvability and non-invertible symmetries. Furthermore, our non-perturbative description provides a non-trivial test for all the flows conjectured by anomaly matching conditions, but so far not-observed by other means.

IDEA: Symmetries constrain RG FLOWS

All Symmetries

Why MINIMAL MODELS? We know the set of all TDL

1 Anomaly matching ⇒ Conjecture flows 2 Can we test it?

1. $\mathcal{M}(p,q)$ 2d Rational CFTs

$c(p,q) = 1 - \frac{6(p-q)^2}{pq}$, finite # primaries $\phi_{(r,s)}$

$\uparrow$

$h_{(r,s)}$

$[\phi_\rho] \times [\phi_\sigma] = \sum_\delta \mathcal{N}_{\rho\sigma}^{\ \ \delta} [\phi_\delta]$ fusion

Topological lines $\equiv$ Symmetries $\{\mathcal{L}_\sigma\}$

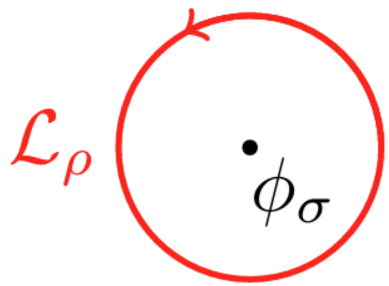
CRUCIAL FACT 1 to 1 correspondence

$\{\mathcal{L}_\sigma\} \leftrightarrow \{\phi_\sigma\}$ $\qquad \mathcal{L}_\rho \times \mathcal{L}_\sigma = \sum_\delta \mathcal{N}_{\rho\sigma}^{\ \ \delta} \mathcal{L}_\delta$

$\not\!\phi$ Non invertible

$\phi_\rho |0\rangle = |\phi_\sigma\rangle$

$$\mathcal{L}_\rho |\phi_\sigma\rangle =$$



$$= \frac{S_{\rho\sigma}}{S_{0\sigma}} |\phi_\sigma\rangle \, .$$

Deformation by relevant primary

$$\mathcal{M}(p,q) + g_\rho \int dx\, \phi_\rho \qquad [h_\rho < 1]$$

GAPPED

$\mathcal{T}_{IR} = \text{TQFT}$

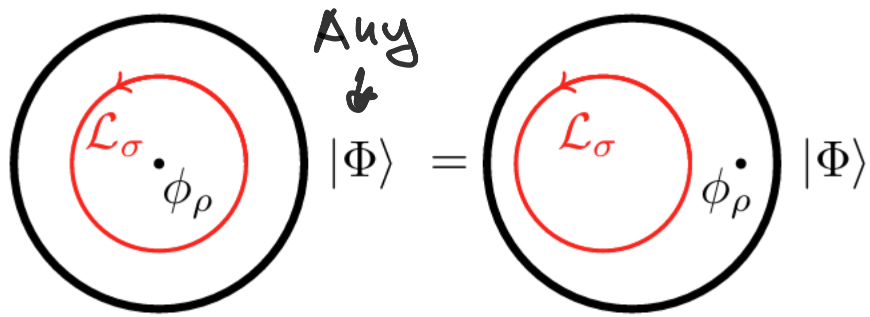
GAPLESS

$\mathcal{T}_{IR} = \text{CFT}$

$$c_{eff} = 1 - \frac{6}{pq}$$ MONOTONICALLY DECREASING (PT)

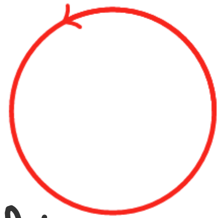
$\boxed{\text{TODAY: } \quad \mathcal{T}_{IR} \text{ GAPLESS} \qquad \mathcal{T}_{IR} = \mathcal{M}(p', q')}$

If $[\mathcal{L}_\rho, \phi_\sigma] \, |\Phi\rangle = 0$



$\Rightarrow$ $\mathcal{L}_\rho$ is preserved

Then

(1) QUANTUM DIMENSION $\langle \mathcal{L}_\rho \rangle = \langle 0 | \mathcal{L}_\rho | 0 \rangle =$ $\mathcal{L}_\rho$  $= d_\rho$

$d_\rho = 1$ iff $\mathcal{L}_\rho$ does not have 't Hooft anomaly

(2) Spin content of Defect Hilbert space

$$Z_{\mathcal{L}_\lambda}(\tau, \bar{\tau}) = \sum_{\rho, \sigma} N^{\lambda}_{\rho\sigma} \, \chi_\rho(\tau) \, \bar{\chi}_\sigma(\bar{\tau})$$

$s_\lambda = \{ h_\rho - h_\sigma \} \bmod \mathbb{Z}$ over $N^{\lambda}_{\rho\sigma} \neq 0$

if anomaly free $s_\lambda = \{0, \frac{1}{2}\}$ otherwise RATIONAL NUM

(1) & (2) ARE RG INVARIANTS!

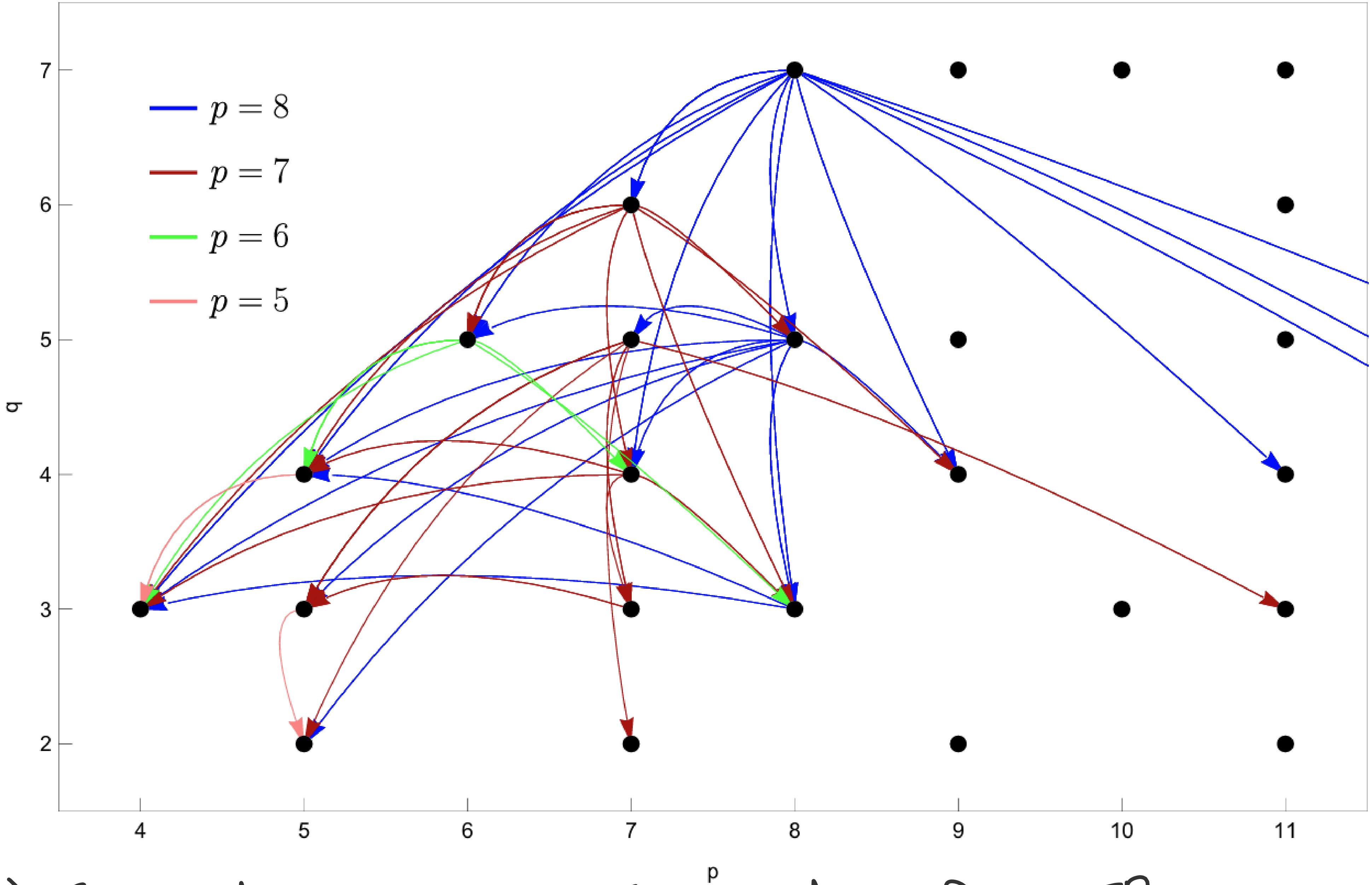
LET US IMPOSE ANOMALY MATCHING !

① Given $\Upsilon_{UV} = \mathcal{M}(p,q)$ for any relevant $\phi_{(r,s)}$ compute $\{\mathcal{L}_\alpha\}^{(r,s)}_{UV}$ commuting w/ $\phi_{(r,s)}$

② Generate Υ_{IR} s.t $c^{eff}_{IR} < c^{eff}_{UV}$

③ Select Υ_{IR} fulfilling anomalies

$$\Rightarrow \quad \mathcal{M}(p,q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p',q')$$



(i) Exclusive procedure (ii) Match from IR

(iii) Multi-field deformation (iv) Control ops generated along flow

Can we study these flows ? TBA for BD $\phi(2,1)$ $\phi(1,5)$ integrable cases

$$f_{\mathrm{R}}(\theta) = \mathrm{i}\alpha' - \mathrm{i}\frac{r}{2}e^{\theta} - \sum_{\sigma=\pm} \sigma \int_{\mathcal{C}_{\mathrm{s}}^{\sigma}} d\theta' \left[\phi(\theta-\theta') L_{\mathrm{R}}^{-\sigma}(\theta') + \chi(\theta-\theta') L_{\mathrm{L}}^{\sigma}(\theta')\right] ,$$

$$f_{\mathrm{L}}(\theta) = -\mathrm{i}\alpha' - \mathrm{i}\frac{r}{2}e^{\theta} + \sum_{\sigma=\pm} \sigma \int_{\mathcal{C}_{\mathrm{s}}^{\sigma}} d\theta' \left[\phi(\theta-\theta') L_{\mathrm{L}}^{\sigma}(\theta') + \chi(\theta-\theta') L_{\mathrm{R}}^{-\sigma}(\theta')\right] ,$$

$$L^{\pm}_{R/L}(\theta) = \log\left(1 + \exp(\pm f_{R/L}(\theta))\right)$$

Scaling function: [G.S. free energy]

$$f_s(r) = \sum_{\sigma=\pm} \frac{3\mathrm{i}r\sigma}{2\pi^2} \int_{\mathcal{C}_{\mathrm{s}}^{\sigma}} d\theta \left[e^{-\theta} L_{\mathrm{L}}^{\sigma}(\theta) - e^{\theta} L_{\mathrm{R}}^{-\sigma}(\theta)\right]$$

For $(2,1)$ $(1,5)$ NLIE follow from $\partial_\theta S(\theta)$ BD.

Known TBA also from $\phi(1,3)$ (SG)

CAN WE EXTEND TO ALL THE NEW FLOWS ?

We bootstrapped ϕ, χ kernels:

$$\phi(\theta) = -\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{i\theta\omega} \frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{2\xi-\kappa}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)},$$

$$\chi(\theta) = -\int_{\mathbb{R}} \frac{d\omega}{2\pi} e^{i\theta\omega} \frac{\sinh\left(\frac{1}{\kappa}\pi\omega\right)\cosh\left(\frac{\kappa-2}{2\kappa}\pi\omega\right)}{\sinh\left(\frac{\xi-1}{\kappa}\pi\omega\right)\cosh\left(\frac{1}{2}\pi\omega\right)},$$

Requiring (i) Analiticity (ii) Asymptotic (iii) CPT (iv) Bulk part. $\phi(r,s)$

$f_S(r \to 0) = c_{UV}$

$f_S(r \to \infty) = c_{IR}$

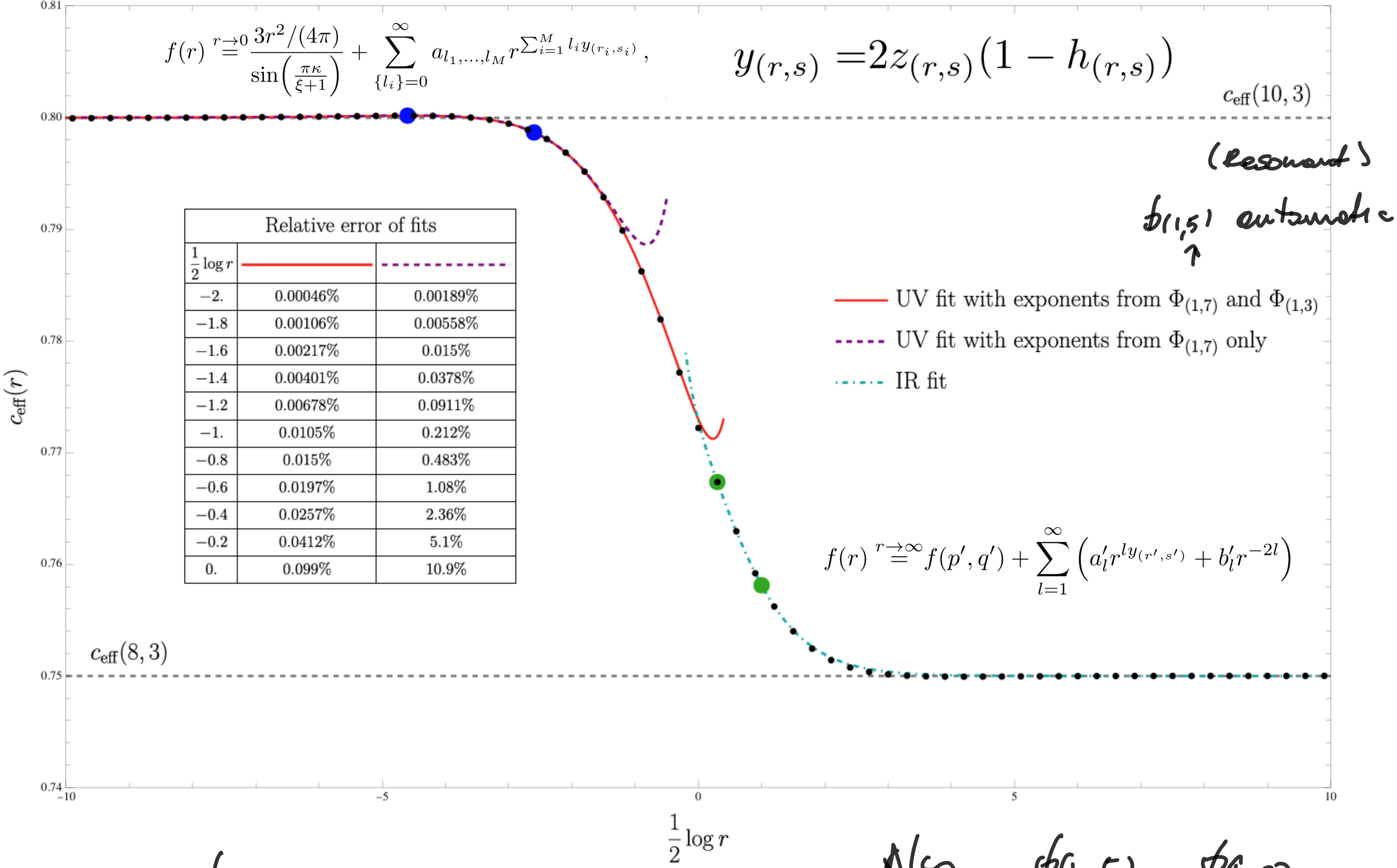
$\Rightarrow$ QUANTIZATION OF κ, ξ

kernels with (i) - (iv) only exist for flows $\mathcal{M}(p,q) \to \mathcal{M}(p',q')$ of non-int !!!

[Experimental fact]

CONJECTURE NEW NON-PERTURBATIVE DESCRIPTION OF THESE FLOWS!

(INTEGRABLE ? PROBABLY NOT!)



$\frac{1}{2}\log r$	UV fit with exponents from $\Phi_{(1,7)}$ and $\Phi_{(1,3)}$ (red solid)	UV fit with exponents from $\Phi_{(1,7)}$ only (purple dashed)
−2.	0.00046%	0.00189%
−1.8	0.00106%	0.00558%
−1.6	0.00217%	0.015%
−1.4	0.00401%	0.0378%
−1.2	0.00678%	0.0911%
−1.	0.0105%	0.212%
−0.8	0.015%	0.483%
−0.6	0.0197%	1.08%
−0.4	0.0257%	2.36%
−0.2	0.0412%	5.1%
0.	0.099%	10.9%

$\mathcal{M}(10,3) \xrightarrow{\phi_{(1,7)}} \mathcal{M}(3,8)$

Also $\phi_{(1,5)}$ $\phi_{(1,3)}$ must be tuned

[MATCHES CFT!]

(1) Conjectured flows $\mathcal{M}(p,q) \xrightarrow{\phi_{(r,s)}} \mathcal{M}(p',q')$

(2) Proposed description for $\downarrow$ via NLIE

Outlook

(i) More tests (CPT, Ham trunc)

(ii) Can we derive NLIE as reduction of known models?

E.g. adding primaries to Liouville?