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ABSTRACT: We explore the connection between super $\mathcal{W}$-algebras ($\mathcal{SW}$-algebras) and G-structures with torsion. The former are realised as symmetry algebras of strings with $\mathcal{N} = (1,0)$ supersymmetry on the worldsheet, while the latter are associated with generic string backgrounds with non-trivial Neveu–Schwarz flux H. In particular, we focus on manifolds featuring Spin(7), G_2, SU(2), and SU(3)-structures. We compare the full quantum algebras with their classical limits, obtained by studying the commutators of superconformal and $\mathcal{W}$-symmetry transformations—which preserve the action of the $(1,0)$ non-linear σ-model. We show that, at first order in the string length scale ℓ_s, the torsion deforms some of the OPE coefficients corresponding to special holonomy through a scalar torsion class.

- *A W-algebras primer*
- *A G-structures primer*
- *A W-symmetries primer*

Results of the paper

Bootstrapping W-algebras (1985)

INFINITE ADDITIONAL SYMMETRIES IN TWO-DIMENSIONAL CONFORMAL QUANTUM FIELD THEORY

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Additional symmetries in two-dimensional conformal field theory generated by spin $s = \frac{1}{2}, 1, \ldots, 3$ currents are investigated. For spins $s = 5/2$ and $s = 3$, the generators of the symmetry form associative algebras with quadratic determining relations. "Minimal models" of conformal field theory with such additional symmetries are considered.

- Symmetries: Virasoro + *new primaries*.
- How does the new algebra close?

Bootstrapping SW-algebras

- SW-algebras are simply supersymmetric W-algebras.

- Symmetry algebras of the worldsheet theory of $\mathcal{N} = (1,0)$ strings.
- Adopting superspace conventions:

$$\mathcal{T}(Z_1)\mathcal{T}(Z_2) \sim \frac{c}{6}\frac{1}{Z_{12}^3} + \frac{3}{2}\frac{\theta_{12}}{Z_{12}^2}\mathcal{T}(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{T}(Z_2) + \frac{1}{2Z_{12}}D\mathcal{T}(Z_2) + \ldots$$

$$\mathcal{T}(Z_1)\mathcal{J}_h(Z_2) \sim \frac{h\,\theta_{12}}{Z_{12}^2}\mathcal{J}_h(Z_2) + \frac{\theta_{12}}{Z_{12}}\partial\mathcal{J}_h(Z_2) + \frac{1}{2\,Z_{12}}D\mathcal{J}_h(Z_2) + \ldots$$

G-structures

- Why G-structures?
  - In supergravity, we want the compactification to preserve susy
  - This requires the existence of at least one globally defined spinor field η
  - Spinor bilinears:

$$\omega_p = \bar{\eta}\gamma_{\mu_1 \ldots \mu_p}\eta$$

- Roughly, more in general:

G-structures → A set of globally defined, no-where vanishing p-forms

G-structures

G-struct.
$\mathrm{O}(d-n)$ $(\sigma_1, \ldots, \sigma_n)$
$\mathrm{Spin}(7)$ (Ψ)
G_2 (φ, ψ)
$\mathrm{SU}(2)$ $(\omega, \Omega^{\pm})$
$\mathrm{SU}(3)$ $(\omega, \Omega^{\pm})$

- Example: SU(3)-structure (includes CY 3-folds)

$$\Omega \wedge \omega = 0\,, \qquad \frac{1}{6}\,\omega \wedge \omega \wedge \omega = \frac{i}{8}\,\Omega \wedge \overline{\Omega}$$

- Torsion (or: NS 3-flux) H:

$$\mathrm{d}\omega = -\frac{3}{4}\,\mathrm{Im}(W_0\,\overline{\Omega}) + W_1 \wedge \omega + W_3$$

$$\mathrm{d}\Omega = W_0\,\omega \wedge \omega + W_2 \wedge \omega + \bar{\vartheta} \wedge \Omega$$

G-structures

G-struct.
$\mathrm{O}(d-n)$
$(\sigma_1, \ldots, \sigma_n)$
$\mathrm{Spin}(7)$
(Ψ)
G_2
(φ, ψ)
$\mathrm{SU}(2)$
$(\omega, \Omega^{\pm})$
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- Example: SU(3)-structure (includes CY 3-folds)

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- Torsion (or: NS 3-flux) H:

$$\mathrm{d}\omega = 0$$
$$\mathrm{d}\Omega = 0$$

→ Special Holonomy!
(CY 3-folds)

Worldsheet CFT and special holonomy

G-struct.	$H=0$
$\mathrm{O}(d-n)$ $(\sigma_1,\ldots,\sigma_n)$	$\mathcal{S}\mathrm{Vir}\oplus\mathrm{Free}^n$
$\mathrm{Spin}(7)$ (Ψ)	$\mathrm{SV}^{\mathrm{Spin}(7)}$ [6]
G_2 (φ,ψ)	$\mathrm{SV}^{\mathrm{G}_2}$ [6]
$\mathrm{SU}(2)$ $(\omega,\Omega^{\pm})$	$\mathrm{Od}(2)$ [4]
$\mathrm{SU}(3)$ $(\omega,\Omega^{\pm})$	$\mathrm{Od}(3)$ [4]

Shatashvili-Vafa algebras (1994)

Odake algebras (1988)

- These algebras are well understood.
- They rely on **free-fields realizations.**
- Usually affected by «pathologies» (e.g. closure up to **null fields** ideals).

Introducing a generic torsion

- Consider the classical worldsheet action:

$$S[X,\Lambda] = \int_\Sigma \frac{\mathrm{d}^{2|1}\zeta}{2\,\ell_s^2} \; M_{ij}(X)\bar{\partial}X^i D X^j \qquad\qquad M_{ij}(X) = G_{ij}(X) + B_{ij}(X)$$

- The action is sensitive to the G-structure via W-symmetry:

$$\delta^\Phi X^i = \frac{\epsilon(\zeta)}{(p-1)!} \, \Phi^i{}_{i_2\ldots i_p}(X) D X^{i_2\ldots i_p}$$

G-structure $(p+1)$-form!

W-symmetries

- For each p-form, we obtain a Noether current

$$\nabla^+\Phi = 0 \longrightarrow \mathcal{J}^{\Phi}_{\text{cl.}} = (-1)^{p-1}\,\widehat{\Phi}$$

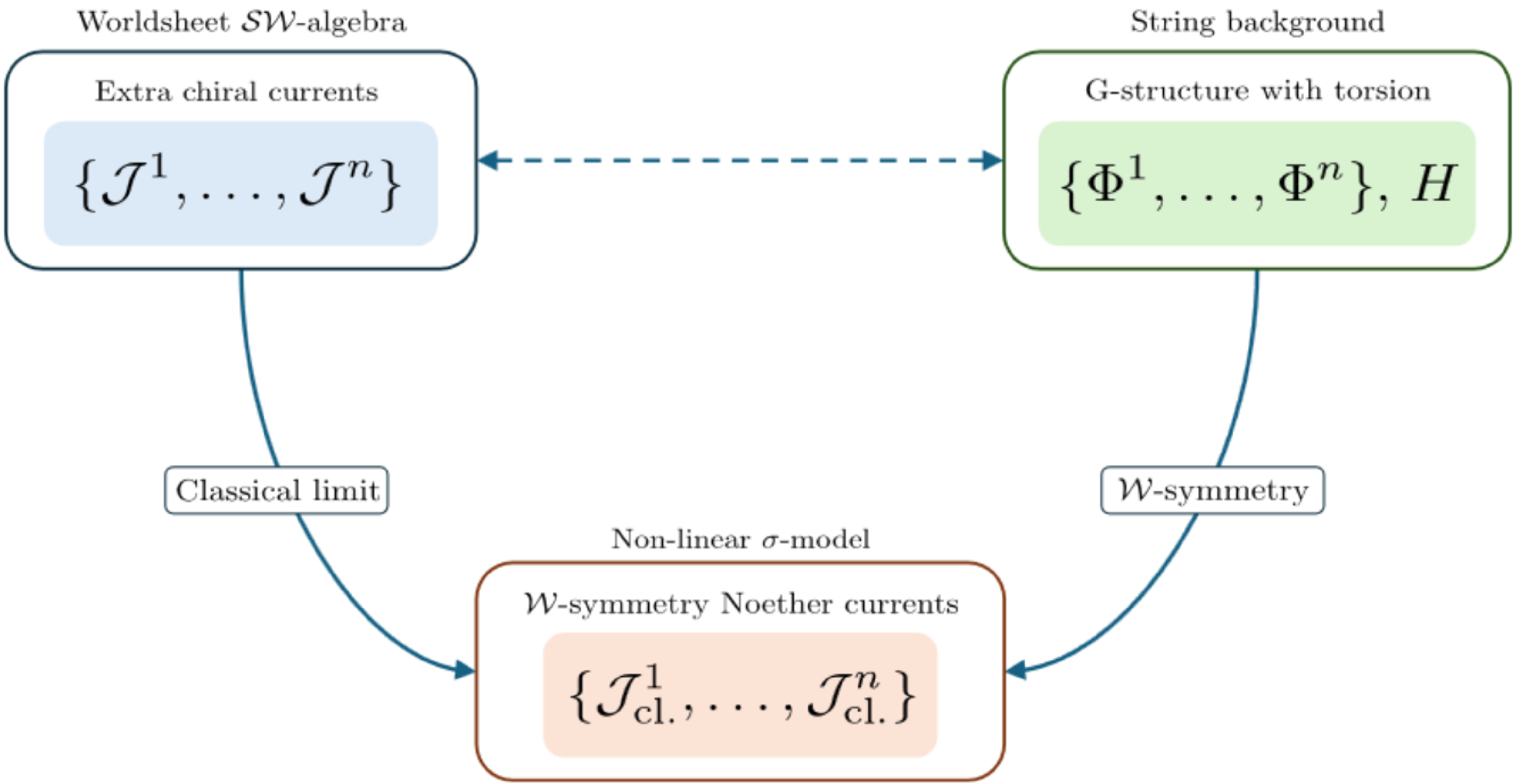
- We can write commutators which translate into classical OPEs

$$[\delta^{\Phi}_{\epsilon_1}, \delta^{\Psi}_{\epsilon_2}]X^i = \delta^{U}_{\epsilon_U} X^i + \delta^{N}_{\epsilon_N} X^i + \delta^{\mathcal{T}V}_{\epsilon_{\mathcal{T}V}} X^i$$

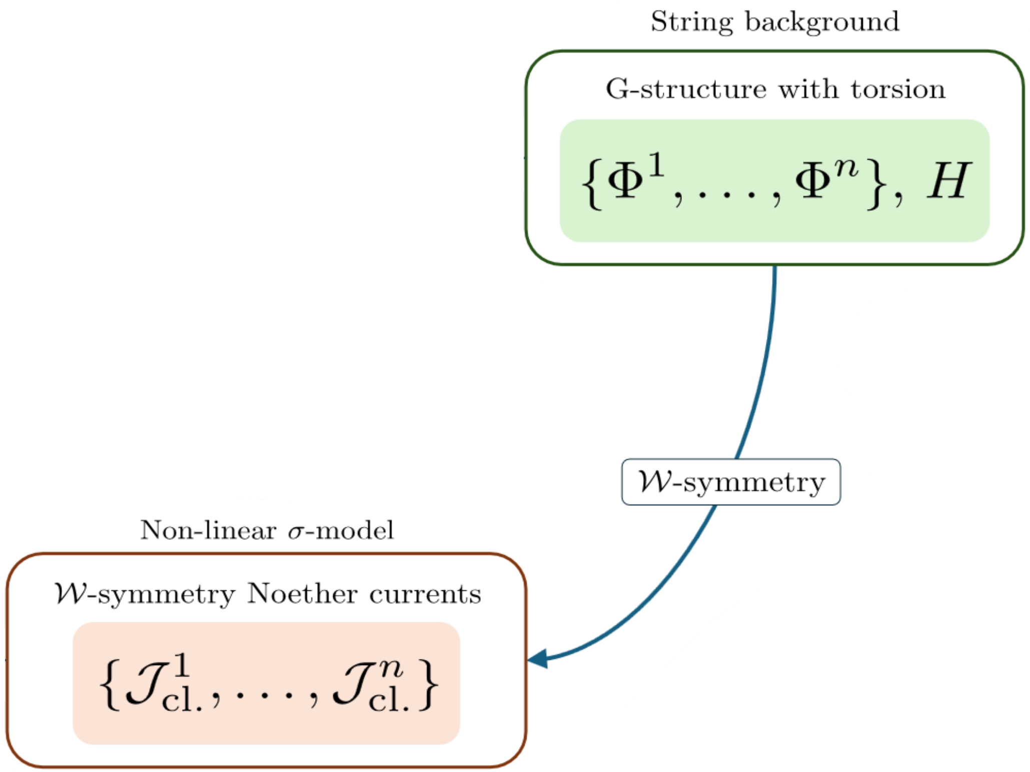
$$\Downarrow$$

$$\mathcal{J}^{\Phi}_{\text{cl.}}(Z_1)\mathcal{J}^{\Psi}_{\text{cl.}}(Z_2) \sim (-1)^{p+1}\, c_U\, \frac{\mathcal{J}^{U}_{\text{cl.}}(Z_2)}{Z_{12}} + (-1)^{p+1}\, c_U \left(\frac{p-1}{p+q-2}\right) \frac{\theta_{12}}{Z_{12}} D\mathcal{J}^{U}_{\text{cl.}}(Z_2)$$
$$+ (-1)^{p}\, c_N\, \frac{\theta_{12}}{Z_{12}} \mathcal{J}^{N}_{\text{cl.}}(Z_2) + c_V \left(\frac{2}{d-(p+q-4)}\right) \frac{\theta_{12}}{Z_{12}} \mathcal{T}_{\text{cl.}}(Z_2)\mathcal{J}^{V}_{\text{cl.}}(Z_2) + \dots .$$

The broad picture



Example: a bunch of vector fields



- The string background is endowed with an $O(d-n)$-**structure**, so we have n vector fields and a generic torsion H.

- To each vector field we associate a **covariantly constant 1-form**:

$$\nabla^+_i \sigma_{I,j} = 0$$

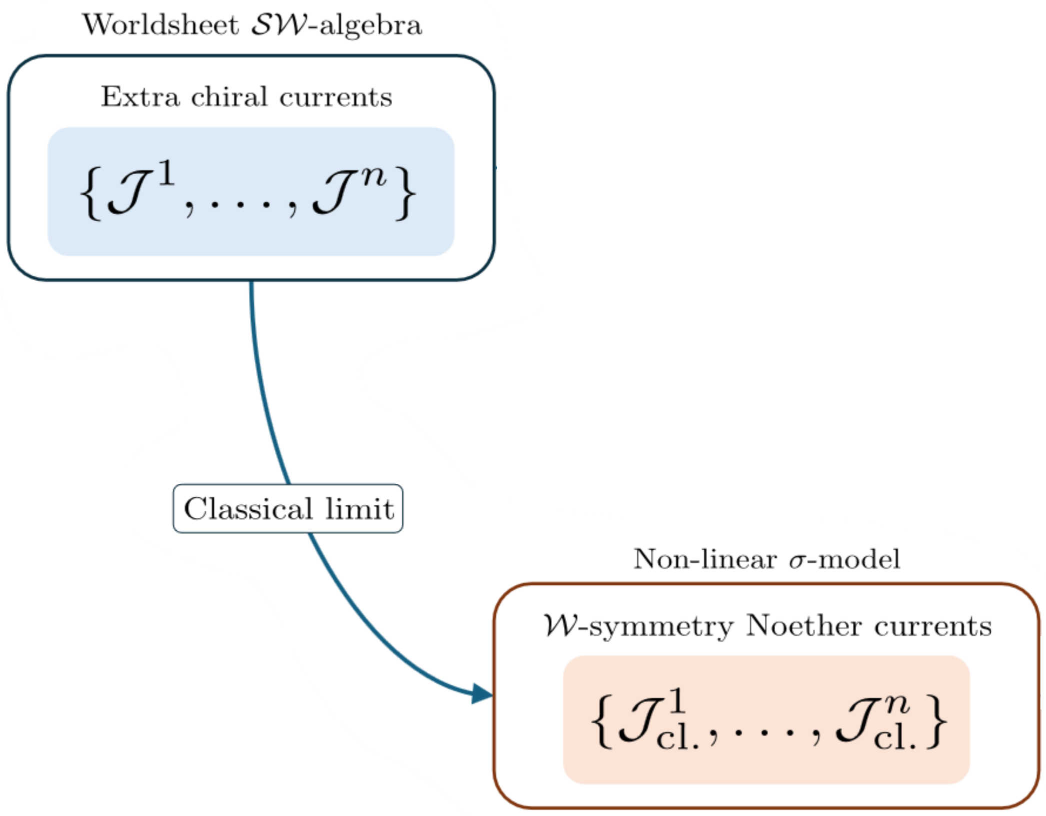
- So we have n new W-symmetry **Noether currents**

$$\mathcal{J}^\sigma_{I,\text{cl.}} = \sigma^i_I(X) DX_i$$

- The **classical OPE** reads:

$$\mathcal{J}^\sigma_{\text{cl.},I}(Z_1)\mathcal{J}^\sigma_{\text{cl.},J}(Z_2) \sim -\frac{\sigma_I \cdot \sigma_J}{Z_{12}} - \ell_s \frac{\theta_{12}}{Z_{12}} \mathcal{J}^{i_{\sigma_J} i_{\sigma_I}(H)}_{\text{cl.}} + \ldots$$

Example: a bunch of vector fields



- We add **a current** for each **vector field**.
- The SW-algebra is generated by:

$$\langle \mathbb{1}, \mathcal{T}, \mathcal{J}^\sigma_1, \ldots, \mathcal{J}^\sigma_n \rangle$$

- The **OPE Ansatz** is:

$$\mathcal{J}^\sigma_I(Z_1)\mathcal{J}^\sigma_J(Z_2) \sim \frac{C_{IJ}}{Z_{12}} + C^K_{IJ}\frac{\theta_{12}}{Z_{12}}\mathcal{J}^\sigma_K + \ldots$$

- We interpret this OPE as the all-loops completion of the classical OPE.

Example: a bunch of vector fields

Worldsheet $\mathcal{SW}$-algebra

Extra chiral currents

$$\{\mathcal{J}^1, \dots, \mathcal{J}^n\}$$

String background

G-structure with torsion

$$\{\Phi^1, \dots, \Phi^n\},\ H$$

$$\mathcal{J}^\sigma_I(Z_1)\mathcal{J}^\sigma_J(Z_2) \sim \frac{C_{IJ}}{Z_{12}} + C_{IJ}^K \frac{\theta_{12}}{Z_{12}} \mathcal{J}^\sigma_K + \dots$$

$$\mathcal{J}^\sigma_{\text{cl.},I}(Z_1)\mathcal{J}^\sigma_{\text{cl.},J}(Z_2) \sim -\frac{\sigma_I \cdot \sigma_J}{Z_{12}} - \ell_s \frac{\theta_{12}}{Z_{12}} \mathcal{J}_{\text{cl.}}^{i_{\sigma_J} i_{\sigma_I}(H)} + \dots$$

$$C_{IJ} = -\sigma_I \cdot \sigma_J + O(\ell_s^2)$$

$$C_{IJ}^K\, \sigma_K^k = -\ell_s H_{ij}{}^k \sigma_I^i \sigma_J^j + O(\ell_s^2)$$

State of the art

G-struct.	$H=0$	$H \neq 0$	Par.	Constraints found
$\mathrm{O}(d-n)$ $(\sigma_1,\ldots,\sigma_n)$	$\mathcal{S}\mathrm{Vir} \oplus \mathrm{Free}^n$	$\mathcal{SW}\left(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2}\right)$	C_{IJ} f_{IJK}	$C_{IJ} = -\sigma_I \cdot \sigma_J + O(\ell_s^2)$ $f_{IJK} = -\ell_s H_{ijk}\sigma_I^i \sigma_J^j \sigma_K^k + O(\ell_s^2)$
$\mathrm{Spin}(7)$ (Ψ)	$\mathrm{SV}^{\mathrm{Spin}(7)}$ [6]	FS [23]	c	–
G_2 (φ,ψ)	$\mathrm{SV}^{\mathrm{G}_2}$ [6]	FG_k [16] $\subset$ Bl [27]	k	$\sqrt{\frac{2}{k}\frac{7k-4}{49k-24}} = \frac{1}{6}\tau_0\ell_s + O(\ell_s^2)$
$\mathrm{SU}(2)$ $(\omega,\Omega^\pm)$	$\mathrm{Od}(2)$ [4]	$\mathcal{SW}\left(\frac{3}{2},1,1,1\right)$	c	$c = 6 + O(\ell_s^2)$
$\mathrm{SU}(3)$ $(\omega,\Omega^\pm)$	$\mathrm{Od}(3)$ [4]	$\mathrm{Od}^\varepsilon(3) \subset \mathcal{SW}(\frac{3}{2},\frac{3}{2},\frac{3}{2},1)$	$v_\pm$	$v_\pm = 4\,w_0^\pm \ell_s + O(\ell_s^2)$ $W_0^\pm = \varepsilon\, w_0^\pm + O(\varepsilon^2,\ell_s^2)$

Take-home message

- Some geometrical features of the string background are encoded in the G-structure and in the torsion.

- These geometrical features are encoded at the CFT level as a symmetry enhancement of Virasoro to a W-algebra.

- The details of this connection are difficult to track down.
  1. Engineer a torsion for exact computations (ex. WZW models)
  2. Work perturbatively in ℓ_s using W-symmetries (ex. this paper)