# Theory of Fundamental Interactions

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## Contents

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Many of the following problems are derived and adapted from a lecture course by Prof. Olaf Lechtenfeld at Leibniz University Hannover (from 2016), which can be found at his webpage:
`https://www.itp.uni-hannover.de/553.html`

## 1.1. Euler–Lagrange Equation and Conserved Currents

Recall the derivation of the Euler–Lagrange equation from a given Lagrangian density $\mathcal{L}$.

**a)** Assume that $\mathcal{L}$ only depends on a field $\phi$ and its derivative $\partial_\mu\phi$, but not explicitly on $x^\mu$: $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$. Compute the variation $\delta S$ of the action $S = \int \mathcal{L}\, \mathrm{d}^4x$ under a general variation $\phi(x) \to \phi'(x) + \delta\phi(x)$. Assume a closed system (surface terms vanish). Express the result in the form $\delta S = \int \mathrm{d}^4x\, f(\phi, \partial_\mu\phi)\, \delta\phi$.

**b)** Starting from the Lagrangian density $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) + \phi\rho$, derive the equation of motion for the field $\phi$.

**c)** Let $J^\mu$ be a conserved current, such that $\partial_\mu J^\mu = 0$. Assuming a closed system, derive from $J^\mu$ an expression for a conserved charge $Q$, for which $\partial Q/\partial t = 0$.

## 1.2. The Field Strength Tensor

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ are expressed in terms of the scalar potential $V$ and the vector potential $\vec{A}$ as

$$\vec{E} = -\nabla V - \partial\vec{A}/\partial t\,, \qquad \vec{B} = \nabla \times \vec{A}\,. \tag{1.1}$$

The potentials are in turn combined into the four-vector $A^\mu = (V; \vec{A})$. Define the antisymmetric tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \tag{1.2}$$

**a)** Express the components $E^i$ and $B^i$ of the electric and magnetic fields in terms of the field strength tensor $F^{\mu\nu}$. Write the matrix $F^{\mu\nu}$ in terms of $E^i$ and $B^i$.

**b)** Show that the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,(\vec{E}^2 - \vec{B}^2) - \rho V + \vec{J}\cdot\vec{A} \tag{1.3}$$

for electrodynamics can be written as

$$\mathcal{L} = -\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - J_\mu A^\mu\,, \tag{1.4}$$

where the sources $\rho$ and $\vec{J}$ are combined in $J^\mu = (\rho, \vec{J})$.

## 1.3. Maxwell's Equations

**a)** Starting with the Lagrangian (1.3), apply the Euler–Lagrange equations for $V$ and the three components of $\vec{A}$. Recover Maxwell's equations from your results.

**b)** Starting with the Lagrangian (1.4), find an equation for $F^{\mu\nu}$ by applying the Euler–Lagrange equations for the four components of $A^\mu$. Verify that the equation is equivalent to Maxwell's equations.

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## 1.4. Local Gauge Transformations

Show that the Schrödinger equation for a charged particle in the presence of an electromagnetic field

$$(\mathrm{i}\partial_t + eV)\psi = \frac{1}{2m}(-\mathrm{i}\partial + e\vec{A})^2\psi \tag{1.5}$$

is invariant under the simultaneous local gauge transformations

$$\begin{aligned} \psi &\to \psi' = \mathrm{e}^{\mathrm{i}\chi}\psi\,, \\ \vec{A} &\to \vec{A}' = \vec{A} + \tfrac{1}{e}(\partial\chi)\,, \\ V &\to V' = V - \tfrac{1}{e}(\partial_t\chi)\,. \end{aligned} \tag{1.6}$$

## 2.1. Relativistic Kinematics

Consider a two-to-two scattering event $\mathrm{a} + \mathrm{b} \to \mathrm{c} + \mathrm{d}$. The four particles a, b, c, and d have masses $m_\mathrm{a}$, $m_\mathrm{b}$, $m_\mathrm{c}$, $m_\mathrm{d}$, and four-momenta $p_\mathrm{a} = (E_\mathrm{a}; \vec{p}_\mathrm{a})$ etc. It is convenient to introduce the *Mandelstam variables*

$$s = (p_\mathrm{a} + p_\mathrm{b})^2 , \qquad t = (p_\mathrm{c} - p_\mathrm{a})^2 , \qquad u = (p_\mathrm{d} - p_\mathrm{a})^2 . \tag{2.1}$$

The theoretical virtue of these variables is that they are Lorentz invariant, i. e. have the same value in all inertial systems. Experimentally, the more accessible parameters are energies and scattering angles.

**a)** Show that $s + t + u = m_\mathrm{a}^2 + m_\mathrm{b}^2 + m_\mathrm{c}^2 + m_\mathrm{d}^2$.

**b)** Find the energy $E_\mathrm{a}$ in the frame where b is at rest.

From now on, consider the center-of-mass frame, where $\vec{p}_\mathrm{a} + \vec{p}_\mathrm{b} = 0$ (this is the relevant case for particle colliders).

**c)** Express $E_\mathrm{a}$ in terms of $s$, $t$, $u$, and the masses.

**d)** Find the total energy $E_\mathrm{tot} = E_\mathrm{a} + E_\mathrm{b} + E_\mathrm{c} + E_\mathrm{d}$ in terms of $s$, $t$, and $u$.

**e)** The scattering angle $\theta$ is defined via $\vec{p}_\mathrm{a} \cdot \vec{p}_\mathrm{c} = |\vec{p}_\mathrm{a}||\vec{p}_\mathrm{c}| \cos\theta$. For negligible masses, express $\theta$ in terms of $s$, $t$, and $u$.

## 2.2. Lie Algebras

Let $V$ be a vector space with a bilinear product $[\cdot,\cdot] : V \times V \to V$ that satisfies the conditions

$$[v, w] + [w, v] = 0 \qquad \text{(antisymmetry)} , \tag{2.2}$$
$$[v, [w, u]] + [u, [v, w]] + [w, [u, v]] = 0 \qquad \text{(Jacobi identity)} , \tag{2.3}$$

for all $v, w, u \in V$. Then $(V, [\cdot,\cdot])$ is a *Lie algebra.* A basis $\{T_a\}$ of the vector space is spanned by the *generators* $T_a$ of the Lie algebra. Evaluating the product on the generators gives rise to the *structure constants* ${f_{ab}}^c$ via

$$[T_a, T_b] = {f_{ab}}^c\, T_c . \tag{2.4}$$

**a)** Consider a three-dimensional vector space $V$ with basis elements $T_1$, $T_2$, and $T_3$. Define the bilinear product as follows:

$$\begin{aligned} [T_1, T_2] &= -[T_2, T_1] = \mathrm{i}T_2 , \\ [T_2, T_3] &= -[T_3, T_2] = \mathrm{i}T_3 , \\ [T_3, T_1] &= -[T_1, T_3] = \mathrm{i}T_1 . \end{aligned} \tag{2.5}$$

Does this define a Lie algebra?

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**b)** What relations do the antisymmetry and the Jacobi identity imply for the structure constants ${f_{ab}}^c$ of a general Lie algebra?

**c)** The exponential map allows to map a Lie algebra element $X$ into a group element $\exp(\mathrm{i}X)$ of the corresponding Lie group. Show that

$$\det(\exp(X)) = \exp(\mathrm{tr}(X)) \tag{2.6}$$

for any matrix $X \in \mathrm{Mat}(n, \mathbb{C})$, where $\mathrm{tr}(X)$ is the trace of $X$.
*Hint:* Proceed in three steps: Diagonal $X$, diagonalizable $X$, and generic $X$.

**d)** Determine the properties of the Lie algebra elements for the Lie groups

$$\begin{aligned} \mathrm{SU}(n) &= \left\{ U \in \mathrm{Mat}(n, \mathbb{C}) \,|\, UU^\dagger = U^\dagger U = 1, \det(U) = 1 \right\}, \\ \mathrm{SO}(n) &= \left\{ O \in \mathrm{Mat}(n, \mathbb{R}) \,|\, OO^\mathsf{T} = O^\mathsf{T} O = 1, \det(O) = 1 \right\}, \end{aligned} \tag{2.7}$$

by using the relation (2.6) and the expansion $\exp X \approx 1 + X + \ldots$.

## 2.3. Non-Abelian Gauge Transformations

Consider a scalar field $\phi$ and a gauge field $A_\mu$ subject to the (non-Abelian) gauge transformations

$$\phi \mapsto \phi^U := U\phi\,, \qquad A_\mu \mapsto A_\mu^U := UA_\mu U^{-1} - \frac{\mathrm{i}}{g}\,(\partial_\mu U)U^{-1}\,. \tag{2.8}$$

Here, $U = U(\vec{x}, t)$ is a smooth function on spacetime that takes values in a matrix representation $R(G)$ of a Lie group $G$. The field $\phi$ transforms as a vector in this representation. The gauge field $A^\mu$ takes values in the Lie algebra associated to $G$, and therefore can be expanded in the generators of $R(G)$.

**a)** Show that the transformations (2.8) form a group.

**b)** Verify that the covariant derivative $D_\mu := (\partial_\mu - \mathrm{i}gA_\mu)$, with $D_\mu^U := (\partial_\mu - \mathrm{i}gA_\mu^U)$, indeed transforms covariantly under (2.8), that is $(D_\mu\phi)^U = U(D_\mu\phi)$, or, in other words, $D_\mu^U = UD_\mu U^{-1}$.

**c)** The field strenght $F_{\mu\nu}$ can be defined by

$$F_{\mu\nu}\phi = \frac{\mathrm{i}}{g}\,[D_\mu, D_\nu]\phi \qquad \text{for all } \phi\,. \tag{2.9}$$

Find the explicit form of $F_{\mu\nu}$ in terms of $\partial_\mu$ and $A_\mu$. Defining $F_{\mu\nu}^U$ in terms of $D_\mu^U$, derive the transformation behavior of $F_{\mu\nu}$ under (2.8).

**d)** Derive the infinitesimal version of (2.8) by writing

$$U = \mathrm{e}^{\mathrm{i}X}\,, \qquad X = \sum_a \omega_a T^a\,, \tag{2.10}$$

where $\omega_a$ are smooth functions of spacetime called *gauge parameters*, and $T^a$ are representation matrices for the generators of the Lie group of $G$.

**e)** *Bonus problem:*
Rewrite the infinitesimal gauge transformation of $A_\mu$ by using the covariant derivative of the adjoint representation.

## 3.1. Dirac's Ansatz

Dirac made the following ansatz for an equation describing the wave function $\psi$ of a relativistic particle:

$$D\psi = 0\,, \qquad D = -\mathrm{i}\left(\frac{\partial}{\partial t} + \alpha_i\,\frac{\partial}{\partial x^i}\right) + \beta m\,. \tag{3.1}$$

**a)** Show that requiring consistency of equation (3.1) with the relativistic energy condition $(\partial_t^2 - \vec{\nabla}^2 + m^2)\psi = 0$ is equivalent to the conditions

$$\alpha_i^2 = 1\,, \qquad \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad (i \neq j)\,, \qquad \alpha_i\beta + \beta\alpha_i = 0\,, \qquad \beta^2 = 1\,. \tag{3.2}$$

**b)** Set $\gamma^i = \beta\alpha^i$, and $\gamma^0 = \beta$. Show that (3.2) is equivalent to the Clifford algebra relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\,. \tag{3.3}$$

Solutions $\psi$ to the Dirac equation are four-component spinors. Under Lorentz rotations, they transform as $\psi \mapsto \Lambda_S\psi$, $\Lambda_S = \exp(\mathrm{i}\theta_{\mu\nu}S^{\mu\nu})$, with generators $S^{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$, where the matrices $\gamma^\mu$ satisfy (3.3). Assume that $\gamma^{0\dagger} = \gamma^0$, and $\gamma^{i\dagger} = -\gamma^i$.

**c)** Show that $\bar{\psi}\psi$ is a Lorentz scalar, where $\bar{\psi} = \psi^\dagger\gamma^0$.
*Hint:* Show first $\gamma^0\gamma^\mu\gamma^0 = \gamma^{\mu\dagger}$, then $\gamma^0\Lambda_S^\dagger\gamma^0 = \Lambda_S^{-1}$.

## 3.2. Plane Wave Solutions

We will investigate plane-wave solutions of the Dirac equation:

$$\begin{aligned} \psi_{\mathrm{u}} &= u(p)\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\,, \qquad & p^\mu &= (E;\vec{p})\,, \qquad & E &= +\sqrt{m^2+\vec{p}^{\,2}}\,, \\ \psi_{\mathrm{v}} &= v(p)\mathrm{e}^{+\mathrm{i}p_\mu x^\mu}\,, \qquad & p^\mu &= (E;-\vec{p})\,, \qquad & E &= +\sqrt{m^2+\vec{p}^{\,2}}\,. \end{aligned} \tag{3.4}$$

**a)** Show that the Dirac equation for the spinors $u(p)$ (particles) and $v(p)$ (antiparticles) becomes

$$(\gamma^\mu p_\mu - m)u(p) = 0\,, \qquad (\gamma^\mu p_\mu + m)v(p) = 0\,. \tag{3.5}$$

For the gamma matrices $\gamma^\mu$, we use the Dirac representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\,, \tag{3.6}$$

where $\mathbf{1}$ is the $2 \times 2$ identity matrix, and $\sigma_i$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.7}$$

**b)** Show that the Pauli matrices satisfy $[\sigma_i, \sigma_j] = 2\mathrm{i}\varepsilon_{ijk}\sigma_k$, and are therefore generators of the Lie algebra $\mathfrak{su}(2)$. Show also that $\sigma_i\sigma_j = \mathrm{i}\varepsilon_{ijk}\sigma_k + \delta_{ij}\mathbf{1}$.

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**c)** Show that $(\vec{\sigma}\cdot\vec{p})\cdot(\vec{\sigma}\cdot\vec{p}) = \vec{p}^{\,2}\,\mathbf{1}$.

**d)** Verify that the Dirac equation is solved by the following spinors:

$$u(p) = \sqrt{E+m}\begin{pmatrix}\xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,\xi\end{pmatrix}, \qquad v(p) = \sqrt{E+m}\begin{pmatrix}\frac{-\vec{\sigma}\cdot\vec{p}}{E+m}\,\chi \\ \chi\end{pmatrix}, \tag{3.8}$$

where $\xi$ and $\chi$ are (at this point arbitrary) constant two-component spinors.

**e)** For $\vec{p} = (0,0,p)$ with $p > 0$, and

$$\xi_1 = \chi_2 = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_2 = \chi_1 = \begin{pmatrix}0\\1\end{pmatrix}, \quad u_i(p) \equiv u(p)\big|_{\xi=\xi_i}, \quad v_i(p) \equiv v(p)\big|_{\chi=\chi_i}, \tag{3.9}$$

verify that

$$u_{1,2} = \begin{pmatrix}\sqrt{E+m}\,\xi_{1,2} \\ \pm\sqrt{E-m}\,\xi_{1,2}\end{pmatrix}, \qquad v_{1,2} = \begin{pmatrix}\pm\sqrt{E-m}\,\chi_{1,2} \\ \sqrt{E+m}\,\chi_{1,2}\end{pmatrix}. \tag{3.10}$$

**f)** The spin operator for Dirac spinors is

$$\vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}. \tag{3.11}$$

Compute the eigenvalues of $S_3 = S_z$ for the spinors (3.10). What is the physical interpretation of the four states?

**g)** Find expressions for the spinors (3.8) in the two limits *(i)* $E = m$ (particle at rest), and *(ii)* $E \gg m$ (relativistic particle). How does the expression in case *(ii)* simplify when $\xi$ and $\chi$ are eigenvectors of $\vec{p}\cdot\vec{\sigma}$?

The four independent solutions of the Dirac equation can be interpreted as different spin states of the electron ($u$) and its antiparticle, the positron ($v$). So far, we have seen this for the special case $\vec{p} = (0,0,p)$. For general $\vec{p}$, we will see in the following that not the spin projection $S_3$, but the *helicity* is a "good" quantum number for solutions to the Dirac equation.

**h)** Show that the Hamiltonian for the states (3.4) takes the form

$$H = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\end{pmatrix}. \tag{3.12}$$

**i)** Show that in general $H$ does not commute with the components of the spin operator (3.11).

**j)** Show that $H$ *does* commute with the helicity operator $h$, defined (for non-zero $\vec{p}$) as

$$h = \frac{2\,\vec{p}\cdot\vec{S}}{|\vec{p}|}\,. \tag{3.13}$$

In the following exercises, we explore three steps towards the so-called Higgs mechanism, which generates mass terms in the Standard Model. First, we study spontaneous symmetry breaking for a discrete symmetry. Secondly, we consider a continuous (global) symmetry and find an example of the Goldstone theorem. Finally, we consider the simplest example of the Abelian Higgs effect.

## 4.1. Spontaneous Symmetry Breaking I: Discrete Symmetry

Consider the following Lagrangian for a real scalar field $\phi$:

$$\mathcal{L} = \mathcal{T} - \mathcal{V}\,, \qquad \mathcal{T} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi\,, \qquad \mathcal{V} = \frac{1}{2}\,\mu^2\phi^2 + \frac{1}{4}\,\lambda\phi^4\,, \tag{4.1}$$

where $\lambda > 0$, but $\mu^2$ may have either sign. $\mathcal{L}$ is invariant under the $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$.

**a)** Assuming a constant field $\phi(x) \equiv \phi$, find the value $\phi = v$ that minimizes the total energy density $\mathcal{T} + \mathcal{V}$. Sketch the potential $\mathcal{V}$ for $\mu^2 > 0$ and for $\mu^2 < 0$.

**b)** Expand the theory around the minima (i) $v = 0$ for $\mu^2 > 0$ and (ii) $v = \sqrt{-\mu^2/\lambda}$ for $\mu^2 < 0$ by setting $\phi(x) = v + \eta(x)$ in $\mathcal{L}$ and expanding in powers of $\eta(x)$.

**c)** Is the original $\mathbb{Z}_2$ symmetry visible in the re-written $\mathcal{L}$? What are the masses of the scalar field $\eta(x)$ in both cases? To identify the mass term, recall the Lagrangian for a massive real scalar field.

We learn that the choice of one of the two equivalent vacua $v = \pm\sqrt{-\mu^2/\lambda}$ for $\mu^2 < 0$ breaks the original $\mathbb{Z}_2$ symmetry. This means that the vacua do not have the symmetry of the original Lagrangian, which is called *spontaneous symmetry breaking*.

## 4.2. Spontaneous Symmetry Breaking II: Goldstone Theorem

For the complex scalar field $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$, we consider the Lagrangian

$$\mathcal{L} = \mathcal{T} - \mathcal{V}\,, \qquad \mathcal{T} = \partial_\mu\phi^*\,\partial^\mu\phi\,, \qquad \mathcal{V} = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2\,, \tag{4.2}$$

which has a continuous global $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ symmetry under $\phi(x) \mapsto \exp(\mathrm{i}\chi)\phi(x)$.

**a)** Find the minima of the total energy density $\mathcal{T} + \mathcal{V}$ for a constant scalar field, for $\mu^2 > 0$ and for $\mu^2 < 0$, and sketch the potential in the $(\phi_1, \phi_2)$ plane.

**b)** Assume that $\mu^2 < 0$. Now, there is an entire circle of equivalent vacua. We pick the vacuum point $\phi_1 = v$, $\phi_2 = 0$, with $v^2 = -\mu^2/\lambda$. To expand around this vacuum, set

$$\phi(x) = \frac{v + \eta(x) + \mathrm{i}\rho(x)}{\sqrt{2}}\,, \tag{4.3}$$

with $\rho$ and $\eta$ real, and expand around $\eta = 0$ and $\rho = 0$. What are the masses of the two real scalar fields $\eta(x)$ and $\rho(x)$?

You should see in this example that spontaneous breaking of the continuous global U(1) symmetry leads to a massless scalar field, called the *Goldstone boson*. Intuitively, the massless particle corresponds to the flat direction of the potential in the vicinity of the vacuum. The massive scalar field describes excitations in the radial (non-flat) direction.

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## 4.3. Spontaneous Symmetry Breaking III: Abelian Higgs Effect

Now, we promote the global U(1) symmetry of the previous problem to a local symmetry (gauge symmetry). This is done in three steps: (i) introduce local gauge transformations

$$\phi(x) \mapsto \exp(\mathrm{i}\chi(x))\phi(x)\,, \tag{4.4}$$

(ii) replace the partial derivative by the gauge covariant derivative:

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}gA_\mu\,, \tag{4.5}$$

and (iii) introduce the kinetic term $F_{\mu\nu}F^{\mu\nu}$ for the Abelian gauge field $A_\mu$. The resulting Lagrangian reads

$$\mathcal{L} = \mathcal{T} - \mathcal{V}\,, \qquad \mathcal{T} = (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \qquad \mathcal{V} = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2\,. \tag{4.6}$$

This situation is a bit more involved, but we can employ the insights gained in the previous problem. The minima of the scalar potential remain the same, for $\mu^2 < 0$ they are given by $|\phi|^2 = v^2 = -\mu^2/\lambda$, and the complex scalar $\phi$ is re-written as in (4.3). With a suitable gauge transformation (4.4), $\phi$ can always be made real, hence we can expand it as

$$\phi(x) = \frac{v + h(x)}{\sqrt{2}}\,, \tag{4.7}$$

with a real field $h(x)$.

**a)** Insert (4.7) into the Lagrangian, and sort the terms corresponding to kinetic terms, mass terms, and interaction terms.

**b)** What do you observe for the scalar field $h$ and for the gauge field $A$? What are their masses? Can one infer the mass of $h$ from the mass of $A_\mu$?

**c)** Since we broke a continuous symmetry, one may ask the following: What happened to the degree of freedom that previously was the massless Goldstone boson? Compare the degrees of freedom before and after the symmetry breaking.

In conclusion, breaking a *local* symmetry evades the Goldstone theorem, i. e. there is no massless scalar field. In addition, the gauge field has acquired a mass term due to the symmetry breaking. Note that a mass term for gauge fields is not gauge invariant, but can still be introduced in a gauge theory by spontaneous symmetry breaking. This phenomenon is the celebrated *Higgs effect*.

## 5.1. Proton Wave Function

In this problem, we consider bound states consisting of up quarks $u$ and down quarks $d$, held together by strong interactions. Because the strong interactions act identically on all quark flavors, and because the two light quarks have similar masses, $m_\mathrm{u} \approx m_\mathrm{d}$, the strong interactions possess an approximate *flavor symmetry* under exchanging $u \leftrightarrow d$. This symmetry is expressed as an invariance under SU(2) *isospin* transformations

$$\begin{pmatrix} u \\ d \end{pmatrix} \mapsto \begin{pmatrix} u' \\ d' \end{pmatrix} = U \begin{pmatrix} u \\ d \end{pmatrix}, \qquad U \in \mathrm{SU}(2)\,, \tag{5.1}$$

where $U$ is a $2 \times 2$ special unitary matrix.[1] The properties of SU(2) transformations are familiar from the theory of spin. We know that the group SU(2) has three generators $T_i = \tau_i/2$, where $\tau_i$ are the three Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.2}$$

such that $U = \exp(\mathrm{i} \sum_i \alpha_i T_i)$ for all $U \in \mathrm{SU}(2)$. The generators satisfy $[T_i, T_j] = \mathrm{i}\varepsilon_{ijk} T_k$. From quantum mechanics, we know that one can find simultaneous eigenstates of the total isospin $T^2$ and the third isospin component $T_3$. Eigenbasis states $|I, I_3\rangle$ are labeled by their eigenvalues under these two operators, such that

$$T^2|I, I_3\rangle = I(I+1)|I, I_3\rangle\,, \qquad T_3|I, I_3\rangle = I_3|I, I_3\rangle\,. \tag{5.3}$$

The two light quarks are the two eigenbasis states $u = |\frac{1}{2}, +\frac{1}{2}\rangle$ and $d = |\frac{1}{2}, -\frac{1}{2}\rangle$. The action of the isospin ladder operators $T_\pm = T_1 \pm \mathrm{i}T_2$ on any isospin eigenstate are given by

$$T_\pm|I, I_3\rangle = \sqrt{I(I+1) - I_3(I_3 \pm 1)}\,|I, I_3 \pm 1\rangle\,, \tag{5.4}$$

such that

$$T_+ u = 0\,, \qquad T_+ d = u\,, \qquad T_- u = d\,, \qquad T_- d = 0\,. \tag{5.5}$$

We want to derive a wave function for a three-quark state. We first study the combined system of two quarks. Then, we will add the third quark.

**a)** The isospin doublet $q = (u, d)^\mathsf{T}$ representation is denoted by $\mathbf{2}$. Decompose the product representation $\mathbf{2} \otimes \mathbf{2}$ into irreducible isospin representations. Mathematically, this is the same as decomposing a state of two spin 1/2 particles into multiplets of $T^2$ and $T_3$ eigenstates. Determine all appearing states and express them in terms of product states $|q_1 q_2\rangle := |q_1\rangle \otimes |q_2\rangle$ with $q_i \in \{u, d\}$. What are the eigenvalues of $T^2$ in the appearing multiplets?

*Hint:* Use the ladder operators $T_\pm$ and the orthogonality of irreducible representations in the product decomposition.[2]

[1] These isospin transformations are not to be confused with *weak isospin* transformations, which only act on left-handed states. Rather, the flavor symmetry becomes the strong isospin symmetry at the level of nucleons (protons and neutrons).

[2] If you are not familiar with SU(2) tensor product decompositions, see for example Chapter VI.3.3 in the lecture notes by N. Borghini at `https://www.physik.uni-bielefeld.de/~borghini/Teaching/Symmetries/Symmetries.pdf`.

**b)** Now, we add the third quark by taking the tensor product of another doublet $\mathbf{2}$ with the direct sum of the triplet $\mathbf{3}$ and the singlet $\mathbf{1}$ found in a). Thus, decompose $\mathbf{3} \otimes \mathbf{2}$ and $\mathbf{1} \otimes \mathbf{2}$ into irreducible isospin multiplets, and express the states in triple products $|q_1 q_2 q_3\rangle := |q_1\rangle \otimes |q_2\rangle \otimes |q_3\rangle$ of $q_i \in \{u, d\}$.

**c)** Investigate the symmetry properties of the states in the isospin quadruplet $\mathbf{4}$ and the two isospin doublets $\mathbf{2}$ under pairwise exchanges $q_i \leftrightarrow q_j$ of the three quark constituents. Denote the two doublets by $\mathbf{2}_\mathrm{S}$ (for symmetric) and $\mathbf{2}_\mathrm{A}$ (for anti-symmetric).

The total wave function of a multi-quark state factorizes as $\psi = \phi_\text{flavor}\, \chi_\text{spin}\, \xi_\text{color}\, \eta_\text{space}$. It is known that the color wave function $\xi_\text{color}$ for all bound states $qqq$ is anti-symmetric under exchange of any two quarks. We furthermore restrict to vanishing angular momentum, such that $\eta_\text{space}$ is symmetric under pairwise quark exchange. The total wave function $\psi$ has to be anti-symmetric under exchange of any two of its constituents.

**d)** What symmetry property must $\phi_\text{flavor}\, \chi_\text{spin}$ obey?

**e)** We have already determined the possible flavor wave functions $\phi_\text{flavor}$. What are the possible spin wave functions $\chi_\text{spin}$ for a three-quark state made of $u$ and $d$ quarks? Recall that isospin and spin have the same mathematical structure.

**f)** Knowing the symmetry properties of the isospin and spin multiplets from c), what are the possibilities to obtain a valid wave function $\phi_\text{flavor}\, \chi_\text{spin}$? Ignore for the moment the required symmetry under $q_1 \leftrightarrow q_3$ and $q_2 \leftrightarrow q_3$ exchange.

**g)** For the proton wave function, both $\phi_\text{flavor}$ and $\chi_\text{spin}$ are built from the mixed-symmetry doublets $\mathbf{2}_\mathrm{S}$ and $\mathbf{2}_\mathrm{A}$. Consider the combinations $(\phi\chi)_\mathrm{S} := \phi(\mathbf{2}_\mathrm{S})\chi(\mathbf{2}_\mathrm{S})$ and $(\phi\chi)_\mathrm{A} := \phi(\mathbf{2}_\mathrm{A})\chi(\mathbf{2}_\mathrm{A})$. How do $(\phi\chi)_\mathrm{S}$ and $(\phi\chi)_\mathrm{A}$ transform under quark exchanges $q_1 \leftrightarrow q_2$ and $q_2 \leftrightarrow q_3$? Find a combination of $(\phi\chi)_\mathrm{S}$ and $(\phi\chi)_\mathrm{A}$ that is totally symmetric under exchanges $q_i \leftrightarrow q_j$, $i, j \in \{1, 2, 3\}$. The proton wave function is given by the $I_3 = +1/2$ component, the $I_3 = -1/2$ component is identified as the neutron wave function.

**Remark:** The combination $\phi(\mathbf{4})\chi(\mathbf{4})$ gives the isospin 3/2 quadruplet of $\Delta$ resonances.

## 5.2. Light Mesons

The light anti-quarks $\bar{u}$ and $\bar{d}$ transform in the anti-fundamental representation $\bar{\mathbf{2}}$ of flavor isospin SU(2). Denoting $\bar{q} = (-\bar{d}, \bar{u})^\mathsf{T}$, their transformation rule

$$\bar{q} \mapsto \bar{q}' = U\bar{q}\,, \qquad U \in \mathrm{SU}(2) \tag{5.6}$$

is the same as for the fundamental quarks $q = (u, d)^\mathsf{T}$.

**a)** Compute the isospin eigenvalues $I_3$ and the action of the ladder operators $T_\pm$ (5.4) for the $\bar{u}$ and $\bar{d}$ states. Compare the result to the case of quarks from Problem 5.1.

**b)** Decompose the tensor product $\mathbf{2} \otimes \bar{\mathbf{2}}$, and express the normalized states in terms of product states $|q\bar{q}\rangle := |q\rangle \otimes |\bar{q}\rangle$. You should find an isospin triplet $\mathbf{3}$ (these are the pions $\pi^\pm$, $\pi^0$) and a singlet $\mathbf{1}$ (this is the $\eta$ meson).

**Remark:** The next-lightest quark is the strange quark $s$. The triplet $(u, d, s)$ enjoys an approximate SU(3) flavor symmetry. Extending the above considerations to this case, many of the hadrons (multi-quark states) found in the 1950s and 1960s can be identified as components of SU(3) flavor multiplets (singlets/triplets/octets/decuplets) as part of the "eightfold way".

## 6.1. Two-Particle Decay: Fermi's Golden Rule

For the decay $1 \to 2 + 3$, where particle 1 is assumed to be at rest, the decay rate is given by

$$\Gamma = \frac{S}{32\pi^2 m_1} \int |\mathcal{M}|^2 \frac{\delta(p_1 - p_2 - p_3)}{\sqrt{m_2^2 + \vec{p}_2^2}\sqrt{m_3^2 + \vec{p}_3^2}} \, \mathrm{d}^3\vec{p}_2 \, \mathrm{d}^3\vec{p}_3 \; . \tag{6.1}$$

Here, $m_i$ is the mass of the $i$'th particle, and $p_i = (E_i, \vec{p}_i)$ its four-momentum. $S$ is a symmetry factor that corrects double-counting for identical particles: $S = 1/2!$ if particles 2 and 3 are identical. The dynamics of the decay process is contained in the amplitude $\mathcal{M} = \mathcal{M}(p_1, p_2, p_3)$, which we assume to be averaged over the spin degrees of freedom of the three particles.

**a)** Verify that formula (6.1) is correct, based on the equations given in the lecture.

**b)** Split the four-dimensional delta function into the temporal and the three-dimensional spatial delta function. Perform the integral over $\vec{p}_3$, using that particle 1 is at rest.

**c)** The amplitude $\mathcal{M}$ depends on all three four-momenta, subject to the delta-function constraint. However, in the rest frame of particle 1, $p_1$ is fixed. Because of the delta function, $\mathcal{M}$ thus only depends on $\vec{p}_2$. Moreover, $\mathcal{M}$ must be a Lorentz scalar, and therefore invariant under rotations. It hence only depends on the magnitude $\vec{p}_2^2$.

For the remaining integral over $\vec{p}_2$, change to spherical coordinates $(r, \theta, \phi)$ and perform the angular integrations over $\theta$ and $\phi$.

**d)** Simplify the remaining integration over $r$ by the substitution

$$u = \sqrt{m_2^2 + r^2} + \sqrt{m_3^2 + r^2} \; . \tag{6.2}$$

Evaluate the final integral with the help of the remaining delta function, and verify that

$$\Gamma = \frac{S|\vec{p}_2|}{8\pi {m_1}^2} \, |\mathcal{M}(\vec{p}_2^2)|^2 \, , \tag{6.3}$$

where the magnitude of $\vec{p}_2$ takes the particular value

$$|\vec{p}_2| = \frac{1}{2m_1} \sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2} \tag{6.4}$$

determined from the conservation laws.

Without knowing the amplitude $\mathcal{M}$, we could evaluate all integrals for the two-body decay, only using the conservation of energy and momentum. Formula (6.3) is called *Fermi's Golden Rule* for two-body decay.

## 6.2. Z-Boson Decay Width

The Standard Model Lagrangian contains an interaction vertex between the Z-boson $Z_\mu$ and a fermion anti-fermion pair $f\bar{f}$. The vertex can be written as

$$\frac{g_2}{2\cos\theta_{\mathrm{W}}}\,\gamma^\mu\Big(c_{\mathrm{V}f}-c_{\mathrm{A}f}\gamma^5\Big)\,. \tag{6.5}$$

Here, $c_{\mathrm{V}}$ and $c_{\mathrm{A}}$ denote the vector and axial-vector couplings of the fermion to the Z-boson. They are given by

$$c_{\mathrm{V}f}=T_{3f}-2Q_f\sin^2\theta_{\mathrm{W}}\,,\qquad c_{\mathrm{A}f}=T_{3f}\,, \tag{6.6}$$

where $T_{3f}$ denotes the third component of the weak isospin, and $Q_f$ the electric charge of the respective fermion $f$. The amplitude $\mathcal{M}_{Z\to f\bar{f}}$ is computed by *(i)* multiplying the vertex (6.5) with the polarization vector $Z_\mu$ of the Z-boson, the fermion wavefunction $f$ (from the right), and the anti-fermion wavefunction (from the left), and *(ii)* averaging over the spin and color degrees of freedom of the particles. In the approximation that $m_f \ll m_{\mathrm{Z}}$, where $m_{\mathrm{Z}}$ is the Z-boson mass, one obtains

$$|\mathcal{M}_{Z\to f\bar{f}}|^2=\frac{N_{\mathrm{c}}}{3}\,\frac{g_2^2}{\cos^2\theta_{\mathrm{W}}}\,m_{\mathrm{Z}}^2\Big(c_{\mathrm{V}f}^2+c_{\mathrm{A}f}^2\Big)\,, \tag{6.7}$$

where $N_{\mathrm{c}}$ is the number of different color states the fermion $f$ can occupy.

**a)** Verify that formula (6.5) is correct, based on the equations given in the lecture.

**b)** What are the possible fermion anti-fermion pairs that a Z-boson can decay to? You should identify four distinct cases for each family of quarks and leptons.

**c)** Using (6.3) in the limit $m_f \ll m_{\mathrm{Z}}$ as well as (6.7), verify that the decay width $\Gamma_{f\bar{f}}$ of the Z-boson into a fermion anti-fermion pair $f\bar{f}$ is given by

$$\Gamma_{f\bar{f}}=\frac{N_{\mathrm{c}}}{48\pi}\,\frac{g_2^2}{\cos^2\theta_{\mathrm{W}}}\,m_{\mathrm{Z}}\Big(c_{\mathrm{V}f}^2+c_{\mathrm{A}f}^2\Big)\,. \tag{6.8}$$

**d)** Evaluate the couplings $c_{\mathrm{V}f}$ and $c_{\mathrm{A}f}$ for the four different cases identified in b), and write the explicit formulas for the four different $\Gamma_{f\bar{f}}$.

**e)** Compute the numerical values for the partial widths $\Gamma_{f\bar{f}}$, as well as the total Z-boson decay width

$$\Gamma_{\mathrm{tot}}=\sum_f \Gamma_{f\bar{f}}\,, \tag{6.9}$$

using $\sin^2\theta_{\mathrm{W}}=0.23$, $g_2=e/\sin\theta_{\mathrm{W}}$ (or $g_2^2\approx 4\pi/30$), and $m_{\mathrm{Z}}=91.19\,\mathrm{GeV}$. Which of the six quarks and six leptons have to be included in the sum?

To compare the numerical results to measurements, one can look up all available experimental data at `https://pdg.lbl.gov` (the Particle Data Group).