Classical separation of variables I

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Abstract

These are the notes to a lecture given on October 24, 2024 at the ZMP seminar at the university of Hamburg on the topic of Liouville integrability and Lax connections applied to the example of the closed Toda chain.

1 Reminder: classical mechanics

Definition 1.1. (i) Let (M, ω) be a symplectic manifold with symplectic (closed, non-degenerate) 2-form ω. Any $f \in C^\infty(M)$ defines $df \in \Omega^1(M)$, which by non-degeneracy of ω defines $X_f \in \Gamma(TM)$. The *Poisson bracket* is

$$\{f, g\} := \omega(X_f, X_g).$$

This is a Lie bracket and a derivation in both arguments, making $C^\infty(M)$ into a Poisson algebra.

(ii) Suppose $\omega = \sum_i dp_i \wedge dq_i$. Then

$$df(Y) = \omega(Y, X_f) = \sum_i (X_f(q_i)Y(p_i) - X_f(p_i)Y(q_i)),$$

so $X_f(p_i) = -\frac{\partial f}{\partial q_i}$ and $X_f(q_i) = \frac{\partial f}{\partial p_i}$ ($\rightarrow$ Hamilton equations). It follows that

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right), \quad \{q_i, q_j\} = \{p_i, p_j\} = 0, \{q_i, p_j\} = \delta_{ij}.$$

(iii) A *classical system* is a symplectic manifold (M, ω) together with a choice of Hamiltonian $H \in C^\infty(M)$. The time evolution is generated by the vector field X_H and observables $f \in C^\infty(M)$ satisfy

$$\dot{f} = X_H(f) = df(X_h) = \omega(X_H, X_f) = \{H, f\}.$$

(iv) Observables $h \in C^\infty(M)$ that Poisson-commute with the Hamiltonian $\{H, h\} = 0$ are *conserved quantities.*

Remark. Generically, classical systems are impossible to solve exactly. However if we have a conserved quantity h, say with constant value c, then the system never leaves $h^{-1}(c)$, which generically has codimension one. **Conclusion**: Gathering conserved quantities cuts down on the effective dimension of the phase space, simplifying the problem.

2 Liouville-Arnold theorem

Theorem 2.1 (Liouville-Arnold)**.** *Let (M,ω) be a symplectic manifold of dimension $2N$. Suppose there exists $h=(h_1,...,h_N): M\to\mathbb{R}^N$ with $\{h_i,h_j\}=0$ and dh_i linearly independent. Let $c\in\mathbb{R}^N$.*

(i) If the level set $h^{-1}(c)$ is connected then $h^{-1}(c)\cong\mathbb{R}^{N-K}\times(S^1)^K$ for some K. If we choose a Hamiltonian $H=f(h_1,...,h_N)$ and let $\theta_1,...,\theta_N$ be the standard coordinates of $\mathbb{R}^{N-K}\times(S^1)^K$, then the EOMs reduce to

$$\dot\theta_i=\omega_i(c),$$

where $\omega_i(c)$ depend only on c and we call (M,ω,H) Liouville integrable.

(ii) Suppose $\omega=d\alpha,\alpha=p_i\wedge dq_i$. Then there exists a canonical transformation $(p_i,q_i)\to(h_i,\theta_i)$. In particular, choosing a Hamiltonian $H=f(h_1,...,h_N)$, the EOMs reduce to

$$\dot h_i=\{H,h_i\}=0,\quad \dot\theta_i=\frac{\partial H}{\partial h_i}.$$

Proof. (i) By the Frobenius theorem, $h^{-1}(c)$ is a smooth submanifold. Further, if the flows are complete, we have an action of the abelian Lie group $\mathbb{R}^N$ on $h^{-1}(c)$:

$$\mathbb{R}^N\times h^{-1}(c)\to h^{-1}(c),\quad (t_1,...,t_N,x)\mapsto(\Phi_{t_1}^{X_{h_1}}\circ\cdots\circ\Phi_{t_N}^{X_{h_N}})(p)$$

This action is locally free by linear independence of dh_i. For dimension reasons, it follows that the action is transitive with discrete stabilizer (picture). The discrete subgroups of $\mathbb{R}^N$ are lattices of the form $\mathbb{Z}^K$. It follows that $h^{-1}(c)\cong\mathbb{R}^N/\mathbb{Z}^K\cong\mathbb{R}^{N-K}\times(S^1)^K$.

(ii) To exhibit the canonical transformation, we define the generating function $S(x):=\int_{\gamma_x}\alpha$ where γ_x is a curve in $h^{-1}(c)$ going from a fixed point x_0 to x. Assuming that $q_1,...,q_N$ parametrize $h^{-1}(c)$, we find

$$\frac{\partial S}{\partial q_i}=\sum_j\int_{q_0}^q\frac{\partial p_j}{\partial q_i}dq_j=p_i$$

Defining $\theta_i:=\frac{\partial S}{\partial h_i}$ gives

$$dS=\sum_i\left(\frac{\partial S}{\partial q_i}dq_i+\frac{\partial S}{\partial h_i}dh_i\right)=\alpha+\sum_i\theta_i dh_i.$$

It follows that $\omega=d\alpha=d(\alpha-dS)=\sum_i dh_i\wedge d\theta_i$, so S defines a canonical transformation. It remains to show that $S(x)$ is independent under homotopy of γ_x (generally has monodromy). By Stokes' theorem (picture), it needs to be shown that $d\alpha|_{Th^{-1}(c)}=\omega|_{Th^{-1}(c)}=0$. But this is true because the tangent space of $h^{-1}(c)$ is generated by X_{h_i} and $\omega(X_{h_i},X_{h_j})=\{h_i,h_j\}=0$. □

Definition 2.2. Let (M,ω) be a Liouville integrable system with compact connected level sets, canonical 1-form $\alpha=\sum_i p_i dq_i$ and generating function $S(p)=\int_{\gamma_p}\alpha$. The level sets have the form $(S^1)^N$, with fundamental cycles $\gamma_1,...,\gamma_N$. Define the *action-angle variables*

$$I_i:=\frac{1}{2\pi}\oint_{\gamma_i}\alpha,\quad \theta_i:=\frac{\partial S}{\partial I_i}.$$

Proposition 2.3. *The angle variables satisfy*

$$\oint_{\gamma_i}d\theta_j=2\pi\delta_{ij}.$$

Proof.

$$\oint_{\gamma_i} d\theta_j = \frac{\partial}{\partial I_j}\oint_{\gamma_i} dS = \frac{\partial}{\partial I_j}\oint_{\gamma_i}\sum_k \left(\frac{\partial S}{\partial q_k}dq_k + \frac{\partial S}{\partial I_k}dI_k\right) = \frac{\partial}{\partial I_j}\oint_{\gamma_i}\alpha = 2\pi\delta_{ij},$$

where we have used that $dI_k|_{Th^{-1}(c)} = 0$. □

Example. (i) The easiest example is the harmonic oscillator with phase space $(\mathbb{R}^2, dp\wedge dq)$ and Hamiltonian $H = \frac{1}{2}(p^2+q^2)$. At a fixed energy $H = \frac{a^2}{2}$, the EOMs reduce to the circular orbit $\dot\theta = 1$ with $p = a\cos\theta, q = a\sin\theta$. Here θ parametrizes a circle → **confinement**.

(ii) Another example is the Hamiltonian $H = \frac{1}{2}(p^2+q^{-2})$. At fixed energy $H = \frac{a^2}{2}$, the EOMs reduce to $\dot\theta = 1$ with $p = \pm a^2\theta(a^2\theta^2+a^{-2})^{-1/2}, q = \pm(a^2\theta^2+a^{-2})^{1/2}$. Here θ parametrizes a line → **scattering**.

(iii) These are the two archetypal integrable systems. A more complicated example is the closed Toda chain with Hamiltonian

$$H = \frac{1}{2}\sum_{i=0}^N p_i^2 + \sum_{i=0}^N e^{q_i - q_{i+1}}, \quad N+1\equiv 0.$$

→ **cosh-type confinement**.

3 Lax connections

While the Liouville-Arnold theorem essentially gives a complete classification of Liouville integrable systems, it does not give a procedure for finding conserved quantities. A common method of obtaining conserved quantities is from *Lax connections*.

Definition 3.1. A *Lax connection* on $\mathbb{R}\times S^1$ is a matrix-valued 1-form $A(\lambda)$ that is meromorphic in λ and flat away from its poles, i.e.

$$\partial_t A_x(\lambda) - \partial_x A_t(\lambda) = [A_t(\lambda), A_x(\lambda)].$$

Proposition 3.2. *Let γ be a curve in $\mathbb{R}\times S^1$ winding once around S^1, starting at (t_0,x_0) and ending at (t_1,x_1) with $x_0 = x_1 = 0$ and $t_0 = t_1$. Letting*

$$L(\lambda) := P\exp\oint_\gamma A(\lambda), \quad M(\lambda) := A_t|_{x=0}(\lambda),$$

Then $\dot L(\lambda) = M(\lambda)L(\lambda) - L(\lambda)M(\lambda)$.

Proof. The defining equations of the holonomy imply

$$\begin{aligned}\dot L(\lambda) &= \frac{\partial}{\partial t_0}P\exp\oint_\gamma A(\lambda) + \frac{\partial}{\partial t_1}P\exp\oint_\gamma A(\lambda)\\ &= -P\exp\oint_\gamma A(\lambda)A_t|_{x=0}(\lambda) + A_t|_{x=0}(\lambda)P\exp\oint_\gamma A(\lambda)\\ &= M(\lambda)L(\lambda) - L(\lambda)M(\lambda).\end{aligned}$$

□

Corollary 3.3. *Letting $T(\lambda) := \operatorname{tr} L(\lambda)$, we have $\dot T(\lambda) = 0$.*

Lemma 3.4. *Define a discrete Lax connection along the closed Toda chain with components*

$$L_i(\lambda) = \begin{pmatrix} 0 & e^{q_i} \\ -e^{-q_i} & \lambda - p_i \end{pmatrix}, \quad M_i(\lambda) = \begin{pmatrix} 0 & -e^{q_{i-1}} \\ e^{-q_i} & -\lambda \end{pmatrix}.$$

Then

$$\dot{L}_i(\lambda) = M_{i+1}(\lambda)L_i(\lambda) - L_i(\lambda)M_i(\lambda),$$

or

$$\begin{pmatrix} 0 & \dot{q}_i e^{q_i} \\ \dot{q}_i e^{-q_i} & -\dot{p}_i \end{pmatrix} = \begin{pmatrix} 0 & p_i e^{q_i} \\ p_i e^{-q_i} & e^{q_i - q_{i+1}} - e^{q_{i-1} - q_i} \end{pmatrix}$$

which is equivalent to the equations of motion.

Proposition 3.5. *Define* $L^{(N)}(\lambda) := L_N(\lambda) \cdots L_0(\lambda)$ *and* $T(\lambda) := \operatorname{tr} L^{(N)}(\lambda)$*. Then* $\dot{T}(\lambda) = 0$.

Proof.

$$\begin{aligned} \dot{T}(\lambda) &= \sum_{i=0}^{N} \operatorname{tr} L_N(\lambda) \cdots L_{i+1}(\lambda)\dot{L}_i(\lambda)L_{i-1}(\lambda) \cdots L_0(\lambda) \\ &= \sum_{i=0}^{N} \operatorname{tr} L_N(\lambda) \cdots L_{i+1}(\lambda)(M_{i+1}(\lambda)L_i(\lambda) - L_i(\lambda)M_i(\lambda))L_{i-1}(\lambda) \cdots L_0(\lambda) = 0. \end{aligned}$$

□

Remark. Writing $T(\lambda) = \sum_{i=0}^{N+1} h_i \lambda^i$, it follows that h_i are conserved quantities. In particular $h_{N+1} = 1$, $h_N = \sum_{i=0}^{N} p_i$, and $H = \frac{1}{2}h_N^2 - h_{N-1}$.

4 Poisson structure

Lax connections provide conserved quantities, but do not ensure that they Poisson-commute. Indeed, the definition of Lax connections never references a Poisson structure.

Proposition 4.1. *For the closed Toda chain, we have*

$$\{L_i(\lambda) \otimes 1, 1 \otimes L_i(\mu)\} = [r(\lambda - \mu), L_i(\lambda) \otimes L_i(\mu)]$$

with the classical r*-matrix* $r(\lambda) = -P/\lambda$*, where* $P = \frac{1}{2}(1 + \sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y + \sigma^z \otimes \sigma^z)$ *and* $\sigma^x, \sigma^y, \sigma^z$ *are the Pauli matrices.*

Proof. The LHS becomes

$$\begin{pmatrix} 0 & \{e^{q_i}, L_i(\mu)\} \\ \{-e^{-q_i}, L_i(\mu)\} & \{\lambda - p_i, L_i(\mu)\} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\{e^{q_i}, p_i\} \\ 0 & 0 & 0 & -\{p_i, e^{q_i}\} \\ 0 & \{e^{-q_i}, p_i\} & \{p_i, e^{-q_i}\} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{q_i} \\ 0 & 0 & 0 & e^{q_i} \\ 0 & -e^{-q_i} & e^{-q_i} & 0 \end{pmatrix}$$

while the RHS becomes

$$\frac{1}{\mu - \lambda} \left[\begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & e^{2q_i} \\ 0 & 0 & -1 & e^{q_i}(\mu - p_i) \\ 0 & -1 & 0 & (\lambda - p_i)e^{q_i} \\ e^{-2q_i} & -e^{-q_i}(\mu - p_i) & -(\lambda - p_i)e^{-q_i} & (\lambda - p_i)(\mu - p_i) \end{pmatrix} \right]$$

□

Corollary 4.2. *We have*

$$\{L^{(N)}(\lambda) \otimes 1, 1 \otimes L^{(N)}(\mu)\} = [r(\lambda - \mu), L^{(N)}(\lambda) \otimes L^{(N)}(\mu)]$$

Proof. We use induction on N:

$$\begin{aligned}
\{L^{(N)}(\lambda) \otimes 1, 1 \otimes L^{(N)}(\mu)\} &= L_N(\lambda) \otimes L_N(\mu)\{L^{(N-1)}(\lambda) \otimes 1, 1 \otimes L^{(N-1)}(\mu)] \\
&+ \{L_N(\lambda) \otimes L_N(\mu)\}L^{(N-1)}(\lambda) \otimes L^{(N-1)}(\mu) \\
&= L_N(\lambda) \otimes L_N(\mu)[r(\lambda - \mu), L^{(N-1)}(\lambda) \otimes L^{(N-1)}(\mu)] \\
&+ [r(\lambda - \mu), L_N(\lambda) \otimes L_N(\mu)]L^{(N-1)}(\lambda) \otimes L^{(N-1)}(\mu) \\
&= [r(\lambda - \mu), L^{(N)}(\lambda) \otimes L^{(N)}(\mu)].
\end{aligned}$$

□

Theorem 4.3. *The closed Toda chain is Liouville integrable:* $\{T(\lambda), T(\mu)\} = 0$.

Remark. It follows action-angle variables exist. However, we still lack the machinery to determine them. For this, we need the *spectral curve*.