ZMP Lectures: Separation of Variables

Lecture 2: Classical SoV: Spectral Curve

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0 Literature and Supplementary Material

- O. Babelon, D. Bernard, M. Talon, *Introduction to Classical Integrable Systems*, 2003
  - Formal developments and motivation: Chapter 5
  - Explicit computation for Toda model: Chapter 6
- E. Sklyanin, *The Quantum Toda Chain*, 1984 (historical paper, but very bare bones)
- E. Sklyanin, *Separation of variables – new trends*, 1995, arXiv:solv-int/9504001 (some motivation and formal developments)
- V. Kuznetsov, F. Nijhoff, E. Sklyanin, *Separation of variables for the Ruijsenaars system*, arXiv:solv-int/9701004 (for Ruijsenaars-Schneider, but very general explanation of the method)
- P. Ryan, *Integrable systems, separation of variables and the Yang-Baxter equation*, PhD thesis, arXiv:2201.12057 [math-ph] (short introduction in chapter 6)

1 Overview of the general idea

Before proceeding with SoV applied to the Toda model, it is useful to briefly spell out the general algorithm (insofar as it exists) for models of this type. The essence of the SoV approach lies in finding a set of so-called *separating equations*

$$\Phi_i\left(\tilde{p}_i, \tilde{q}_i, \{H_j\}\right) = 0\,, \qquad i = 1, \ldots, N\,, \tag{1}$$

for a set of canonical variables $(\tilde{p}_i, \tilde{q}_i)$, which allow us to relate each momentum to its conjugate position variable in terms of the integrals of motion H_j. One way of seeing that this is useful is to consider the very common (albeit not general) case $\dot{\tilde{q}}_i = \tilde{p}_i$. In this case, after solving for the momentum $\tilde{p}_i$, we can use this relation to obtain an ODE, which is easily solved

$$\dot{\tilde{q}}_i = \tilde{p}_i\left(\tilde{q}_i, \{H_j = \text{const}\}\right) \implies t(\tilde{q}_i) = \int \frac{d\tilde{q}_i}{\tilde{p}_i\left(\tilde{q}_i, \{H_j = \text{const}\}\right)}\,. \tag{2}$$

In the general case, this result can also be achieved by considering the generating function of the canonical transformation to action-angle variables, which splits into a sum of single-particle generating functions, determined by ODEs.

The question then is how to actually obtain such a set of variables and their separating equations. For this let us therefore consider the Lax formulation of a (at this stage fairly generic) integrable model. Given a generic monodromy matrix $M(\lambda)$ (or Lax, it doesn't really matter as long as it describes the equations of motion, and evolves by conjugation) so far, we have constructed our conserved quantities in terms of the trace $T(\lambda) = \operatorname{tr} M(\lambda)$. However, we may alternatively consider the set of eigenvalues $\Lambda(\lambda)$, which are also conserved in time, given by the roots of the characteristic polynomial

$$\det(M(\lambda) - \Lambda(\lambda)\mathbb{1}) = 0\,. \tag{3}$$

This defines the *spectral curve* $\Sigma = \{(\lambda, \Lambda(\lambda)) \,|\, (3)\}$, which will be the main object of this lecture. Since we are imposing one complex constraint on two complex variables, this is a smooth curve of complex dimension one, i.e. a Riemann surface. One benefit of this parametrization is that we can more-or-less automatically obtain a set of separated (i.e. obeying separating equations) variables by choosing points that lie on the spectral curve, since evaluating eq. (3) then gives the separating equation. A convenient choice for our separated variables turns out to be the poles of (one of) the eigenvectors of $M(\lambda)$ (given a fixed normalization, then called Baker-Akhiezer functions), and their associated eigenvalues. To see why this choice is good, consider that the spectral curve itself (by construction) contains no information about the dynamics of the theory, seeing as it is conserved. Ultimately the complete data describing the theory is encoded in $M(\lambda)$, meaning that in order to reconstruct the dynamics, we should also consider its eigenvectors (since that is then equivalent to knowing $M(\lambda)$). However fully determining these functions is in some sense overkill, since we know (again, by construction) that these are meromorphic functions on the spectral curve. It should therefore be sufficient to understand their behavior at the poles. While still somewhat arbitrary, this gives some motivation for why this might be a good choice for separated variables. From here one has also opened the door to the theory of Riemann surfaces via the spectral curve, which enables the use of numerous tools for solving the EoMs, notably the *Abel map* (c.f. lecture 3).

Note that we cannot expect this strategy to work in general, since there are still some potential points of failure. For one thing, it is not guaranteed that, for a given normalization, there exist the right number of poles corresponding to the degrees of freedom of our system. For another, since we have made no reference to the Poisson structure, we have no reason to expect these variables to be canonical. Miraculously, as we will see in the case of the Toda chain, the method is nonetheless successful in many cases if one can find the right choice of normalization. In the absence of a more precise explanation, it is therefore also called *Sklyanin's magic trick*.

2 Recap of the closed Toda chain

As a reminder, the closed Toda model consists of a "chain" of N particles (with periodic boundary conditions) described by the Hamiltonian

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2} + e^{q_n - q_{n+1}}\,, \qquad q_{N+1} \equiv q_1\,. \tag{4}$$

Its equations of motion then read

$$\dot{q}_n = p_n\,, \qquad \dot{p}_n = e^{q_{n-1}-q_n} - e^{q_n - q_{n+1}}\,, \tag{5}$$

which can be reformulated in terms of a (discrete) Lax pair

$$L_n(\lambda) = \begin{pmatrix} 0 & e^{q_n} \\ -e^{-q_n} & \lambda - p_n \end{pmatrix}\,, \quad M_n(\lambda) = \begin{pmatrix} 0 & -e^{q_n} \\ e^{-q_{n+1}} & -\lambda \end{pmatrix}\,, \quad \lambda : \text{spectral parameter}, \tag{6}$$

satisfying the "flatness condition" $\dot{L}_n = M_{n+1}L_n - L_n M_n$ (c.f. Lukas' notes from the previous lecture). From this, we can define the monodromy matrix $M(\lambda)$ (note the absence of the index, distinguishing it from M_n in eq. (6), which we will no longer use) and the transfer matrix $T(\lambda)$

$$M(\lambda) \equiv L_N(\lambda)L_{N-1}(\lambda)\dots L_1(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}\,, \qquad T(\lambda) = \operatorname{tr} M(\lambda) \equiv \sum_{k=0}^{N} H_k \lambda^k\,, \tag{7}$$

which is the generating function of the integrals of motion H_k. In particular, we have $H_{N-1} = -P$ (the total momentum), and $H_{N-2} = P^2/2 - H$, while $H_N = 1$.

Let us record some useful properties of the monodromy matrix $M(\lambda)$. First, it is easy to check at the level of the Lax matrices L_n, that $\det M(\lambda) = 1$. In addition, an explicit computation reveals the degree of the polynomials A, B, C, D to be

$$A(\lambda) \sim \lambda^{N-2} + \dots\,, \quad B(\lambda) \sim \lambda^{N-1} + \dots\,, \quad C(\lambda) \sim \lambda^{N-1} + \dots\,, \quad D(\lambda) = \lambda^N + \dots \tag{8}$$

Aside from their conservation, the Liouville-Arnold theorem also requires the integrals H_k to be in involution (i.e. to Poisson commute), which turns out to be equivalent to the existence of a classical r-matrix, here given explicitly by $r_{12}(\lambda) = -\frac{1}{\lambda}P_{12}$, expressing the Poisson bracket of the components of L_n via

$$\{L_{1,n}(\lambda), L_{2,n}(\mu)\} \equiv \{L_n(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L_n(\mu)\} = [r_{12}(\lambda - \mu), L_n(\lambda) \otimes L_n(\mu)]\,. \tag{9}$$

Here the Poisson bracket is to be understood component-wise, and P_{12} is the permutation matrix which exchanges the two factors of $\mathbb{C}^2$, given explicitly by

$$P_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

One can then show that the same relation also holds for the monodromy matrix $M(\lambda)$.

3 Separation of Variables for the Closed Toda Model

Let us now try and apply the SoV to the Toda model. Since we are dealing with a 2×2 matrix, we can compute the spectral curve explicitly by expanding the determinant

$$\det(M(\lambda) - \Lambda(\lambda)\mathbb{1}) = \det M + \Lambda(\lambda)^2 - \Lambda(\lambda)\,\mathrm{tr}\, M(\lambda) = 0 \iff T(\lambda) = \Lambda(\lambda) + \frac{1}{\Lambda(\lambda)}. \tag{11}$$

Furthermore, one can easily show (by normalizing the first component to 1), that the poles of the Baker-Akhiezer functions $\psi(\lambda)$ are given precisely by the zeros of $B(\lambda)$. We therefore define our new variables (γ_r, Λ_r) by

$$B(\lambda) = \gamma_N \prod_{r=1}^{N-1} (\lambda - \gamma_r), \tag{12}$$

as well as

$$\Lambda_r = \Lambda(\gamma_r), \qquad r = 1, \dots, N-1, \qquad D(\lambda) = \lambda^N + \Lambda_N \lambda^{N-1} + \dots \tag{13}$$

In particular, since at $\lambda = \gamma_r$, M becomes lower triangular, the eigenvalues are simply given by A, and D. Here we choose $\Lambda_r = D(\gamma_r)$. The variables γ_N, Λ_N encode the center of mass motion of the system (since $\Lambda_N = H_{N-1} = -P$) and therefore decouple, so we will not consider them further. By plugging these variables into our spectral curve (11), we obtain the separating equation

$$T(\gamma_r) = \Lambda_r + \frac{1}{\Lambda_r}. \tag{14}$$

What we have yet to determine are the Poisson brackets of our new "separated" variables. Using the expression (9) (for $M(\lambda)$), we find that the new variables (γ_r, Λ_r) are (almost) canonical, i.e. we that have

$$\{\gamma_r, \gamma_s\} = \{\Lambda_r, \Lambda_s\} = 0, \qquad \{\gamma_r, \Lambda_s\} = \delta_{rs}\Lambda_s \iff \{\gamma_r, \log \Lambda_s\} = \delta_{rs}. \tag{15}$$

To show this, we expand eq. (9) using the fact that conjugation by P_{12} exchanges the order of the tensor product, and $P_{12}^2 = \mathbb{1}$

$$\{M^{(1)}(\lambda), M^{(2)}(\mu)\} = [r_{12}(\lambda-\mu), M(\lambda)\otimes M(\mu)] \\ = r_{12}(\lambda-\mu)\,(M(\lambda)\otimes M(\mu) - M(\mu)\otimes M(\lambda))\,. \tag{16}$$

Since multiplying with P_{12} from the left simply exchanges the second and third row, we can immediately read off

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = \{D(\lambda), D(\mu)\} = 0\,. \tag{17}$$

From here the first two equations follow from the Leibniz rule and evaluation at γ_r respectively, while the third requires a combination of the two applied to $\{B(\lambda), D(\mu)\}$.

We can also deduce the (generalized) equations of motion for the γ's, i.e. their Poisson bracket with the transfer matrix $T(\lambda)$ (which contains the Hamiltonian, thereby justifying the name EoM)

$$\{T(\lambda), \gamma_s\} = \sqrt{P(\gamma_s)}\prod_{r\neq s}\frac{\lambda-\gamma_r}{\gamma_s-\gamma_r}\,, \qquad P(\lambda) = T(\lambda)^2 - 4\,. \tag{18}$$

Evidently these equations of motion are still very much coupled and nonlinear. However, by this new choice of variables, we have now mapped the problem of solving the EoMs on the full $2N$-dimensional phase space, to that of the dynamics of $N-1$ points on a Riemann surface. In order to better understand their motion, we first need to determine the topology of the spectral curve Σ. To do so, it is useful to slightly reparameterize it by completing the square and rearranging slightly, which gives

$$\mu^2 = P(\lambda)\,, \qquad \mu = 2\Lambda - T(\lambda)\,, \qquad P(\lambda) = T(\lambda)^2 - 4\,. \tag{19}$$

Let us now compute the genus of the spectral curve $\Sigma = \{(\mu,\lambda) \mid \mu^2 = P(\lambda)\}$. Clearly, for any nonzero value of $P(\lambda)$, there are always two solutions $\mu_\pm = \pm\sqrt{P(\lambda)}$, such that we start with a double cover of the complex plane. However when $P(\lambda) = 0$, there is only one solution, meaning that we should identify the roots on both "sheets", and connect them by branch cuts, across which a smooth curve is transported to the other branch. By a slight deformation of this picture, it is analogous to gluing together the two planes (with discs cut out) with a cylindrical tube for every pair of branch points connected by a branch cut (one also needs to flip one of the sheets to account for orientation). One can also show by a change of coordinates that the equations of motion are smooth at infinity, such that we can consider the compactification where we add a point "at each infinity" respectively (equivalently, we could have started with $\mu \in \mathbb{CP}^1$). Since $P(\lambda)$ is of degree $2N$ by construction, the result is a compact Riemann surface of genus $N-1$.

Next, we would like to understand the motion of our separated variables γ_r on the spectral curve. A first (and quite nontrivial) observation, is that the variables γ_r are all real. This involves showing that they are related to the eigenvalues of a Hermitian matrix, but goes beyond the scope of this lecture (c.f. chapter 6.8 of Babelon et al.). It

is therefore sufficient to study the spectral curve for real values of λ. From eq. (19), or equivalently eq. (11), we see that there are two regimes given by

$$\begin{cases} |T(\lambda)| < 2 \implies \Lambda \in [1, \infty)\,, & \text{allowed zone} \\ |T(\lambda)| > 2 \implies \Lambda \in \mathrm{U}(1)\,, & \text{forbidden zone}\,. \end{cases} \tag{20}$$

Since by construction the Λ_r are real when the γ_r are real, this means that the γ_r must be confined to the allowed zones. From here we can make a collection of observations which will give us the qualitative motion of the γ_r on the spectral curve.

1. All roots of $P(\lambda)$ are real, meaning that all branch points lie on the real line, and we have exactly $N-1$ (bounded) allowed regions separated by branch cuts, along with one at infinity.

2. Each bounded allowed region contains exactly one γ_r.

3. The vector field $\{T(\lambda), \bullet\}$ (for generic values of λ) is non-singular in the allowed zones. This is not immediately obvious, since the rhs of eq. (18) appears to vanish at the branch points. However this is just an artifact of our choice of coordinate, since we are essentially only considering the projection of a circle onto the allowed interval. It can be shown by a change of coordinates that $\{T(\lambda), \bullet\}$ has no fixed points anywhere on the circle, and is therefore non-singular.

With this knowledge, we conclude that the γ_r must undergo periodic motion around the allowed zones, which is equivalent to saying that they travel along the a-cycles of the spectral curve Σ.

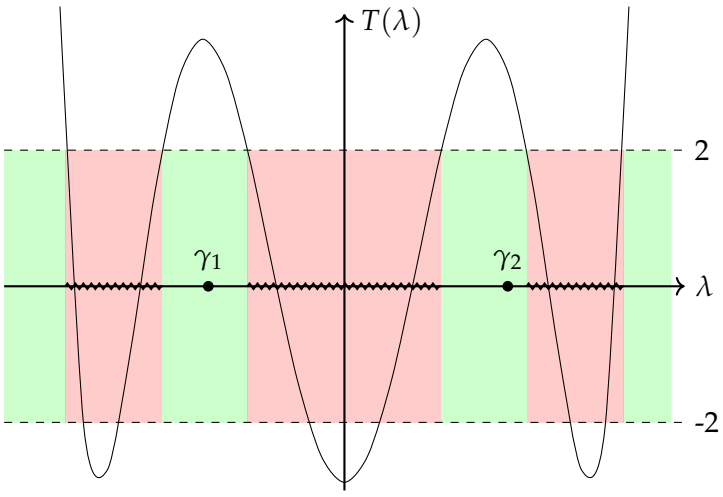

FIGURE: Real part of the spectral curve for $N = 3$ depicted by plotting $T(\lambda)$, split into forbidden (red) and allowed (green) zones. The branch cuts lie in the forbidden zones (which is just convention), while the γ_r are confined to the allowed zones.