# Classical SoV 3: Action-Angle Variables

## 0. Prerequisites

We defined new variables $\gamma_r(p_n, q_n)$, $\Lambda_r(p_n, q_n)$ that satisfy

$$\{\gamma_r, \gamma_s\} = 0, \quad \{\Lambda_r, \Lambda_s\} = 0, \quad \{\gamma_r, \log \Lambda_s\} = \delta_{r,s}$$

and obey the following equation of motion

$$\{T(\lambda), \gamma_s\} = \sqrt{T(\gamma_s)^2 - 4} \prod_{\substack{r=1 \\ r\neq s}}^{N-1} \frac{\lambda - \gamma_r}{\gamma_s - \gamma_r} \qquad \text{(EOM)}$$

$T(\lambda)$ is the transfer matrix (function) that satisfies

1) $$T(\lambda) = \sum_{k=0}^{N} \lambda^k \underbrace{H_{N-k}}_{\text{conserved quantity}} = \lambda^N - \underbrace{P}_{\substack{=\Lambda_N,\text{ thus a} \\ \text{conserved quantity and} \\ \text{can be neglected}}} \lambda^{N-1} + \underbrace{H_2}_{=\frac{1}{2}P^2 - H} \lambda^{N-2} + \dots$$

Hamiltonian

2) For $r = 1, \dots, N-1$: $T(\gamma_r) = \Lambda_r + \Lambda_r^{-1}$, or equivalently

$$\mu_r^2 := (2\Lambda_r - T(\gamma_r))^2 = T(\gamma_r)^2 - 4 =: P(\gamma_r)$$

3) Zeros $(\lambda_1, \dots, \lambda_{2N})$ of $P(\lambda)$ are real and satisfy
$\lambda_{2k} < \gamma_k < \lambda_{2k+1}$, $k = 1, \dots, N-1$

Define spectral curve $\Sigma = \{(\lambda, \mu) \in \mathbb{C}^2 \mid \mu^2 = P(\lambda)\}$

1) $\Sigma$ is a Riemann surface

2) Motion of $\gamma_r$, $r = 1, \dots, N-1$ described by $\{T(\lambda), \cdot\}$ is smooth and confined by $[\lambda_{2k}, \lambda_{2k+1}]$

3) Thus, motion is periodic and homotopic to the a-cyles of $\Sigma$.

# 1. Abel Map

Aim: Define another chart of variables such that motion $\gamma_1, \dots, \gamma_{N-1}$ becomes linear

Abel map: Let $Q_1, \dots, Q_{N-1}$ ; $P_1, \dots, P_{N-1}$ be points on $\Sigma$. Then, let $\Theta = (\Theta_1, \dots, \Theta_{N-1})$ be

$$\Theta_r = \sum_{k=1}^{N-1} \int_{Q_k}^{P_k} \frac{d\lambda}{\sqrt{P(\lambda)}} \lambda^{r-1} \quad , \quad r = 1, \dots, N-1$$

The $Q_i$'s are understood to be fixed initial points, and the $P_i$'s are subject to the motion on $\Sigma$, so locally $P_k = \gamma_k$

Proposition: The Abel map linearizes the EOM:

$$\{T(\lambda), \Theta_r\} = \lambda^{r-1}$$

With $T(\lambda) = \sum_{k=0}^{N} \lambda^k H_{N-k}$, this implies:

$$\{H_{N-k}, \Theta_r\} = \delta_{k, r-1} = \delta_{k+1, r}$$

And particularly with $H_2 = \frac{1}{2} p^2 - H$,

$$\{H_2, \Theta_r\} = - \{H, \Theta_r\} = \dot{\Theta}_r = \text{const.}$$

Proof: $\{T(\lambda), \theta_r\} = \sum_{k=1}^{N-1} \{T(\lambda), \int^{\gamma_k} \frac{d\lambda}{\sqrt{P(\lambda)}} \lambda^{r-1}\}$

$$= \sum_{k=1}^{N-1} \{T(\lambda), \gamma_k\} \cdot \frac{\lambda^{r-1}}{\sqrt{P(\lambda)}}\Big|_{\lambda=\gamma_k}$$

$$= \sum_{k=1}^{N-1} \sqrt{P(\gamma_k)} \prod_{r \neq k} \frac{\lambda - \gamma_r}{\gamma_k - \gamma_r} \frac{\gamma_k^{r-1}}{\sqrt{P(\gamma_k)}}$$

$$= \sum_{k=1}^{N-1} \gamma_k^{r-1} \prod_{r \neq k} \frac{\lambda - \gamma_r}{\gamma_k - \gamma_r} =: \tau_r(\lambda)$$

1) $\tau_r(\lambda)$ is a polynomial of degree $N-2$

2) For $s = 1, \dots, N-1$ : $\tau_r(\gamma_s) = \sum_{k=1}^{N-1} \gamma_k^{r-1} \delta_{k,s} = \gamma_s^{r-1}$

$$\longrightarrow \quad \tau_r(\lambda) = \lambda^{r-1}$$

$\square$

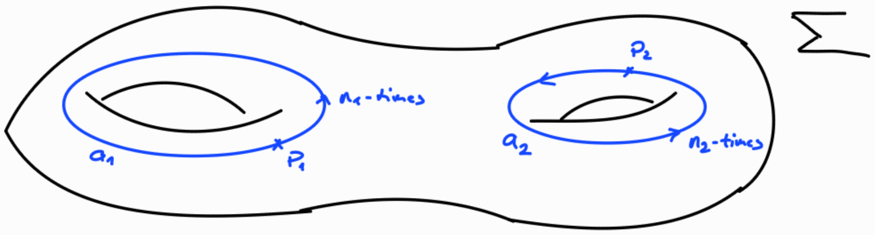
## 2. A-Cycles

We know motion of $P_k$'s is periodic under $\{T(\lambda), \cdot\}$, this introduces periodicity in the variables $\theta_r$.

Let's say, $P_k$'s move $n_k$-times around cyles $a_k$, then:

$$\theta_r = \sum_{k=1}^{N-1} \left( \int_{a_k}^{P_k} + n_k \oint_{a_k} \right) \frac{d\lambda}{\sqrt{P(\lambda)}} \lambda^{r-1}$$

$$= \tilde{\theta}_r + \sum_{k=1}^{N-1} n_k \alpha_{rk} \quad , \text{with } \alpha_{rk} := \oint_{a_k} \frac{d\lambda}{\sqrt{P(\lambda)}} \lambda^{r-1}$$



$\theta_r$ and $\tilde{\theta}_r$ correspond to the same configuration, thus identify

$$\theta_r \sim \theta_r + \sum_{k=1}^{N-1} n_k \alpha_{rk}$$

This can be done naturally by introducing:

$$\varphi_r = 2\pi \sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \theta_{r'} \quad , \text{ where } \sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \alpha_{r's} = \delta_{rs}$$

Then $\varphi_r \sim 2\pi \sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \left( \theta_{r'} + \sum_{k=1}^{N-1} n_k \alpha_{r'k} \right)$

$$= \varphi_r + 2\pi \sum_{k=1}^{N-1} n_k \underbrace{\sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \alpha_{r'k}}_{=\delta_{rk}} = \varphi_r + 2\pi n_r$$

New variables have identification $\varphi_r \sim \varphi_r + 2\pi$.
These parameterize a real torus $T_{N-1} = (S^1)^{N-1}$.

<u>Comment:</u> We can define Normalized Abelian differentials (of the first kind) by

$$\omega_r = \sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \frac{\lambda^{r'-1}}{\sqrt{P(\lambda)}} d\lambda$$

They satisfy: $\oint_{a_s} \omega_r = \sum_{r'=1}^{N-1} \alpha^{-1}_{rr'} \underbrace{\oint_{a_s} \frac{\lambda^{r'-1}}{\sqrt{P(\lambda)}} d\lambda}_{\alpha_{r's}} = \delta_{r,s}$

The $\omega_r$'s form a basis of $H^{(1,0)}(\Sigma)$, the space of holomorphic 1-forms on $\Sigma$.
Thus, the Hodge number satisfies $\dim H^{(1,0)}(\Sigma) = N-1 = g$ genus.

With those, the Abel map becomes:

$$\varphi_r = 2\pi \sum_{k=1}^{N-1} \int_{Q_k}^{P_k} \omega_r \quad , \quad r = 1, \dots, N-1$$

They define the <u>angle variables</u>.

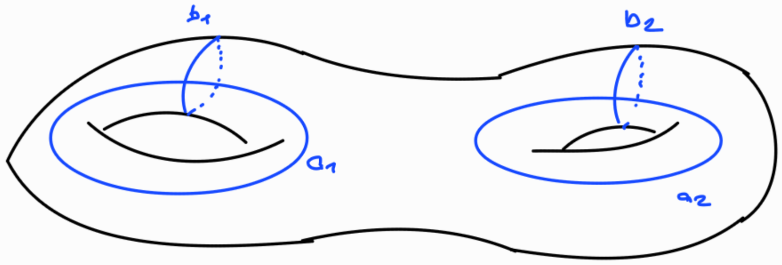
# 3. Complexifying the Spectral Curve : B-Cycles

We complexify the solutions of the EOM

→ Points $P_1, \dots, P_{N-1}$ can be everywhere on $\Sigma$
(they are not constraint onto the a-cycles anymore)

→ Angle variables $\varphi_r$ become complex.

$\Sigma$ has now twice as many closed curves




Assume $b_s$ intersects only $a_s$, and only once.
Moving $P_s$ around $b_s$ brings it back to $P_s$,
but changes $\varphi_r$ by

$$\varphi_r \longrightarrow \varphi_r + 2\pi \oint_{b_s} \omega_r =: \varphi_r + \underbrace{B_{rs}}_{\text{some complex value}}$$

Thus, we obtain the following identifications of the vector $\varphi = (\varphi_1, \dots, \varphi_{N-1}) \in \mathbb{C}^{N-1}$, when moving around $a_s$ and $b_s$ for $s = 1, \dots, N-1$, respectively.

$$(*) \quad \begin{cases} \varphi \sim \varphi + 2\pi\, e_s \\ \varphi \sim \varphi + B \cdot e_s \end{cases} \quad , \text{ for } \quad s = 1, \dots, N-1$$

with $e_s = (0, \dots, 0, \overset{s}{1}, 0, \dots, 0) \in \mathbb{C}^{N-1}$ and $B \cdot e_s = (B_{1s}, \dots, B_{N-1,s})$.

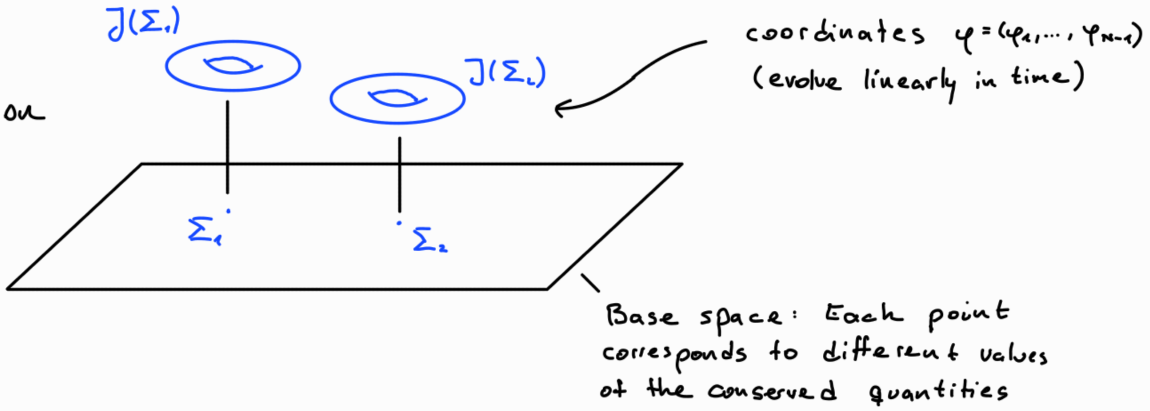
**Definition:** The torus $J(\Sigma)$ parameterized by $\varphi \in \mathbb{C}^{N-1}$ with identifications (*) is called Jacobian.

$$J(\Sigma) = \mathbb{C}^{N-1} \big/ (2\pi \mathbb{Z}^{N-1} + B \cdot \mathbb{Z}^{N-1})$$

To conclude, we arrive at the following picture

For each set of values of the conserved quantities $(H_2, \dots, H_N)$, we can define a spectral curve $\Sigma$ via $T(\lambda)$. Using the Abel map, we introduce coordinates that evolve linearly in time and live on the torus $J(\Sigma)$.

torus - fibration




## 4. Action Variables

We want to find action variables $I_r$, such that

$$\{I_r, I_s\} = 0, \quad \{\varphi_r, \varphi_s\} = 0, \quad \{\varphi_r, I_s\} = \delta_{rs}$$

We define: $$I_r = \frac{1}{2\pi} \oint_{a_r} d\lambda \log \Lambda$$

$$S(\gamma, I) = \frac{1}{2\pi} \sum_{k=1}^{N-1} \int_{Q_k}^{P_k} d\lambda \log \Lambda$$

with $\Lambda(\lambda) = \frac{T(\lambda)}{2} + \sqrt{\frac{T(\lambda)^2}{4} - 1}$, s.t. $\Lambda(\gamma_r) = \Lambda_r$

**Proposition:** $S(\gamma, I)$ is generating function for change of variables $\gamma, \lambda \longrightarrow \varphi, I$

That means:

i) $\frac{\partial}{\partial \gamma_r} S = \frac{1}{2\pi} \log \Lambda_r$

ii) $\partial / \partial I_r \; S = \frac{1}{2\pi} \varphi_r$

iii) Chart of var. $(I_1, \dots, I_{N-1}) \longleftarrow (H_2, \dots, H_N)$ is invertible.

If the conditions are true, then

- $\{I_r, I_s\} = 0$, $I$ depends only on $T(\lambda)$ not on $\gamma_r$.
- $\{\varphi_r, \varphi_s\} = (2\pi)^2 \frac{\partial}{\partial I_r} \frac{\partial}{\partial I_s} \{S, S\} = 0$
- $\{\varphi_r, I_s\} = 2\pi \left\{ \frac{\partial}{\partial I_r} S, I_s \right\} = 2\pi \frac{\partial}{\partial I_r}\left(\frac{\partial S}{\partial \gamma_t}\right) \{\gamma_t, I_s\}$

$$= \frac{\partial}{\partial I_r} (\log \Lambda_t) \underbrace{\{\gamma_t, \log \Lambda_{t'}\}}_{= \delta_{tt'}} \frac{\partial I_s}{\partial \log \Lambda_{t'}} = \frac{\partial \log \Lambda_t}{\partial I_r} \frac{\partial I_s}{\partial \log \Lambda_t} = \delta_{rs}$$

**Proof:** i) $\frac{\partial}{\partial \gamma_r} S = \frac{\partial}{\partial \gamma_r} \sum_{k=1}^{N-1} \int^{\gamma_k} d\lambda \log \Lambda(\lambda)$

$$= \sum_{k=1}^{N-1} \frac{\partial \gamma_k}{\partial \gamma_r} \log \Lambda(\gamma_r) = \log \Lambda_r$$

ii) Using $\delta_{rs} = \frac{\partial I_s}{\partial I_r} = \frac{1}{2\pi} \oint_{a_s} d\lambda \frac{\partial \log \Lambda}{\partial I_r} = \oint_{a_s} \omega_r$

$$\longrightarrow \omega_r = \frac{1}{2\pi} \frac{\partial \log \Lambda}{\partial I_r} d\lambda \quad \text{(norm. Abelian diff.)}$$

$$\frac{\partial S}{\partial I_r} = \frac{1}{2\pi} \sum_{k=1}^{N-1} \int_{Q_k}^{P_k} d\lambda \frac{\partial \log \Lambda}{\partial I_r} = \sum_{k=1}^{N-1} \int_{Q_k}^{P_k} \omega_r = \frac{1}{2\pi} \varphi_r$$

iii) $\omega_r = \frac{1}{2\pi} \frac{\partial \log \Lambda}{\partial I_r} d\Lambda = \frac{1}{2\pi} \underbrace{\frac{1}{\Lambda} \frac{\partial T}{\partial \Lambda}}_{= 1/\sqrt{P(\lambda)}} \frac{\partial T}{\partial I_r} d\Lambda$

$$= \frac{1}{2\pi} \sum_{k=1}^{N-1} \frac{\partial H_{N+1-k}}{\partial I_r} \frac{\lambda^{k+1}}{\sqrt{P(\lambda)}} d\lambda$$

$$\rightarrow \delta_{rs} = \oint_{a_s} \omega_r = \frac{1}{2\pi} \sum_{k=1}^{N-1} \frac{\partial H_{N+1-k}}{\partial I_r} \oint_{a_s} \frac{\lambda^{k+1}}{\sqrt{P(\lambda)}} d\lambda$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N-1} \frac{\partial H_{N+1-k}}{\partial I_r} \cdot \alpha_{ks} \rightarrow \alpha_{ks}^{-1} = \frac{1}{2\pi} \frac{\partial H_{N+1-k}}{\partial I_r} \quad \text{invertable!}$$