ZMP Separation of Variables: Quantum Toda chain — Torben Skrzypek

1. Previously... classical Toda chain

- $H := \sum_{n=1}^{N} \frac{p_n^2}{2} + e^{q_n - q_{n+1}}$, $q_{N+1} = q_N$
- $\{ q_n, p_m \} = \delta_{n,m}$
- Lax pair: $L_n(\lambda) := \begin{pmatrix} 0 & e^{q_n} \\ -e^{-q_n} & \lambda - p_n \end{pmatrix}$, $M_n(\lambda) := \begin{pmatrix} 0 & -e^{q_{n-1}} \\ e^{-q_n} & -\lambda \end{pmatrix}$

$\hookrightarrow$ EoM $\Longleftrightarrow$ $\dot{L}_n(\lambda) = M_{n+1}(\lambda) L_n(\lambda) - L_n(\lambda) M_n(\lambda)$

- Monodromy Matrix: $M = L_N(\lambda) L_{N-1}(\lambda) \dots L_1(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$
- $T(\lambda) := \operatorname{tr} M(\lambda) = A(\lambda) + D(\lambda) = \lambda^N - P\lambda^{N-1} + \left(\frac{P^2}{2} - H\right)\lambda^{N-2} + \dots$

  Conserved quantity and generator of commuting charges. Proof requires the formula

- $\{ L_n(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes L_n(\mu) \} = [R(\lambda - \mu), L_n(\lambda) \otimes L_n(\mu)]$ (A)

  where $R(\lambda) = -\frac{1}{\lambda} P$, ie. $R(\lambda): v \otimes w \mapsto -\frac{1}{\lambda} w \otimes v$.

  $\Rightarrow \{ T(\lambda), T(\mu) \} = 0$ $\square$

- Strategy for solution (SoV)

1. Identify $B(\lambda) = \gamma_N \sum_{r=1}^{N-1} (\lambda - \gamma_r)$

$$\Lambda_r = e^{\nu_r} = D(\gamma_r)$$

$\Rightarrow \{ \gamma_r, \nu_r \} = \delta_{rs}$

$\{ T(\lambda), \gamma_s \} = \sqrt{T^2(\lambda) - 4} \prod_{r \neq s} \frac{\lambda - \gamma_r}{\gamma_s - \gamma_r}$

2. Use Abel-Jacobi Map to action-angle-coordinates $\{ I_r, \phi_r \}$
3. Reconstruct p_n, q_n

② Quantisation

- As usual: $q_n, p_m, \{\cdot,\cdot\} \rightarrow \hat{q}_n, \hat{p}_m, \{\cdot,\cdot\}$

$$\{q_n, p_m\} = \delta_{nm} \rightarrow [\hat{q}_n, \hat{p}_m] = i\hbar\,\delta_{nm}$$

suggesting $\hat{p}_n = -i\hbar\,\partial_{q_n}$ on $\psi(q_i)$

- Quantum Toda chain

Same expressions for $\hat{H}, \hat{L}_n, \hat{M}, \hat{T}, \dots$ (will drop ^)

acting on Hilbert space $\mathcal{H} = (L^2(\mathbb{R}))^{\otimes N}$

- Proof of integrability

DO NOT WRITE $\{ \ast \rightarrow [L_n(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes L_n(\mu)] = i\hbar\,[R(\lambda-\mu), \text{"}L_n(\lambda)\otimes L_n(\mu)\text{"}] \}$

Ordering ambiguous, so have to redo calculation

for convenience: drop n, introduce 12 for $L\otimes\mathbb{1}$, $\mathbb{1}\otimes L$

$$\rightarrow L_1(\lambda)L_2(\mu) - L_2(\mu)L_1(\lambda) = i\hbar\,R(\lambda-\mu)L_1(\lambda)L_2(\mu) - i\hbar\,L_2(\mu)L_1(\lambda)R(\lambda-\mu)$$

Introduce "quantum R-matrix"

$$\mathcal{R}(\lambda) = \lambda\,\mathrm{id} - i\hbar\,P, \text{ i.e. } \mathcal{R}(\lambda): v\otimes w \mapsto \lambda\, v\otimes w - i\hbar\, w\otimes v$$

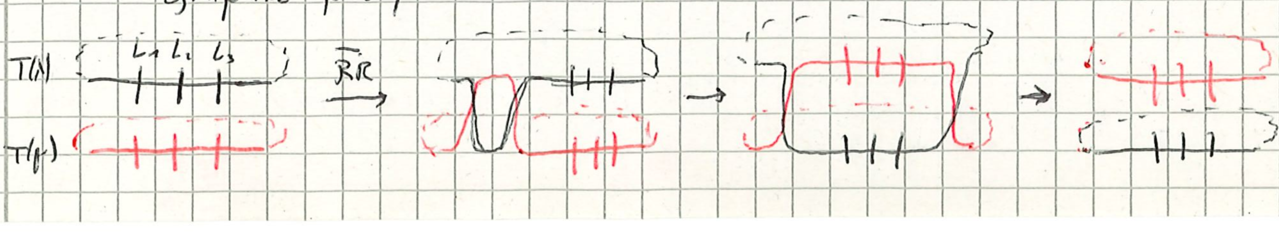
to find the $\boxed{\text{RLL-relation}}$

$$\mathcal{R}(\lambda-\mu)\,L_1(\lambda)\,L_2(\mu) = L_2(\mu)\,L_1(\lambda)\,\mathcal{R}(\lambda-\mu)$$

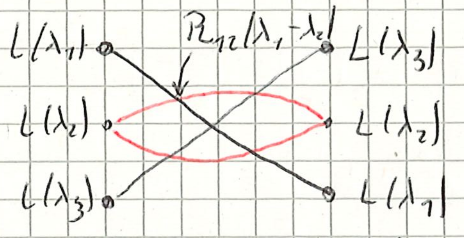
which again holds for $M(\lambda) \rightarrow 16$ relations between A, B, C, D

$$\begin{aligned}
T(\lambda)T(\mu) &= \mathrm{tr}_{\alpha\beta}\,[M_\alpha(\lambda)\,M_\beta(\mu)] \\
&= \frac{1}{(\lambda-\mu)^2+\hbar^2}\,\mathrm{tr}_{\alpha\beta}\,[\bar{\mathcal{R}}(\lambda-\mu)\,\mathcal{R}(\lambda-\mu)\,M_\alpha(\lambda)\,M_\beta(\mu)] \\
&= \frac{1}{(\lambda-\mu)^2+\hbar^2}\,\mathrm{tr}_{\alpha\beta}\,[\bar{\mathcal{R}}(\lambda-\mu)\,M_\beta(\mu)\,M_\alpha(\lambda)\,R(\lambda-\mu)] \\
&= T(\mu)\,T(\lambda) \implies [T(\lambda), T(\mu)] = 0 \qquad \square
\end{aligned}$$

Graphic proof



Aside: $\boxed{\text{Yang-Baxter-equation}}$



$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1-\lambda_3)R_{23}(\lambda_2-\lambda_3)$$
$$=$$
both ways match! $$R_{23}(\lambda_2-\lambda_3)R_{13}(\lambda_1-\lambda_3)R_{12}(\lambda_1-\lambda_2)$$

$\Rightarrow$ "Scattering picture" of integrability

③ Solution via SoV

We want to follow the same strategy as before

- $B(\lambda) = \hat{y}_N \prod_{r=1}^{N-1}(\lambda - \hat{y}_r)$, $\{\hat{y}_r\}$ = commuting, self-adjoint ops.
- $\hat{\Lambda}_r = D(\hat{y}_r) := \sum_{n=0}^{N} \hat{y}_r^n \hat{\alpha}_n$
- Caveat: $\hat{y}_N = e^{\hat{q}_N} \to \hat{\Lambda}_N = p_N$ (will be sloppy)
- We want to show "$[y_r, \Lambda_s] = i\hbar \Lambda_s \delta_{r,s}$"

From ☆: $(\lambda-\mu)D(\lambda)B(\mu) - i\hbar B(\lambda)D(\mu) = (\lambda-\mu-i\hbar)B(\mu)D(\lambda)$

for $\lambda = y_j$ $(y_j-\mu)\Lambda_j B(\mu) = (y_j-\mu-i\hbar)B(\mu)\Lambda_j$

$\Rightarrow \Lambda_j B(\mu) = B(\mu)\Big|_{y_j \to y_j - i\hbar}$

$\Rightarrow \Lambda_j f(y_1, \ldots, y_j, \ldots y_{N-1}) = f(y_1, \ldots, y_j - i\hbar, \ldots y_{N-1})$

for $\forall$ symmetric functions f of $\{y_i\}$

We recognise a structure $y_i \cong q_i$ $\Lambda_j \cong \exp(p_j)$ in generic QM model.

- We now choose a basis $\psi(x_1, \ldots, x_N)$

s.t.: $y_j \psi(x_1, \ldots, x_N) = x_j \psi(x_1, \ldots, x_N)$

$\Lambda_j \psi(x_1, \ldots, x_N) = i^{-N} \psi(x_1, \ldots, x_j - i\hbar, \ldots, x_N)$

What is the inner product of this basis?

$$\langle f, g \rangle = \int_{-\infty}^{\infty} dx^N \, \mu(x_1, \dots, x_{N-1}) \, \bar{f}(x_1, \dots, x_N) \, g(x_1, \dots, x_N)$$

Technical aside: Conjugation

Investigate $T^+(\lambda) = T(\bar{\lambda})$ (without inversion in aux. space)

$$\Rightarrow \gamma_j^+ = \gamma_j$$

$$\Rightarrow \Lambda_j^+ = \prod_{k \neq j}^{N-1} \frac{\lambda_j - \lambda_k - i\hbar}{\lambda_k - \lambda_j} \Lambda_j =: c_j \Lambda_j$$

Inner product should satisfy $\langle \Lambda_j^+ f, g \rangle = \langle f, \Lambda_j g \rangle$

$\rightarrow$ Consistency condition on measure μ

$$\langle f, \Lambda g \rangle = \int dx \, \mu(x) \, \bar{f}(x) \, g(x - i\hbar) = \int dx \, \mu(x + i\hbar) \, \bar{f}(x + i\hbar) \, g(x)$$

$$\parallel$$

$$\langle \Lambda^+ f, g \rangle = \int dx \, \mu(x) \, \bar{c} \, \bar{f}(x + i\hbar) \, g(x)$$

$$\Rightarrow \mu(x_1, \dots, x_j + i\hbar, \dots, x_{N-1}) = \prod_{k \neq j}^{N-1} \frac{x_j - x_k + i\hbar}{x_k - x_j} \mu(x_1, \dots, x_{N-1})$$

which is solved by$^{(AA)}$ $\mu(x_1, \dots, x_{N-1}) = \prod_{j<k=1}^{N-1} \frac{1}{\left| \Gamma\left(\frac{x_j - x_k}{i\hbar} \right) \right|^2}$.

We found a convenient basis on which $T(\lambda)$ acts naturally.

Now find eigenstates / values

$$T(\lambda_j) \Psi(x_1, \dots, x_N) = \Lambda_j \Psi(x_1, \dots, x_N) + \Lambda_j^{-1} \Psi(x_1, \dots, x_N)$$

$$= i^N \Psi(x_1, \dots, x_j + i\hbar, \dots, x_N) + i^{-N} \Psi(x_1, \dots, x_j - i\hbar, \dots, x_N)$$

Finite difference equations in single variables

Solve by SEPARATION OF VARIABLES $\Psi(x_1, \dots, x_N) = \prod_{j=1}^{N} \Psi_j(x_j)$

$$\Rightarrow T(\lambda_j) \Psi_j(x_j) = i^N \Psi(x_j + i\hbar) + i^{-N} \Psi(x_j - i\hbar)$$

$$\boxed{\text{Baxter TQ-relation}}$$

Further impose asymptotic $T(\lambda) \xrightarrow[\lambda \to \infty]{} \lambda^N$ ...

(4) Things we do not have time for

- Solve Baxter TQ - Relation

  see Gutzwiller 80/81 for small N

  see Kozlowski Teschner 10 for connection to

  Nekrasov Shatashvili 09 → TBA, NLIE from gauge theory

- Reconstruct the wavefunction in terms of q_n

$$\Phi(q_n) = \int dx^N \, \mu(x_m) \, K(q_n, x_m) \prod_{k=1}^{N} \Psi_k(x_k)$$

for discussion of Kernel K see Kozlowski 13, also for:

- Technical details: - unitary equivalence
  - treatment of (y_N, p_N)
  - twisting, open, ...

Outlook: Paul Ryan -

$(\cancel{A}\cancel{A}) =$

$$\mu = \prod_{j\neq k} \frac{1}{\Gamma\left(\frac{x_j - x_k}{i\hbar}\right)} \xrightarrow{x_j \to x_j + i\hbar} \prod_{k\neq j} \frac{1}{\Gamma\left(\frac{x_j - x_k}{i\hbar}+1\right)\Gamma\left(\frac{x_k - x_j}{i\hbar}-1\right)} \prod \text{Rest}$$

$$= \frac{\left(\frac{x_k - x_j}{i\hbar} - 1\right)}{\left(\frac{x_j - x_k}{i\hbar}\right)} \mu = \frac{x_j - x_k + i\hbar}{x_k - x_j} \mu$$

□