1. Recall Quantum Toda

$$M = \mathcal{L}_L(u) \dots \mathcal{L}_1(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

$T(u) = \operatorname{tr} M(u) \sim$ conserved quantities.
$= A + D.$

SoV $\to$ roots of $B(u) = \hat{\gamma}_L \prod_{\alpha=1}^{L-1} (u - \hat{\gamma}_\alpha)$

In $\hat{\gamma}_\alpha$ basis, eigenfns of $T(u)$ factorise.

$$\hat{\gamma}_\alpha \Psi(x_1, \dots, x_L) = x_\alpha \Psi(x_1, \dots, x_L)$$

$$\Psi(x_1, \dots, x_L) = \prod_{\alpha=1}^{L} Q(x_\alpha)$$

- $T(u) Q(u) = i^L Q(u + i\hbar) + i^{-L} Q(u - i\hbar).$
- $A(u)$, $D(u)$ acted as shift operators.

$$A(\hat{\gamma}_\alpha) \Psi(x_1, \dots, x_L) = i^{-L} \Psi(x_1, \dots, x_\alpha - i\hbar, \dots)$$

$$D(\hat{\gamma}_\alpha) \Psi(x_1, \dots, x_L) = i^{L} \Psi(x_1, \dots, x_\alpha + i\hbar, \dots).$$

Why this worked so well?

$B(u)$, hence $\hat{\gamma}_\alpha$, self-adjoint ops. on $\mathcal{H}$.
Spectral thm; complete basis of eigenvectors.

$\Downarrow$

Leads to huge simplification.

Now explore fin-dim. integrable system. Such observations fail.

2. su(2) XXX spin chain, spin 1/2.

$$R(u) = u - \hbar P$$

$$\mathcal{L}(u) = \begin{pmatrix} u - \hbar E_{11} & -\hbar E_{21} \\ -\hbar E_{12} & u - \hbar E_{22} \end{pmatrix}$$

$$E_{11} = 1/2 + S_z$$
$$E_{22} = 1/2 - S_z$$
$$E_{12} = S^+, \quad E_{21} = S^-.$$

Put YBE here.

Can we repeat from before?

$B(u) \sim S^-$, nilpotent, Non-diag.

Bad.

Way out: twist.

$$M(u) \to M(u) G$$

G → any 2×2 matrix.

Preserves all commutation relns.

$$[R(u), G \otimes G] = 0$$

$$B(u) \to B(u) + \lambda A(u).$$

$A(u)$ → $u^L + \ldots$

⇒ $B(u)$ no longer nilpotent.

But still non-diag! Solution: Inhomogens.

$$\mathcal{L}_\alpha(u) \to \mathcal{L}_\alpha(u - \theta_\alpha), \quad \theta_\alpha \in \mathbb{C}$$

$$\theta_\alpha - \theta_\beta \notin \hbar \mathbb{Z}.$$

3. Now B diagonalisable, $B(u) = \prod_{\alpha=1}^{L} (u - \hat{X}_\alpha)$.

We can repeat what we did for Toda.

<u>Differences</u>: $X_\alpha \in \{\theta_\alpha, \theta_\alpha + \hbar\}$.

$$\langle x | A(\hat{X}_\alpha) = Q_\theta(X_\alpha) \langle \ldots, X_\alpha + \hbar, \ldots |$$

$$\langle x | D(\hat{X}_\alpha) = Q_\theta(X_\alpha) \langle \ldots, X_\alpha - \hbar, \ldots |$$

$$Q_\theta(u) = \prod (u - \theta_\alpha).$$

$$\langle x | \Psi \rangle = \prod_{\alpha=1}^{L} Q(X_\alpha)$$

$\sim$ $Q(u) = \lambda^{u/\hbar} \times \text{poly}$. ↓ Fixed by BAE.

$$T(u) Q(u) = Q_\theta(u-\hbar) Q(u+\hbar) - Q_\theta(u) Q(u+\hbar).$$

[1304.5011]

Now go higher rank: $\underset{3\times 3}{M(u)} \sim \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ & \ddots & \end{pmatrix}$

$$B(u) = \prod_{\alpha=1}^{3L} (u - X_\alpha)$$, not just M_{12}, M_{13}.

$B(u) =$ ~~$M_{23} M_{12} M_{23} - M_{23}$~~

$$M_{23} (M_{12} M_{23} - M_{22} M_{13}) + M_{13} (M_{11} M_{23} - M_{21} M_{13})$$

classically, + quantum shifts.

Shift operator $A(u) = ~~T~~ M_{23}^{-1} (~~M_{11}~~ ~~M_{32}~~ - ~~M_{31}~~ M_{12})$.

$T(u) = M_{11} + M_{22} + M_{33}$, as in Toda

even if we repeat the same story, very unclear how to proceed.

$T(u)$ does not act on $\langle x|$ basis by simple shifts.

Fin-dim even worse...

- Quantum B by Sklyanin 1992, toda 1985.

~~Absolutely~~ No progress on how to solve integrable systems of high rank.

For spectral problem: Q-operator methods.
* Representation theory.

But building eigenfns still v. difficult...

Progress: 2016 ; Gromov, Levkovich-Maslyuk, Sizov.
2018 ; Maillet, Niccoli.
2018+2020 ; Combined both into rep
Me + D. Volin. theoretic framework.

4. $t(u)\,Q(u) = Q_\theta(u-\hbar)\,Q(u+\hbar) + Q_\theta(u)\,Q(u-\hbar).$

MN 1.

Idea of MN:

Reference vector $\langle W|$

Suppose for some set J the set of vectors

$$\langle W| \prod_{j\in J} T(w_j)/N_j \quad \text{forms a basis.}$$

(normalisation)

Then wave functions of $T(u)$ automatically factorise

$$T(u)|\Psi\rangle = t(u)|\Psi\rangle$$

$$\Rightarrow \langle W| \prod_{j\in J} T(w_j)/N_j |\Psi\rangle$$

$$= \prod_{j\in J} t(w_j)/N_j \; \langle W|\Psi\rangle$$

So can we choose w_j, N_j, $\langle W|\Psi\rangle$ to get smth. nice?

Pick $w_j = \theta_j$, $N_j = Q_\theta(\theta_j - \hbar)$

$$\Rightarrow t(\theta_j)/N_j = \frac{Q(\theta_j+\hbar)}{Q(\theta_j)}$$

$$\langle W | \prod_{j=1}^{L} \left(\frac{T(\theta_j)}{Q_\theta(\theta_j - \hbar)} \right)^{n_j} | \Psi \rangle \quad , \; n_j \in \{0, 1\}$$

$\langle n_1, \ldots, n_L |$

$$= \langle W | \Psi \rangle \prod_{j=1}^{L} \left(\frac{Q(\theta_j + \hbar)}{Q(\theta_j)} \right)^{n_j}$$

Now put $\langle W | \Psi \rangle = \prod_{j=1}^{L} Q(\theta_j)$

$$\Rightarrow \boxed{\langle n_1, \ldots, n_L | \Psi \rangle = \prod_{j=1}^{L} Q(\theta_j + \hbar n_j)}$$

Wave function matches what we get in basis of B!

Can directly show that $\langle n_1, \ldots, n_L |$ diagonalises B. Need to choose W as.

$\langle W | = \langle 0, \ldots, 0 | =: \langle x_1 = \theta_1, \ldots, x_L = \theta_L |$

Now $\langle W | T(\theta_\alpha) = \langle \theta_1, \ldots, \theta_L | (\underbrace{A(\theta_\alpha)}_{} + D(\theta_\alpha))$

$A(x_\alpha) + D(x_\alpha)$

$= Q_\theta(x_\alpha - \hbar) \langle \ldots, \theta_\alpha + \hbar, \ldots |$

$+ \underbrace{Q_\theta(x_\alpha)}_{=0} \langle \ldots, \theta_\alpha - \hbar, \ldots |$ $\quad \sim \langle \ldots, \theta_\alpha + \hbar, \ldots |$.

And repeat.

5. Now we can try to generalise.

1. Higher spin, 2. Higher rank.

Spin-s rep. Can repeat the Toda logic.

$X_\alpha \in \{\theta_\alpha, \theta_\alpha + \hbar, \ldots, \theta_\alpha + \hbar 2S\}$.

$$T Q = Q_\theta(u - \hbar 2s) Q(u+\hbar) + Q_\theta(u) Q(u-\hbar).$$

$$\Psi = \prod_{\alpha=1}^{L} Q_1(X_\alpha).$$

Can we repeat MN trick?

spin 1: $Q(\theta_\alpha)$, $Q(\theta_\alpha + \hbar)$, $Q(\theta_\alpha + 2\hbar)$.

$$\langle W | \Psi \rangle = \prod_{\alpha=1}^{L} Q(\theta_\alpha).$$

$$\langle W | T(\theta_\alpha) | \Psi \rangle \sim Q(\theta_\alpha + \hbar)$$

But cannot get $Q(\theta_\alpha + 2\hbar)$.

need to evaluate at $\theta_\alpha + \hbar$ for TQ to produce $\theta_\alpha + 2\hbar$.

* But then we get a linear combination because second term will also contribute.

* Solution: More transfer matrices.

* Fusion

* Do on separate board and dont erase.

$a^2(1-P)(1-P)$
$= a^2(1-2P+P^2)$
$= 2a^2(1-P)$

$$R_{ab}(u-v)\, M_a(u)\, M_b(v)$$
$$= M_b(v)\, M_a(u)\, R_{ab}(u-v)$$

on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{H}$.
$\downarrow$ a, $\downarrow$ b

(Maybe skip depending on time)

$$R_{ab}(u-v) = (u-v)1 - \hbar P$$

at ~~...~~, $v = u+\hbar$, $R_{ab} = \sqrt{2}\hbar P^+_{ab}$
$\downarrow$
~~...~~ symmetriser.

$$P^+(x\otimes y) = \frac{1}{\sqrt{2}}(x\otimes y + y\otimes x)$$

$(P^+)^2 = P^+$.

$$\Rightarrow P^+_{ab}\, M_a(u)\, M_b(u+\hbar)$$
$$= M_b(u+\hbar)\, M_a(u)\, P^+_{ab}.$$

$\Rightarrow P^+_{ab}\, M_a(u)\, M_b(u+\hbar)$ is a linear operator on $\mathrm{sym}^2(\mathbb{C}^2)$.

i.e. $[P^+_{ab}\, M_a(u)\, M_b(u+\hbar),\, P^+_{ab}] = 0$.

Follows non-trivially from YBE.

Denote $M_{\square\!\square}(u) = P^+_{ab}\, M_a(u)\, M_b(u+\hbar)$.

Now : $\exists\ R_{\square\square,\square\square}(u) : \mathrm{Sym}^2(\mathbb{C}^2) \otimes \mathrm{Sym}^2(\mathbb{C}^2) \circlearrowleft$

s.t. R invertible.

$$R_{\square\square,\square\square}(u-v)\, M_{\square\square}(u)\, M_{\square\square}(v)$$
$$= M_{\square\square}(v)\, M_{\square\square}(u)\, R_{\square\square,\square\square}(u-v)$$

also
$$R_{\square\square,\square}(u-v)\, M_{\square\square}(u)\, M_{\square}(v)$$
$$= M_{\square}(v)\, M_{\square\square}(u)\, R_{\square\square,\square}(u-v).$$

For any Young diagrams λ, μ.

$$[T_\lambda(u), T_\mu(v)] = 0.$$

→ Huge commuting family of operators.

Now back to SoV

8. For us, examine $T_{\square}$.

Satisfies TQ:

$$T_{\square} Q(u) = Q_\theta(u-2\hbar s) Q_\theta(u-2\hbar s-\hbar) \frac{Q_+(u+2\hbar)}{Q_+(u)}$$

$$+ Q_\theta(\ldots)$$

$$\Rightarrow T_{\square}(\theta_\alpha) = \# \frac{Q_+(\theta_\alpha+2\hbar)}{Q_+(\theta_\alpha)}$$

For spin s; also need $T_{\underbrace{\square\square\square\square}_{r}}$, $r \leq s$.

$$\langle W | \prod_{\alpha=1}^{L} T_{\underbrace{\square\square\square\square}_{r_\alpha}}(\theta_\alpha).$$

Main point: unlike with Toda, this approach generalises very nicely to higher rank.

GL(N)

Quantum Lax

$$\mathcal{L}^\alpha_{ij}(u) = \delta_{ij} u - \hbar E_{ji}$$

$\searrow$ some representation of gl(N).

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

*9. GL(N) Rep. Theory, Highest weight. Fin. dim irreps.

$GL(N)$, $E_{i,j}$, $i,j \in \{1, \dots, N\}$

$gl(N)$ $[E_{i,j}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$.

HW reps characterised by $[\lambda_1, \dots, \lambda_N]$
$\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$.

$E_{ij} |HWS\rangle =$ ~~$\lambda_i \delta_{ij}$~~ 0, $i > j$
$E_{ii} |HWS\rangle = \lambda_i |HWS\rangle$.

Also a $|LWS\rangle$

$E_{ij} |LWS\rangle = 0$, $i < j$
$E_{ii} |LWS\rangle = \lambda_{N+1-i} |LWS\rangle$.

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* Bijection between states in ~~NN~~ finite-dim irreps. and GT patterns.

$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots \geq \lambda_N$

$\lambda_{11} \geq \lambda_{22} \geq \lambda_{33} \dots$

$\lambda_{21} \geq \lambda_{32}$

λ_{31} ...

(with $\lambda_1 \geq \lambda_{11}$, $\lambda_2 \geq \lambda_{22}$, $\lambda_{11} \geq \lambda_{21}$, $\lambda_{33} \geq \lambda_{32}$, $\lambda_{21} \geq \lambda_{31}$, $\lambda_{32} \geq \dots$)

all $\lambda_{ij} \in \mathbb{Z}_{\geq 0}$

__Example__ $[2,0]$ rep of $gl(2)$: "spin 1 rep". 3 states. $\{1, 0, -1\}$

$2 \quad 0$

* → $\begin{matrix} 2 & & 0 \\ & 2 & \end{matrix}$ $\quad \begin{matrix} 2 & & 0 \\ & 1 & \end{matrix}$ $\quad \begin{matrix} 2 & & 0 \\ & 0 & \end{matrix}$

For higher rank, more intricate ways to fill up.

10.

* Diagonalising B. ~ 10.

* Eigenvalues of $B(u)$ are related to GT patterns.

$$\langle x | B(u) = \prod_{\alpha=1}^{L} \prod_{i,j \in GT} (u - \theta_\alpha - \hbar \lambda_{ij}^{\alpha}) \langle 1 |$$

λ_{ij}^{α} form a GT pattern.

We now show how to build eigenvectors of B.

For simplicity, we restrict to reps

$$[\underbrace{S, \dots, S}_{A}, 0, \dots, 0]$$

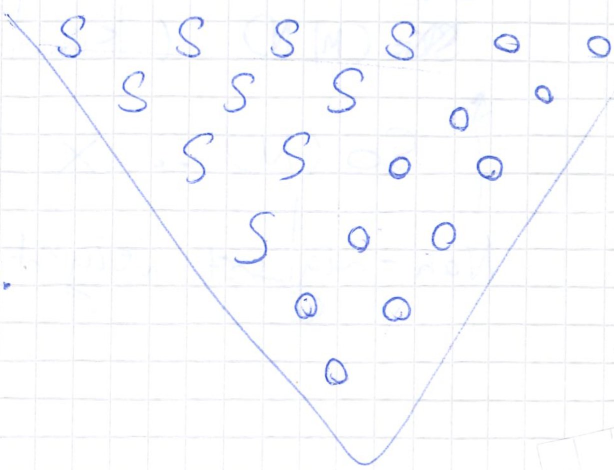
GL(6)

S S S S 0 0
S S S * *
S S * *
S * *
* *
*

$\langle 1^{\alpha} |$

$\langle 0 |$ SoV vacuum.

$\langle 0 | T$  fills by adding boxes.



11. TQ: ~~We saw~~ Now an N-th order difference eqn.

$Q_1, \ldots, Q_N$.

$$T_\lambda(\theta_\alpha) = \frac{\det_{1 \le j,k \le A} Q_k(\theta_\alpha^{th} + (\lambda_j - j + 1))}{\det Q_k(\theta_\alpha^{th}(-j+1))}$$

⇒ Wave function is a product of dets.

Finally; more general reps require ratios of transfer matrices.

But end result is very simple.

Product of ascending dets.

__Outlook:__ HW reps (even infinite-dim). ✓ GL(N)

$\langle \Psi_A | \Psi_B \rangle$ ~ (new tools) (✓ + lecture?)

GL(M|N) ~~SU~~ (kind of ...)(unsatisfactory...).

↑ SO(N) ... × (Sklyanin).

Non-highest weight: (T → Q).