Q-operators from quantum group representations

Chris Raymond

Abstract

Notes for the talk given in the ZMP seminar, Winter semester 24/25. These notes are not self-contained. Significant amounts of background material are assumed and were discussed in earlier seminars. Typos may certainly be present. Statements should be checked against the references below. The idea of the talk was to give students an introduction to the representation theory of quantum groups at generic q, and then describe conceptually how to extract information about quantum integrable systems from representation theoretic ideas.

For references, I used "A guide to quantum groups" by Chari & Pressley, as well as "Lectures on Quantum Groups" by Jantzen, and the lecture notes arXiv:2106.05252 [math.QA] by Etingof and Semenyakin. I also found the papers of Hernandez very useful, but they are mathematically heavy.

Introduction

Given some lattice model on an $M \times N$ lattice, we attach to each site of the lattice a vector space V that is often given additional structure. In this case, and many others, we want this vector space to also be a Hopf algebra representation.

The goal is to construct a commuting family of parameter-dependent matrices $t(z)$, the transfer matrices, that act as elements of $\mathrm{End}(V^{\otimes N})$. The idea being that one can determine all states of the integrable system by action of the transfer matrices on a reference state.

Suppose we are given an R-matrix $R \in \mathrm{End}(V \otimes V)$ that solves the Yang-Baxter equation

$$R_{12}(z)R_{13}(w)R_{23}(z-w) = R_{23}(z-w)R_{13}(w)R_{12}(z),$$

where the indices label the spaces being acted on in the standard way. Then, we can construct an L-operator $L(z) \in \mathrm{End}(V \otimes V^{\otimes N})$ given by

$$L(z) = R_{01}(z)R_{02}(z)\cdots R_{0N}(z),$$

satisfying

$$R_{12}(z-w)L_1(z)L_2(w) = L_2(w)L_1(z)R_{12}(z-w)$$

as operators in $\mathrm{End}(V \otimes V \otimes V^{\otimes N})$.

From these L-operators, we define a transfer matrix as

$$t(z) = \mathrm{tr}_{V_0} L(z) \in \mathrm{End}(V^{\otimes N}).$$

The physical quantity of interest is the partition function of the lattice. This is given by

$$\mathcal{Z} = \mathrm{tr}_{V^{\otimes N}} t(u)^M.$$

To compute such a trace, we need knowledge of the eigenvalues of $t(z)$. This, in essence, is the central question we attempt to answer.

In his study of the eight vertex model, where the vector spaces attached to the sites are two-dimensional, Baxter showed that the eigenvalues of $t(z)$ on eigenvectors constructed via the Bethe ansatz have the form

$$\lambda(z) = A(z)\frac{Q(zq^2)}{Q(z)} + D(z)\frac{Q(zq^{-2})}{Q(z)},$$

which we have seen before in this seminar. The functions $A(z)$ and $D(z)$ are essentially universal, and so really the eigenvalues are controlled by the function $Q(z)$. Our goal will be to see that these functions are given by traces of particular operators, called Q-operators, on quantum group representations. Moreover, we want to motivate their construction from an R-matrix.

Quantum groups

We won't have time to give an exhaustive introduction to quantum groups. As such, we will focus heavily on a "very nice" example, namely $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$. Historically speaking, quantum groups were developed to provide a mathematical structure that produces R-matrices solving the Yang-Baxter equation. However, a uniform mathematical definition of "quantum group" is elusive. The Drinfeld-Jimbo quantum groups arise as quantum deformations of universal enveloping algebras of Lie algebras.

We will work over $\mathbb{C}$, and we will treat $\mathsf{q} \in \mathbb{C}$ as generic (not a root of unity). As an associative unital algebra, $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ is generated by $\{e, f, k^{\pm 1}\}$ satisfying

$$kk^{-1} = k^{-1}k = 1, \qquad kek^{-1} = \mathsf{q}^2 e, \qquad kfk^{-1} = \mathsf{q}^{-2} f, \qquad ef - fe = \frac{k^2 - k^{-2}}{\mathsf{q} - \mathsf{q}^{-1}}.$$

Loosely speaking, we have exponentiated the Cartan element in $\mathsf{U}(\mathfrak{sl}_2)$, $h \mapsto k = \mathsf{q}^h$, thereby introducing a deformation parameter q to the structure of $\mathsf{U}(\mathfrak{sl}_2)$.

Importantly from a mathematical perspective, this algebra satisfies two very nice properties:

(a) $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ contains no nontrivial zero divisors, i.e. there is no non-zero element $a \in \mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ such that $ax = 0$ for some non-zero $x \in \mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$.

(b) $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ admits a PBW basis of the form

$$\{f^r k^n e^s \mid r, s \in \mathbb{Z}_{\geqslant 0},\ n \in \mathbb{Z}\}.$$

Using this, we can start to study highest-weight modules (I will use the terms modules and representations interchangeably). By looking at the defining relations, we see that $\mathsf{U}_\mathsf{q}(\mathfrak{b}) := \langle k^{\pm 1}, e \rangle$ defines a Borel subalgebra, and $\langle e \rangle$ generates a nilpotent subalgebra contained in the Borel.

We proceed to define highest-weight modules in the usual way. Let $\bar{M}$ be the trivial 1-dimensional representation of the nilpotent subalgebra generated by e. That is, $\bar{M} = \operatorname{span}_\mathbb{C}\{v_0\}$, then we have that the equation $ev_0 = 0$ totally determines the representation of this subalgebra. We can upgrade (induce) this to a representation of $\mathsf{U}_\mathsf{q}(\mathfrak{b})$ by choosing an action of k on v_0. Let $\lambda \in \mathbb{C}$, and denote by $\bar{M}_\lambda := \operatorname{span}\{v_0\}$, where $ev_0 = 0$ and $k^{\pm 1} v_0 = \lambda^{\pm 1} v_0$.

We now have a 1-parameter family of modules over the Borel subalgebra. Any such module $\bar{M}_\lambda$ can finally be lifted to a $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$-module by inducing again. Formally, we define

$$M_\lambda := \operatorname{Ind}_{\mathsf{U}_\mathsf{q}(\mathfrak{b})}^{\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)} \bar{M}_\lambda = \mathsf{U}_\mathsf{q}(\mathfrak{sl}_2) \otimes_{\mathsf{U}_\mathsf{q}(\mathfrak{b})} \mathbb{C} v_0.$$

If this notation is unfamiliar, the tensor product "over" $\mathsf{U}_\mathsf{q}(\mathfrak{b})$ means that we can take elements of $\mathsf{U}_\mathsf{q}(\mathfrak{b})$ past the tensor product and act them on v_0.

Aside: This is known as the induced module construction and is a common way of constructing modules for many algebras. One sees it most often . In the seminar there was a helpful question asking why we go through this effort to define a highest-weight module, when the usual notion one sees is so simple to write down. This is done to ensure the existence of the resulting module. By starting with the 1-dimensional module, the resulting infinite-dimensional module will exist. This is important because we cannot simply check existence by determining the matrices in the infinite dimensional case.

Using the PBW theorem, every element of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ can be written as a sum of monomials $f^r k^n e^s$. The tensor product over $\mathsf{U}_\mathsf{q}(\mathfrak{b})$ means we can pull any powers of e or k through to act on v_0, and we know that action. Using this, we see that M_λ has a basis given by $\{v_r := f^r v_0 \mid r \in \mathbb{Z}_{\geqslant 0}\}$, in particular, M_λ is infinite-dimensional. We get additional facts:

(a) Weight decomposition: $M_\lambda = \bigoplus_\mu M_{(\mu)}$ where

$$M_{(\mu)} = \{v \in M_\lambda \mid kv = \mu v\}.$$

(b) Each weight space $M_{(\mu)}$ is 1-dimensional.

The module action of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ on M_λ is determined from the the action given above for $k^{\pm 1}$ and e, and the defining relations of the algebra. We have that

$$kv_n = \lambda\mathsf{q}^{-2n}v_n, \qquad fv_n = v_{n+1}, \qquad ev_n = \begin{cases} 0, & \text{if } n = 0, \\ [n]\frac{\lambda\mathsf{q}^{1-n} - \lambda^{-1}\mathsf{q}^{n-1}}{\mathsf{q}-\mathsf{q}^{-1}}v_{n-1}, & \text{otherwise,} \end{cases}$$

where

$$[n] = \frac{\lambda\mathsf{q}^n - \lambda^{-1}\mathsf{q}^{-n}}{\mathsf{q}-\mathsf{q}^{-1}}$$

[[draw a picture of the strand of vectors descending from the highest weight state with module action of the generators]]

The modules M_λ are indecomposable, but they are not all irreducible.

If $\lambda = \pm\mathsf{q}^n$ for some $n \in \mathbb{Z}_{\geqslant 0}$, then it is reducible. That is, there is a proper submodule, and the module does not decompose as a direct sum.

One can (and should) verify that if $\lambda = \pm\mathsf{q}^n$, then $ev_{n+1} = 0$. That is, the vector $v_{n+1} \in M_{\mathsf{q}^n}$ generates a submodule that is isomorphic to $M_{\mathsf{q}^{-n-1}}$. The quotient module $V_n \cong M_{\mathsf{q}^n}/M_{\mathsf{q}^{-n-1}}$ is a finite-dimensional irreducible representation of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$.

[[using the picture from before, show the submodule and how the quotient module is then something finite dimensional]].

The module V_n has a basis $\{f^i v_0 \mid i = 0, \ldots, n\}$, which implies $\dim(V_n) = n+1$.

In fact, all finite-dimensional $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$-modules are isomorphic to V_n for some n, or a twist of some V_n where the action of the algebra is modified by the automorphism $k \mapsto -k,\ e \mapsto -e,\ f \mapsto f$. What we see here is that finite-dimensional irreps of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ are essentially the same as those of $\mathsf{U}(\mathfrak{sl}_2)$ (there are two non-isomorphic finite "strand modules", rather than one, per positive integer).

Furthermore, we have the following:

(a) One can prove that finite-dimensional $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$-modules are completely reducible.

(b) $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$ is a Hopf algebra. This implies significantly more structure: coproduct, counit, antipode. This means that the category of finite-dimensional modules is something called a braided tensor category. This is a fascinating theory but is outside the scope of the talk.

Recalling our motivation, the key thing we wanted was an R-matrix. For two finite-dimensional modules V, W, we should have some $R_{V,W}$ that defines an isomorphism of $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$-modules

$$R_{V,W} : V \otimes W \to W \otimes V.$$

In an ideal world, we would have $R \in \mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$. However, this turns out not to be the case. For $\mathsf{U}_\mathsf{q}(\mathfrak{sl}_2)$, Drinfeld constructed

$$R = \sum_{n=0}^{\infty} \frac{\mathsf{q}-\mathsf{q}^{-1}}{[n]!}\mathsf{q}^{n(n-1)/2}(fk \otimes k^{-1}e)^n \mathsf{q}^{2(h\otimes h)} \in \overline{\mathsf{U}_\mathsf{q}(\mathfrak{b}^-) \otimes \mathsf{U}_\mathsf{q}(\mathfrak{b}^+)},$$

where $\mathfrak{b}^+$ is the positive Borel we defined earlier, and $\mathfrak{b}^-$ is the opposite Borel, using f instead of e. The overline denotes the algebraic completion as we neglect questions of convergence of the infinite series and simply consider an enlarged setting.

This seems bad but R defined as above is a finite series when acting on any tensor product of finite-dimensional modules. This implies that for any two such modules, R has a matrix realisation.

Suppose we take two copies of the 2-dimensional representation V_1, known as the fundamental representation. We consider $V_1 \otimes V_1$. This space is 4-dimensional. Relative to the basis $\{v_0 \otimes v_0,\ v_0 \otimes v_1,\ v_1 \otimes v_0,\ v_1 \otimes v_1\}$, we can express R as the matrix

$$R_{V_1,V_1} = \begin{pmatrix} \mathsf{q}^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \mathsf{q}^{-1} - \mathsf{q} & 1 & 0 \\ 0 & 0 & 0 & \mathsf{q}^{-1} \end{pmatrix}.$$

One can verify that this indeed satisfies all the properties expected of the R-matrix. However, we also immediately verify that it is not dependent on a spectral parameter...

Affine quantum groups

To obtain an R-matrix that depends on a spectral parameter, we need to generalise our setting to affine quantum groups. Recall the affine Lie algebra corresponding to $\mathfrak{sl}(2)$, which we denote by $\widehat{\mathfrak{sl}}_2 = \mathfrak{sl}_2 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathsf{k}$. We want to define some notion of $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$.

For $\mathsf{U_q}(\mathfrak{sl}_2)$, we used essentially the Chevalley generators of the Lie algebra, and there is a general presentation of $\mathsf{U_q}(\mathfrak{g})$ using these generators. There is an identical description for $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$, given by introducing the zeroth, or imaginary root, in the usual way. However, we will use a slightly different choice of basis to define this algebra, known as the loop presentation.

The algebra $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$ is generated by $\{x_r^{\pm},\ h_s,\ k^{\pm 1},\ c^{\pm 1/2} \mid r \in \mathbb{Z},\ s \in \mathbb{Z} \setminus \{0\}\}$, satisfying that $c^{\pm 1/2}$ is central, and that

$$kk^{-1} = k^{-1}k = 1, \qquad k^{\pm 1} h_s = h_s k^{\pm 1}, \qquad k x_r^{\pm} k^{-1} = q^{\pm 2} x_r^{\pm}$$

$$[h_s, x_r^{\pm}] = \pm \frac{1}{s}[2s] c^{\mp |r|/2} x_{r+s}^{\pm},$$

$$[x_r^+, x_s^-] = \frac{c^{(r+s)/2}\phi_{r+s}^+ - c^{-(r-s)/2}\phi_{r+s}^-}{\mathsf{q} - \mathsf{q}^{-1}},$$

along with further relations that are given in an explicit form in [arXiv:1104.1891 [math.QA]]. We should think of $c \propto \mathsf{q}^{\mathsf{k}}$, where k is the central element in $\widehat{\mathfrak{sl}}_2$. We also introduced the modes $\phi_{\pm r}^{\pm}$ are determined by comparison of formal series with the expansion

$$\sum_{r=0}^{\infty} \phi_{\pm r}^{\pm} u^{\pm r} = k^{\pm 1} \exp\left(\pm (\mathsf{q} - \mathsf{q}^{-1}) \sum_{s=1}^{\infty} h_{\pm s} u^{\pm s} \right).$$

Let $\widehat{\mathsf{U}}^{\pm}$ (resp. $\widehat{\mathsf{U}}^0$) be the subalgebra generated by $x^{\pm r}$ (resp. $\phi_r^{\pm}$) for $r \in \mathbb{Z}$. Then, we have that $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2) = \widehat{\mathsf{U}}^- . \widehat{\mathsf{U}}^0 . \widehat{\mathsf{U}}^+$.

We are going to limit ourselves to considering representations where $c^{\pm 1/2}$ acts as the identity. This time, we can skip the details that we included for $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$ and simply write down how we want a highest-weight representation to work.

Depending on your familiarity with the construction, it might be a useful exercise to go through the same steps that we used to define the infinite-dimensional $\mathsf{U_q}(\mathfrak{sl}_2)$-modules but in this new setting.

We will say a $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$-module M_Φ is highest-weight if there is a highest weight vector $v \in M_\Phi$ that satisfies

$$x_r^+ v = 0, \qquad \phi_r^{\pm} v = \Phi_r^{\pm} v, \qquad c^{\pm 1/2} v = v$$

for some complex numbers $\Phi = (\Phi_1^+,\ \Phi_{-1}^-,\ \Phi_2^+,\ \ldots)$. We have that

$$\Phi_r^+ = 0,\ \text{for } r < 0, \qquad \Phi_r^- = 0,\ \text{for } r > 0, \qquad \Phi_0^+ \Phi_0^- = 1.$$

[[Draw a picture of the module acting on the highest-weight state]]

These modules M_Φ are isomorphic to the quotient $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)/\langle x_r^+,\ \phi_r^{\pm} - \Phi_r^{\pm}.1,\ c^{\pm 1/2} - 1 \rangle$.

As $\mathsf{U_q}(\mathfrak{sl}_2) \subset \mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$, we can view M_Φ as a $\mathsf{U_q}(\mathfrak{sl}_2)$-module. it will have a rather complicated decomposition, but we can show that the weight spaces are 1-dimensional.

Similarly to the non-affine case, the module M_Φ is not necessarily irreducible and has a unique irreducible quotient, which we will denote by V_Φ. Every irreducible highest-weight $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$-module is isomorphic to some V_Φ for an appropriate choice of Φ.

These modules are all still infinite-dimensional. However, there is a way of obtaining finite-dimensional $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$-modules from V_Φ. We use the evaluation map, which is a parameter-dependent ($z \in \mathbb{C}$) linear map

$$\mathrm{ev}_z : \mathsf{U_q}(\widehat{\mathfrak{sl}}_2) \to \mathsf{U_q}(\mathfrak{sl}_2)$$

defined by

$$\mathrm{ev}_z(x_r^+) = \mathsf{q}^{-r} z^{-r} k^r e, \qquad \mathrm{ev}_z(x_r^-) = \mathsf{q}^{-r} z^{-r} f k^r,$$

for all $r \in \mathbb{Z}$.

Note that ev_z acts as the identity on $\mathsf{U_q}(\mathfrak{sl}_2) \subset \mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$. This collapses the picture we drew on the board to something finite-dimensional, but scaled w.r.t q in such a way that the loop structure of $\widehat{\mathfrak{sl}}_2$ is not thrown away.

If we think about this carefully, what we have done is give a way of lifting the $\mathsf{U_q}(\mathfrak{sl}_2)$-modules V_n to $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$-modules, by pulling back along the map ev_z. As these modules are related to the evaluation map (which is evaluating the variable t in $\widehat{\mathfrak{sl}}_2$ at some value of z), we creatively call them *evaluation modules*, and denote them by $V_n(z) =: \mathrm{ev}_z(V_\Phi)$, for some Φ.

The image of $v_0 \in V_n$ is a highest-weight vector, satisfying

$$\phi_s^{\pm} v_0 = z^s \mathsf{q}^{s(n-1)}(\mathsf{q}^n - \mathsf{q}^{-n}) v_0, \qquad h_s v_0 = z^{-s} \mathsf{q}^{-s} \frac{[ns]}{s} v_0,$$

and we get a basis for $V_n(z)$ given by $\{v_0, \ldots, v_n\}$, where

$$k v_m = \mathsf{q}^{n-2m} v_m, \qquad x_1^+ v_m = [n-m+1] v_{m-1}, \qquad x_1^- v_m = [m+1] v_{m+1}.$$

These relations are sufficient to determine the module action.

To each evaluation representation, we can attach a polynomial (known as a Drinfeld polynomial) in some complex variable u. Let $P_{n,z}(u)$ be the polynomial corresponding to $V_n(z)$, defined by

$$P_{n,z}(u) = \prod_{s=1}^{n} (1 - z^{-1} \mathsf{q}^{n-2s+1} u)$$

It is convenient to introduce notation $Y_{z^m \mathsf{q}^n} = (1 - z^{-m} \mathsf{q}^{-n} u)$.

The Drinfeld polynomials can in fact be used to classify finite-dimensional $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$-modules. The set of roots of any given polynomial is called a q-segment of length n and centre z. While we will make use of these polynomials shortly when talking about characters of modules, we won't delve into the theory of q-segments and their centres, but I mention them here because these terms appear widely in the literature in this area.

Tensor products of evaluation modules produce almost all finite-dimensional representations of $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$ of type 1, which is the largest class of finite-dimensional modules for an affine quantum group. Type 1 is a technical criterion that requires $c^{\pm 1/2}$ to act as 1, as well as some finiteness conditions. The Drinfeld polynomial of a tensor product of evaluation modules is the product of their Drinfeld polynomials.

Now, $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$ also has a universal R-matrix associated with it, that again does not live in $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$ but rather an algebraic completion of the tensor product of Borel subalgebras. That is, it is again some complicated infinite series in the generators. Closed and recursive formulas for the R matrix are particularly ugly but can be found in the literature. However, using the same logic as before, we can attempt to evaluate such an infinite series on a tensor product of finite-dimensional representations where it truncates to something finite or is convergent.

Note: We need to take some care here. In the $\mathsf{U_q}(\mathfrak{sl}_2)$ world, our finite-dimensional representations were completely reducible. This means that the action of any R-matrix on a tensor product of finite-dimensional representations was well-behaved. This is not true for the evaluation representations of $\mathsf{U_q}(\widehat{\mathfrak{sl}}_2)$. In this setting, although the tensor product will again be finite-dimensional, it may be reducible but indecomposable. One can show that, in cases where this happens, the R-matrix formal series does not necessarily truncate to a finite series, and furthermore may not converge.

We will again consider fundamental (evaluation) representations $V_1(z) \times V_1(w)$, which are 2-dimensional. In this case, we can compute that

$$R_{V_1 \otimes V_1}(z/w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\mathsf{q}(z/w-1)}{z/w-\mathsf{q}^2} & \frac{1-\mathsf{q}^2}{z/w-\mathsf{q}^2} & 0 \\ 0 & \frac{z/w(1-\mathsf{q}^2)}{z/w-\mathsf{q}^2} & \frac{\mathsf{q}(z/w-1)}{z/w-\mathsf{q}^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we change variables $z \mapsto \mathsf{e}^u,\ w \mapsto \mathsf{e}^v$, then we have an R-matrix $R(u-v)$ with spectral parameter.

Using this, we can define transfer matrices and so on. Suppose we have some transfer matrix, how to get the spectrum?

Given a weight module over an algebra, a character is a series in a formal variable that captures information about the dimension of the weight spaces (multiplicities of the weights). This is usually constructed by choosing a formal variable (we will use q), and exponentiating elements of the Cartan subalgebra (the algebra that acts diagonally on the weight spaces) using this variable, then taking a trace over the module. The result is a series in q, where powers of the variable q give the weights, and the coefficient describes the dimension of the corresponding weight space.

If $V_n(z)$ is an irreducible $\mathsf{U}_\mathsf{q}(\widehat{\mathfrak{sl}}_2)$-module with polynomial

$$m_+ = \prod_{s=1}^{n} Y_{z\mathsf{q}^{n-2s+1}}.$$

The corresponding q-character is the series

$$\chi_\mathsf{q}(V_n(z)) = m_+ \sum_{i=0}^{n} \prod_{j=1}^{i} A^{-1}_{z\mathsf{q}^{n-2j+2}}, \text{ where } A_{z\mathsf{q}} = Y_{z\mathsf{q}^{-1}} Y_{z\mathsf{q}}.$$

A surprising result by Frenkel and Hernandez [arXiv:1308.3444 [math.QA]] is that since the transfer matrix $t(z)$ acts diagonally on the highest-weight state, and has nice commutation relations with the generators of the algebra, then we can write the eigenvalues of $t(z)$ on a given irreducible representation, say W, as a change of variables of the q-character of W. For the fundamental representation $V_1(z)$, we have that

$$\chi_\mathsf{q}(V_1(z)) = Y_{z\mathsf{q}} + Y^{-1}_{z\mathsf{q}^{-1}}.$$

Frenkel and Hernandez then give a concrete construction of a function $Q(z)$, such that by sending

$$Y_z \mapsto \frac{Q(z\mathsf{q}^{-1})}{Q(z\mathsf{q})},$$

one can transform the character into an expression for the eigenvalues of $t(z)$. That is, for the fundamental representation, we have that the eigenvalues are

$$\lambda(z) = A(z)\frac{Q(z\mathsf{q}^2)}{Q(z)} + D(z)\frac{Q(z\mathsf{q}^{-2})}{Q(z)},$$

up to the coefficient functions $A(z)$ and $D(z)$, which are also determined from the representation theory directly.

Now, where does the TQ relation above actually come from? That is, we have an equation

$$\lambda(z)Q(z) = A(z)Q(z\mathsf{q}^2) + D(z)Q(z\mathsf{q}^{-2}),$$

but this we should think of as coming from a trace over operators $t(z)\hat{Q}(z)$.

The Q-functions are given by traces over Q-operators, which themselves are constructed by taking a trace over an R-matrix, but where we take an infinite-dimensional representation known as an oscillator representation. These require more time than we have to go into, but if we are careful, we can take the tensor product of an oscillator representation with a finite-dimensional irreducible, and still obtain a meaningful R-matrix action that we can trace over to obtain an operator. More information on oscillator representations can be found in arXiv:2412.14811 [math-ph] and related references.