1.

Functional SoV

- What is "Functional"?
- Is it different from SoV à la Sklyanin?
- What is it good for? (a lot!)

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* Review of a previous IMP

$SU(2)$ XXX spin $1/2$ chain

R-matrix $R(u) = u - \hbar P$

$$\mathcal{L}(u) = \begin{pmatrix} u - \hbar E_{11} & -\hbar E_{21} \\ -\hbar E_{12} & u - \hbar E_{22} \end{pmatrix} \quad \text{Lax}$$

$$= u\mathbb{I} - \hbar \mathbb{E}^t$$

$$E_{11} = 1/2 + S_z, \quad E_{22} = 1/2 - S_z$$
$$E_{12} = S^+, \quad E_{21} = S^-.$$

Quantum monodromy matrix

$$M(u) = \mathcal{L}_1(u - \theta_1) \ldots \mathcal{L}_L(u - \theta_L)\, G$$

twist.

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

inhomogeneity $\in \mathbb{C}$. $\theta_\alpha - \theta_\beta \notin \hbar\mathbb{Z}$.

$G = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 1/\lambda_1 \end{pmatrix}$

$$t(u) = \mathrm{tr}\, M = A + D.$$

$$[t(u), t(v)] = 0 \Rightarrow \text{commuting quantities.}$$

* Spectrum of $t(u)$ in Baxter eqn

$$Q_\theta(u) = \prod_{\alpha=1}^{L} (u - \theta_\alpha)$$

$$Q_\theta(u - \hbar)\, Q(u + \hbar) - t(u)\, Q(u) + Q_\theta(u)\, Q(u - \hbar) = 0$$

2. We constructed a dual basis

$$\langle x| = \langle \theta_1 + n_1 \hbar, \ldots, \theta_L + n_L \hbar| = \langle x_1, \ldots, x_L| \quad , \quad n_\alpha \in \{0, 1\}$$

$$\langle x | \Psi \rangle = \prod_{\alpha=1}^{L} Q(x_\alpha) \qquad \text{SoV basis.}$$

Ψ: eigenvectors of $t(u)$

$\langle x|$ built using method of Maillet, Niccoli.

$$\langle x| = \langle W| \prod_{\alpha=1}^{L} \left(\frac{t(\theta_\alpha)}{Q_\theta(\theta_\alpha - \hbar)} \right)^{n_\alpha}$$

$\langle W|$: reference state.

$$\Rightarrow \langle x | \Psi \rangle = \prod_{\alpha=1}^{L} \left(\frac{t(\theta_\alpha)}{Q_\theta(\theta_\alpha - \hbar)} \right)^{n_\alpha} \langle W | \Psi \rangle$$

$\Rightarrow$ ~~con~~ normalise $|\Psi\rangle$ s.t. $\langle W|\Psi\rangle = \prod_{\alpha=1}^{L} Q(\theta_\alpha)$

\+ use TQ

$$t(\theta_\alpha) = Q_\theta(\theta_\alpha - \hbar) \frac{Q(\theta_\alpha + \hbar)}{Q(\theta_\alpha)}$$

$$\Rightarrow \langle x | \Psi \rangle = \prod_{\alpha=1}^{L} Q(\theta_\alpha + n_\alpha \hbar).$$

This completes the review.

Modulo mild technical details, very general method.

["so where is the trouble?"]

Answer : eigenvectors of $t(u)$ does not buy us much.

* What do we typically try to compute in QM?

(1) – spectrum of conserved quantities, e.g. E.

(2) – wave function ~~useless on its own~~

$\Downarrow$

(3) – expectation values, correlation functions.

eg in spin chain $\langle \Omega | O(n) | \Omega \rangle$

one-point fn

$O(n)$: local operator on site n.

Ω: vacuum

or include excited states,

or say $\langle \Psi_* | \{A, B, C, D\} | \Psi' \rangle$.

[wave function useless on its own.
Used as ingredient in other objects

but also need scalar product on space of states.

3. (1) - powerful Q-operator methods (see previous IMP).

(2) - we have this ✓

(3) - need scalar produce.

Have $\langle x|$ s.t. $\langle x|\Psi\rangle$ nice.

Need $|x\rangle$ s.t. $\langle\Psi|x\rangle$ nice.

to Compute scalar prod

$$\langle\Psi'|\Psi\rangle = \sum_{x,x'} \langle\Psi'|x'\rangle\langle x|\Psi\rangle M_{x,x'}.$$

$|x'\rangle$ and $\langle x|$ need not be related.

We would like to make $M_{x,x'}$ simple,

e.g. $M_{x,x'} \sim \delta_{xx'}$.

But how?.

$$\langle x| \sim \langle W| \prod_{\alpha=1}^{L} \left(\frac{t(\theta_\alpha)}{N_\alpha}\right)^{n_\alpha}$$

how to choose

$$|x'\rangle \sim \prod_{\alpha=1}^{L} \left(\frac{t(w_\alpha)}{N'_\alpha}\right)^{n_\alpha} |w'\rangle.$$

Can directly compute $\langle x|x'\rangle$, but long and tedious.

* Functional SoV to the rescue.

4. <u>Laguerre Polynomials</u>

$$xy'' + (1-x)y' + ny = 0 \quad , \quad n \in \mathbb{Z}_{\geq 0}$$

Polynomial solutions $P_n(x)$, $P_0(x) = 1$

$P_1(x) = -x+1$

$\vdots$

Polynomials orthogonal wrt inner product

Define linear form $\langle f \rangle = \int_0^\infty dx \, e^{-x} f(x)$

$\Rightarrow \langle P_n P_m \rangle \propto \delta_{nm}$.

- View as QM system – n – energy levels
  $P_n(x)$ – wave function in basis x.
  inner product on space of states. ~~scalar~~

- How to construct scalar product?
  Rewrite ODE as linear operator

$$O_n = x \frac{d^2}{dx^2} + (1-x)\frac{d}{dx} + n \quad \Rightarrow \quad O_n P_n = 0.$$

Now look for linear forms s.t. O_n is self-adjoint.

$$\langle f \rangle = \int_\gamma dx \, \mu(x) f(x)$$

such that

contour γ
measure μ

$$\langle f \, O_n g \rangle = \langle g \, O_n f \rangle.$$

Requirements on μ, γ : integral should converge for all Polynomials.
+ self-adjointness.
(= integration by parts).

Now consider

$$0 = \langle P_n (O_n - O_m) P_m \rangle$$
$$= (n-m) \langle P_n P_m \rangle$$

$\Rightarrow$ for $n \neq m$, $\langle P_n P_m \rangle = 0$

$\Rightarrow$ $\langle P_n P_m \rangle \propto \delta_{nm}$

$\Rightarrow$ scalar product such that two solutions orthogonal.

Now we generalise to integrable spin chain.

5. Baxter eqn for state $|\Psi_A\rangle$

$$Q_\theta(u-\hbar)\,Q_A(u+\hbar) - t_A(u)\,Q_A(u) + Q_\theta(u)\,Q_A(u-\hbar) = 0.$$

Introduce linear operator

$$\mathcal{O}_A = Q_\theta(u-\hbar)\,D^2 - t_A(u) + Q_\theta(u)\,D^{-2}$$

$$D^{\pm 2} f(u) = f(u \pm \hbar).$$

Now we repeat the same games.

Want

$$\langle f \rangle = \int_\gamma du\, \mu(u)\, f(u)$$

s.t. $\langle f\, \mathcal{O}\, g \rangle = \langle g\, \mathcal{O}\, f \rangle.$

⚠ Finite-diff eqn, not ODE.

=> Integration by parts replaced by contour deformation requirement.

More precisely

$$\langle f\, Q_\theta\, D^{-2} g(\cancel{u}) \rangle = \langle g\, Q_\theta(u-\hbar)\, D^2 \cancel{g} (u) \rangle.$$

$$\int_\gamma du\, \mu(u)\, f(u)\, Q_\theta(u)\, g(u-\hbar)$$

$$= \int_\gamma du\, \mu(u)\, g(u)\, Q_\theta(u-\hbar)\, f(u+\hbar).$$

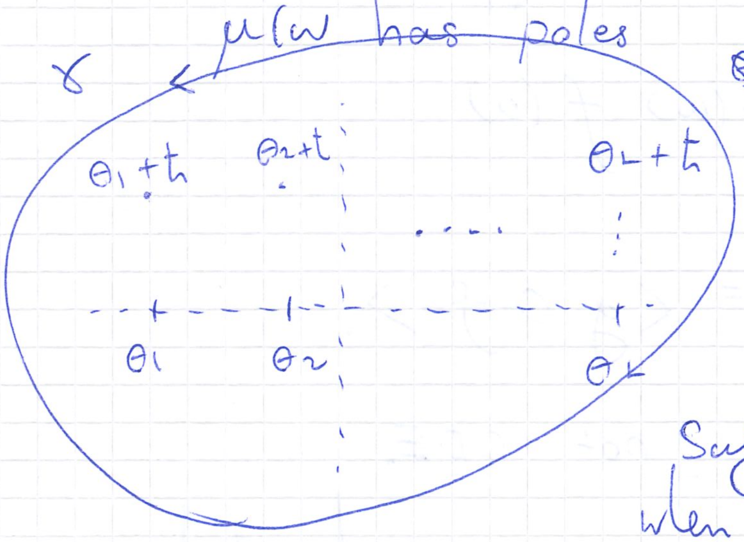
Suppose we can freely deform contour.

$\Rightarrow$ Require: $\mu(u+\hbar)\, Q_\theta(u+\hbar) = \mu(u)\, Q_\theta(u-\hbar)$

solved by $\mu(u) = \dfrac{\rho(u)}{Q_\theta(u-\hbar)\, Q_\theta(u)}$ ← periodic in $\hbar$.

Assume $\rho(u)$ entire.

Now what about contour?



~~θ₁,...θ_L+ħ~~

Just choose γ to be a large circle.

Sufficiently large so that when we deform contour we never cross poles in measure.

$\Rightarrow \langle f \rangle_\rho = \oint du \, \dfrac{\rho(u)}{Q_\theta(u)\, Q_\theta(u-\hbar)} f(u).$

– Notice it still depends on ρ.

6. Now proceed as in Laguerre.

$$\langle Q_A (O_A - O_B) Q_B \rangle = 0$$

$$= \langle Q_A (t_B(u) - t_A(u)) Q_B \rangle = 0.$$

$t(u) = \text{poly} \Rightarrow$ cannot just pull out as in Laguerre.

For current model

$$t_A(u) = \left(\lambda + \frac{1}{\lambda}\right) u^L + \sum_{\beta=1}^{L} u^{\beta-1} \underset{\text{integrals of motion}}{I_\beta^A}$$

$\Rightarrow$ get $\displaystyle\sum_{\beta=1}^{L} \langle Q_A u^{\beta-1} Q_B \rangle (I_\beta^A - I_\beta^B) = 0.$

Now we choose ρ : cancel some poles

$$\rho_\alpha(u) = \prod_{\beta \neq \alpha} \left(1 - e^{\frac{2\pi i}{\hbar}(u - \theta_\beta)}\right)$$

$$\langle f \rangle_\alpha = \oint du \frac{\rho_\alpha(u)}{Q_\theta(u-\hbar) Q_\theta(u)} f(u)$$

$$\sim \# f(\theta_\alpha) + \# f(\theta_\alpha + \hbar)$$

by residues.

$\Rightarrow L$ such $\langle f \rangle_\alpha$.

$\Rightarrow$ Linear system

$$\sum_{\alpha=1}^{L} \langle Q_A u^{\beta-1} Q_B \rangle_\alpha \, (I^A_\beta - I^B_\beta) = 0$$

$$M_{AB} \cdot I = 0 \, ,$$

$$(M_{AB})_{\alpha\beta} = \langle Q_A u^{\beta-1} Q_B \rangle_\alpha$$

$$(I)_\beta = I^A_\beta - I^B_\beta \, .$$

$t(u)$ non-degnen spectrum

$\Rightarrow$ For $A \neq B$, at least one $I^A_\beta - I^B_\beta \neq 0$

$\Rightarrow$ $\det M_{A\beta} \propto \delta_{AB}$.

$\Rightarrow$ This provides a natural scalar product.

$$\det M_{AB} = \oint du_1 \ldots du_L \prod_{\beta > \alpha}^{\mu_1(u) \ldots \mu_L(u)} (u_\beta - u_\alpha) \times \prod_{\alpha=1}^{L} Q_A(u_\alpha) \, Q_B(u_\alpha) \, .$$

$$= \sum_{n_\alpha \in \{0,1\}} M_{n_1, \ldots, n_L} \underbrace{Q_A(\theta_\alpha + n_\alpha \hbar)}_{\propto \langle \Psi^\dagger_B | x' \rangle} \boxed{Q_B(\theta_\alpha + n_\alpha \hbar)}_{\langle x | \Psi \rangle} .$$

$$\sim \langle \Psi_A | \Psi_B \rangle$$

$\Rightarrow$ naturally given $|x'\rangle$ so that measure diagonal.

Hence "Functional" SoV.

7. Now we compute expectation values.

* Recall QM perturbation theory

– Hamiltonian depends on param. p, $H|\Psi\rangle = E|\Psi\rangle$

$$\frac{\langle\Psi|\frac{\partial H}{\partial p}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\partial E}{\partial p}.$$

In spin chain: $p = \theta_\alpha$, twist λ, ...

$$\mathcal{O}Q = 0 \quad , \text{ shift } p \to p + \delta p$$

$$\Rightarrow \quad (\mathcal{O} + \delta\mathcal{O})(Q + \delta Q) = 0$$

$$\Rightarrow \quad \langle Q(\mathcal{O} + \delta\mathcal{O})(Q + \delta Q)\rangle_\alpha = 0.$$

expand to first order

$$= \langle Q \underbrace{\mathcal{O} Q}_{=0} \rangle_\alpha + \langle Q\,\delta\mathcal{O}\,Q\rangle_\alpha + \langle \underbrace{Q\mathcal{O}}_{=0}\,\delta Q\rangle_\alpha = 0$$

$\Rightarrow$ In limit of small variation get

$$\langle Q\,\partial_p\mathcal{O}\,Q\rangle_\alpha = 0.$$

Now consider example, $p = \theta_\alpha$.

$$0 = Q_\theta(u-\hbar) D^{+2} - \underbrace{t(u)}_{\substack{\parallel \\ u^L(\lambda+\frac{1}{\lambda}) + \sum_{\beta=1}^{L} I_\beta u^{\beta-1}}} + Q_\theta(u) D^{-2}$$

$$\Rightarrow \partial_\theta 0 = Q_\theta'(u-\hbar) D^2 - \sum_{\beta=1}^{L} \partial_\theta I_\beta u^{\beta-1} + Q_\theta'(u) D^{-2}$$

$$\Rightarrow \sum_{\beta=1}^{L} \partial_\theta I_\beta \langle Q u^{\beta-1} Q \rangle_\alpha$$

$$= \langle Q Y Q \rangle_\alpha, \quad Y = Q_\theta'(u-\hbar) D^2 + Q_\theta'(u) D^{-2}.$$

Now have an inhomogeneous linear system solvable by Cramers rule.

$$\Rightarrow \partial_\theta I_\beta = \frac{\det \tilde{M}_\beta}{\det M} = \frac{\det \tilde{M}_\beta}{\langle \Psi | \Psi \rangle}$$

$$M_{\alpha\beta} = \langle Q u^{\beta-1} Q \rangle_\alpha$$

$$\Rightarrow \underline{\langle \Psi | \partial_\theta \hat{I}_\beta | \Psi \rangle} = \det \tilde{M}_\beta.$$

so what have we computed? What is $\partial_\theta \hat{I}_\beta$?

8. $$t(u) = u^L \operatorname{tr}(G) + u^{L-1} \sum_{\alpha=1}^{L} \left(-\hbar \operatorname{tr}(\mathbb{E}^{\alpha,t} G) - \theta_\alpha \operatorname{tr}(G)\right) + O(u^{L-2}).$$

$$\left[\begin{array}{l} t(u) = \operatorname{tr}\left(\mathcal{L}_1(u-\theta_1) \cdots \mathcal{L}_L(u-\theta_L) G\right) \\ \mathcal{L}_\alpha(u) = u - \hbar \mathbb{E}^{\alpha,t}, \quad \mathbb{E}^\alpha = \begin{pmatrix} E^\alpha_{11} & E^\alpha_{12} \\ E^\alpha_{21} & E^\alpha_{22} \end{pmatrix}. \end{array}\right]$$

Now take $\dfrac{\partial}{\partial \theta_\alpha} t(u)$. $\to$ α'th site removed

$\mathcal{L}_\alpha(u-\theta_\alpha) \longrightarrow -1$

$$\Rightarrow \frac{\partial t(u)}{\partial \theta_\alpha} = -u^{L-1} \operatorname{tr}(G) - u^{L-2} \sum_{\beta \neq \alpha} \left(-\hbar \operatorname{tr}(\mathbb{E}^{\beta,t} G) - \theta_\beta \operatorname{tr}(G)\right)$$

$$\Rightarrow t(u) + u \frac{\partial t(u)}{\partial \theta_\alpha} = u^{L-1} \left(-\hbar \operatorname{tr}(\mathbb{E}^{\alpha,t} G) - \theta_\alpha \operatorname{tr}(G)\right)$$

$$t(u) = \operatorname{tr}(G) u^L + I_{L-1} u^{L-1} + I_{L-2} u^{L-2} + \ldots$$

$\Rightarrow$ get final relation

$$\operatorname{tr}(\mathbb{E}^{\alpha,t} G) = \hbar \partial_\theta \hat{I}_{L-2} + \hbar \hat{I}_{L-1} + \hbar \theta_\alpha \operatorname{tr}(G)$$

$$= \lambda E^\alpha_{11} + \frac{1}{\lambda} E^\alpha_{22} = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right) + S^\alpha_z \left(\lambda - \frac{1}{\lambda}\right).$$

$\Rightarrow$ end result $\langle \Psi | S_z^\alpha | \Psi \rangle$

$= \det \tilde{M}_{L-2}$ + explicitly computed numbers.

Results can be extended:

$\langle \tilde{\Psi}_A | A, B, D | \Psi_B \rangle$

= dets in Qs.

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correlation functions in planar $\mathcal{N}=4$ SYM.