Separation of Variables & Langlands Correspondance

Plan

(1) SoV transform sl_2 WZW $\leftrightarrow$ Liouville theory
(2) quantum analytic Langland & SoV
(3) Comments on possible extensions.

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(1) SoV transform sl_2 WZW $\leftrightarrow$ Liouville theory

(i) H_3^+ model

Let us consider a CFT based on the (affine) Lie algebra $\mathfrak{g}$

Introduce ($\dim \mathfrak{g}$) holomorphic $\hat{\mathfrak{g}}$ currents $J^a(z)$

with OPE
$$J^a(z) J^b(w) \sim \frac{k\, K^{ab}}{(z-w)^2} + \frac{f^{ab}_c\, J^c(w)}{z-w} + O(1)$$

K^{ab}: Killing form, f^{ab}_c structure const, k: level

Eg. for sl_2

$$J^0(y)J^0(z) = \frac{\frac{k}{2}}{(y-z)^2} + O(1)\,, \quad J^0(y)J^\pm(z) = \frac{\pm J^\pm(z)}{y-z} + O(1)\,,$$
$$J^\pm(y)J^\pm(z) = O(1)\,, \quad J^+(y)J^-(z) = \frac{k}{(y-z)^2} + \frac{2J^0(z)}{y-z} + O(1)\,.$$

Sugawara construction of S.E. tensor $T = \dfrac{K_{ab}(J^a J^b)}{2(k+g)}$

is a Virasoro field with $c = \dfrac{k \dim \mathfrak{g}}{k+g}$

g: dual coxeter number

$$\left(T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{(z-w)} + O(1) \right)$$

s.t.
$$T(z) J^a(w) \sim \frac{\partial}{\partial w} \frac{1}{z-w} J^a(w) + O(1)$$

J^a has conf. dim 1.

The modes of $J^a(z_0)$, $J^a_n(z_0) = \frac{1}{2\pi i}\oint_{z_0} dy\, (y-z_0)^n J(y)$

satisfy the affine $\hat{\mathfrak{g}}$ algebra
$$[J^a_m, J^b_n] = f^{ab}_c J^c_{m+n} + k\, m\, K^{ab} \delta_{m+n,0}$$

Given R rep. of g, affine primary is defined by its OPE

$$J^a(z)\,\Phi^R(w) \sim -\frac{t^a\,\Phi^R(w)}{z-w} + O(1)\,,\quad t^a:\ \text{generator of } g \text{ in } R$$

$\Phi^R(w)$ is Virasoro primary with $\Delta_R = C_2(R)/2(k+g)$

WI and KZ equations

$$\left[J^a(w) \xrightarrow[w\to\infty]{} O(1/w^2) \right]$$

$$\oint_\infty dy\, \varepsilon(y) \left\langle J^a(y) \prod_{i=1}^N \Phi^{R_i}(z_i) \right\rangle = 0 \qquad \varepsilon(y) \xrightarrow[y\to\infty]{} O(1)$$

$$\left\langle J^a(w) \prod_{i=1}^N \Phi^{R_i}(z_i) \right\rangle = -\sum \frac{\mathcal{D}^{R_i}(t^a)}{w-z_i} \left\langle \prod_{i=1}^N \Phi^{R_i}(z_i) \right\rangle \qquad \varepsilon(y) = \frac{1}{(y-w)^n}$$

global WI: $\displaystyle \sum \mathcal{D}^{R_i}(t^a) \left\langle \prod_{i=1}^N \Phi^{R_i}(z_i) \right\rangle = 0 \qquad \varepsilon(y) = 1$

Applying local WI to a corr. of affine primaries

Knizhnik - Zamolodchikov equations:

$$\left[(k+g)\frac{\partial}{\partial z_i} + H_i \right] \left\langle \prod_{i=1}^N \Phi^{R_i}(z_i) \right\rangle = 0$$

H_i : Gaudin Hamiltonians [integrable model associated to g]

$$H_i = \sum_{j\neq i} \frac{K_{ab}\,\mathcal{D}^{R_i}(t^a)\,\mathcal{D}^{R_j}(t^b)}{z_i - z_j}$$

$$\boxed{H_3^+ \text{ WZW} : \widehat{sl}_2 \quad = \quad \text{Liouville} : \text{Virasoro}}$$

Symplest model with $\widehat{sl}_2$ symmetry and continuum spectrum

spectrum of operators labelled by sl_2 spin j

$$G^{(n)}(\underline{x}, \underline{z}; \underline{j}) := \left\langle \prod_{i=1}^N \Phi^{j_k}(x_k, z_k) \right\rangle := \int \mathcal{D}_h\, e^{-S[h]} \prod_{i=1}^n \phi^{j_k}(h(z_k); x_k)$$

$$S[h] = \frac{k}{\pi}\int dx\, \partial\phi\,\bar\partial\phi + |\bar\partial \gamma|^2 e^{2\phi} \qquad h = \begin{pmatrix} e^{-\phi} + |\gamma|^2 e^{\phi} & \bar\gamma e^{\phi} \\ \gamma e^{\phi} & e^{\phi} \end{pmatrix}$$

Coset model on Euclidean $AdS_3 = H_3^+ = Sl_2(\mathbb{C})/SU(2)$

$\phi^j(h|x) = \frac{2j+1}{\pi}\left((1,-x)\cdot h\cdot \begin{pmatrix}1\\-x\end{pmatrix}\right)^{2j}$ affine primaries

$\Rightarrow$ Satisfy sl_2 KZ-equations

sl_2 generators in rep j_k:

$$\mathcal{D}_r^0 = x_r\partial_{x_r} - j_r \,,\quad \mathcal{D}_r^+ = \partial_{x_r}\,,\quad \mathcal{D}_r^- = -x_r^2\partial_{x_r} + 2j_r x_r$$

$$\Big[\left(\frac{1}{k+2}\right)\frac{\partial}{\partial z_i} - \underbrace{\sum_{i\neq j}\frac{2\mathcal{D}_i^0\mathcal{D}_j^0 + \mathcal{D}_i^+\mathcal{D}_j^- + \mathcal{D}_i^-\mathcal{D}_j^+}{z_i-z_j}}_{\text{Gaudin Hamiltonians}}\Big]\Big\langle\prod_{i=1}^N \Phi^{j_i}(z_i|x_i)\Big\rangle = 0$$

+ Global WI $\quad \sum_{r=1}^n \partial_{x_r} G^{(n)} = 0,\quad \sum_{r=1}^n (x_r\partial_{x_r} - j_r)G^{(n)} = 0,\quad \sum_{r=1}^n(-x_r^2\partial_{x_r} + 2j_r x_r)G^{(n)} = 0$

(ii) Liouville theory and BPZ equations

$$\Phi^{(n)}(z) = \Big\langle\prod_{k=1}^n V_{\alpha_k}(z_k)\Big\rangle = \int\mathcal{D}\phi\; e^{-S_L[\phi]}\prod_{k=1}^N e^{2\alpha_k\phi(z_k)}$$

$S_L[\phi] = \frac{1}{4\pi}\int_{\mathbb{C}} dx\, |\nabla\phi(x)|^2 + 4\pi\mu e^{2b\phi(x)}$ $\qquad V_{\alpha_k}$ Virasoro primary

$$c = 1+6Q^2 \;;\; Q = b+b^{-1} \;;\; \Delta(\alpha_k) = \alpha_k(Q-\alpha_k)$$

Generically only global sl_2 invariance (same WI as above)

But for $\alpha_k = -\frac{1}{2b}$ $\quad V_{-1/2b}$ is a degenerate rep of Vir.

$V_{-1/2b} = V_{\langle 1,2\rangle}$ $\Rightarrow$ It is annihilated by $(b^2L_{-1}^2 + L_{-2})V_{-1/2b} = 0$

Henceforth $\Phi^{(n|m)}(z,y) = \Big\langle\prod_{i=1}^n V_{\alpha_i}(z_i)\prod_{j=1}^m V_{-\frac{1}{2b}}(y_j)\Big\rangle$

satisfies $\boxed{\text{BPZ equations}}$ $\quad \mathcal{D}_r^{BPZ}\,\Phi^{(n,m)}(z,y) = 0$ $\quad r = 1,\dots,m$

$$\mathcal{D}_r^{BPZ} = b^2\frac{\partial^2}{\partial y_r^2} + \sum_{\substack{s\neq r\\ s=1}}^m \frac{1}{y_r-y_s}\frac{\partial}{\partial y_s} + \frac{\Delta_{-1/2b}}{(y_r-y_s)^2} + \sum_{i=1}^n \frac{1}{y_r-z_i}\frac{\partial}{\partial z_i} + \frac{\Delta_{\alpha_s}}{(y_r-z_s)^2}$$

$s=1$

$$+ \sum_{i=1}^{n} \frac{1}{y_r - z_i} \frac{\partial}{\partial z_i} + \frac{\Delta \alpha_s}{(y_r - z_s)^2}$$

(iii) SoV H_3^+ WZW <=> Liouville theory

Claim: There exist a separation of Variable transform between Conformal blocks of H_3^+ and of Liouville theory.

$$G^{(n)}(\underline{z},\underline{x}) = \int dy_1 \cdots dy_{n-2}\, K^{SoV}(\underline{x},\underline{y})\, \Phi^{(n|n-2)}(\underline{z},\underline{y})$$

H_3^+ correlators can be expressed by corr. functions of Liouville theory w/ degenerate insertions!

$\boxed{\text{Step 1}}$ Express primaries of $\hat{sl}_2$ in μ-basis

$$\Phi^j(z,\mu) = \frac{1}{\pi}|\mu|^{2j+2}\int_{\mathbb{C}} d^2x\, e^{\mu x - \bar{\mu}\bar{x}}\, \Phi^j(z,x) \quad \text{(Fourier)}$$

$$G^{(n)}(\underline{z},\underline{\mu}) = \left\langle \prod_{r=1}^{n} \Phi^{j_r}(z_r,\mu_r)\right\rangle$$

$\boxed{\text{Step 2}}$ Sklyanin change of Variables

$$(\mu_1,\dots,\mu_n) \to (y_1,\dots,y_{n-2},u)$$

$$\boxed{\sum_{i=1}^{n}\mu_i = 0}$$

$$\boxed{\sum_{r=1}^{n}\frac{\mu_r}{t-z_r} = u\,\frac{\prod_{j=1}^{n-2}(t-y_j)}{\prod_{s=1}^{n}(t-z_s)}}$$

$$b^2 = -\frac{1}{(k+2)}\ ; \quad \alpha_k = b(j_k+1) + \frac{1}{2b}$$

$$\Delta_{j_k} = \frac{1}{2} + \frac{1}{4b^2} + \Delta\alpha_k$$

Then one can show by direct computation that the system of n KZ equations + Global WI

$$\begin{cases} \left(\frac{\partial}{\partial z_k} + \mathcal{H}_k^{(n)}\right) \mathcal{G}^{(n)}(\underline{z},\underline{\mu}) = 0 \\ \left(\sum \mathcal{D}_r^{+,-,0}\right) \mathcal{G}^{(n)}(\underline{z},\underline{\mu}) = 0 \end{cases}$$

is equivalent under the Sklyanin SoV to

$$\begin{cases} \mathcal{D}_r^{BPZ}\, \Phi^{(n|m)}(\underline{z},\underline{y}) = 0 \\ \left(\sum D_r^{+,-,0}\right) \Phi^{(n|m)}(\underline{z},\underline{y}) = 0 \end{cases}$$

up to twisting by $\Theta = \prod_{r=1}^{n} (z_r - z_s)^{1/2b^2} \prod_{r\neq s} (y_r - y_s)^{-1/2b^2} \prod_{r,s} (z_r - y_s)^{-1/2b^2}$

$$\Rightarrow \mathcal{G}^{(n)}(\underline{z},\underline{\mu}) = \Theta\, \Phi^{(n|m)}(\underline{z},\underline{y})$$

By Fourier transformation one then finds:

$$\mathcal{G}^{(n)}(\underline{z},\underline{x}) = \int dy_1 \cdots dy_{n-2}\, K^{SoV}(\underline{y},\underline{x})\, \Phi^{(n|m)}(\underline{z},\underline{y})$$

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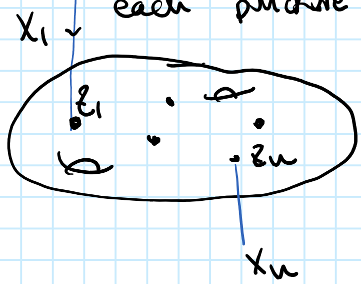
(2) Relation to Langlands

Punctured Riemann Surface $\mathcal{C}_{g,n}$

punctures @ z_i



Geometric Langlands

$\boxed{{}^L\mathfrak{g}\text{-local systems on } \mathcal{C}} \longleftrightarrow \mathcal{D}\text{-mod on } \mathrm{Bun}_G(\mathcal{C})$

$(\mathcal{E}, \nabla)$

loc ${}^LG$-bundle

Flat connection on $\mathcal{E}$

locally constant Vector bundle $\cong \mathbb{C}^n$ carrying an action of ${}^L\mathfrak{g}$

Set of differential ops acting on $\mathrm{Bun}_G(\mathcal{C})$

(moduli of Vector bundle on $\mathcal{C}$ with fibers $\cong G$

Beilinson-Drinfeld: Turn ∇ in OPER FORM

Fix $\mathfrak{g} = sl_2$; $\nabla \sim dz\left(\partial_y + \begin{pmatrix} 0 & t(y) \\ 1 & 0 \end{pmatrix}\right)$

with $t(z)$ transforming projectively (Schwartzian der)

local systems are then determined by the flatness condition of the connection

$(\partial^2_{y_k} + t(y_k))\, \chi_k(y_k) = 0$ (locally for each puncture)

and $\mathcal{D}$-module is Hitchin integrable system!

so for sl_2 Gaudin model $\mathcal{H}_r \Psi = E_r \Psi$

$\mathcal{H}_r$ at each puncture depends on the periodic var. x

So $\Rightarrow$ Langlands is correspondence between

Whittaker system $\xleftrightarrow{\text{Law}}$ Gaudin model

$(\partial^2_y + t(y))\chi(y) = 0$ $\qquad$ $\mathcal{H}_r \Psi = E_r \Psi$

Existence of Complete spectrum of these ops

$b^{-2} \to 0$ $\qquad$ $\mathcal{H}\, \Psi(z,x) = \chi(y)\, \Psi(z,x)$ $\qquad$ $k \to -2$

this is nothing but classical limit

BPZ-equation Liouville local system $\qquad$ KZ-equation H_3^+ WZW

$\mathcal{D}_r^{BPZ}\, \Psi^{(n|m)}(\underline{z}, \underline{y}) = 0$ $\qquad$ $\left(\frac{1}{k+2}\partial_z + \mathcal{H}_r\right)\Psi^{(n)}(\underline{z}, \underline{x})$

Flatness condition $\qquad$ D-opers

This allowed us e.g. to construct explicitly the Hecke operator in CFT, study stratified structure of moduli space etc...

I can talk @ length about this, maybe in a research talk on my work w/ Jörg but for today I just want to convey the key message. G.L. systematize the SoV transform.

Start with the sl_2 connection

∇_{sl_2} is NOT in oper form

$$\mathcal{D}^0 = x_r \partial_{x_r}$$
etc...

$$\nabla_{sl_2} = J(y) = \sum_{r=1}^{n} \frac{1}{y - z_r} \begin{pmatrix} \mathcal{D}^0_r & \mathcal{D}^+_r \\ \mathcal{D}^-_r & -\mathcal{D}^0_r \end{pmatrix}$$

Bringing it to oper form via gauge transform.

$$\begin{pmatrix} 0 & t(z) \\ 1 & 0 \end{pmatrix} = \underset{\underset{Sl_2}{\Uparrow}}{A} \, J(y) \, A^{-1}$$

can be shown to lead directly to Sklyanin SoV

Idea $J^-(y) = \sum \frac{1}{y - z_r} \partial_{x_r}$

$$\begin{pmatrix} * & * \\ \sum \frac{\partial_{x_r}}{y - z_r} & * \end{pmatrix} \sim \begin{pmatrix} 0 & t(z) \\ 1 & 0 \end{pmatrix}$$

$\xrightarrow{\text{Fourier transform}} \sum_{r=1}^{n} \frac{\mu_r}{y - z_r}$ a multiplication op

Leads directly to the $t(z)$ of BPZ and gives for free

leads directly to the t(z) of BPZ and gives for free
K^{SOV} to be the kernel of the diagonalization

So it can be in principle repeated for any $\mathfrak{g}$!

Eg sl_3 $J(y) = \sum \frac{1}{y - z_r} \begin{pmatrix} J^r_{11} & J^r_{12} & J^r_{13} \\ J^r_{21} & J^r_{22} & J^r_{23} \\ J^r_{31} & J^r_{32} & J^r_{33} \end{pmatrix}$

$$\boxed{\partial_y^3 - w_2(y)\partial_y - w_3(y) = 0}$$

OPER equation

$\downarrow$ OPER FORM

$$\begin{pmatrix} 0 & w_2(y) & w_3(y) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

the kernel to go from $J(y)$ to

gives K^{SoV} for $sl_3 \longleftrightarrow toda_2$

with BPZ equation for $(3,1)$ (null vector @ 3^{rd} level)