Musterlösung Aufgabe 6

a) $\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\Psi\bar{\Psi}$

Euler-Lagrange: $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\right) - \frac{\partial\mathcal{L}}{\partial\Psi} = 0$

$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = i\bar{\Psi}\gamma^\mu \qquad \frac{\partial\mathcal{L}}{\partial\Psi} = m\bar{\Psi}$ ($\Psi, \bar{\Psi}$ als unabh. Variable auffassen!)

$\Rightarrow \partial_\mu(i\bar{\Psi}\gamma^\mu) - m\bar{\Psi} = 0$ ("adjungierte Dirac-Gl.")

zur "normalen" Dirac-Gleichung:

$\gamma^{\mu+} = \gamma^0\gamma^\mu\gamma^0 \quad , \bar{\Psi} = \Psi^+\gamma^0$

$i\partial_\mu\Psi^+\gamma^0\gamma^\mu - m\Psi^+\gamma^0 = 0 \quad | \leftarrow \gamma^0$

$\Rightarrow i\partial_\mu\Psi^+\gamma^{\mu+} - m\Psi^+ = 0 \quad | (\ )^+$

$\Rightarrow i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$

$$\boxed{(i\not\partial - m)\Psi = 0}$$

Für skalares Feld:

reell: $\mathcal{L} = \frac{1}{2}[\partial_\mu\varphi\partial^\mu\varphi - m^2\varphi^2] = 0$

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = \partial_\mu\partial^\mu\varphi = \Box\varphi$

$\frac{\partial\mathcal{L}}{\partial\varphi} = -m^2\varphi \qquad \Rightarrow \boxed{(\Box + m^2)\varphi = 0}$ Klein-Gordon

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Komplex: $\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi^*) - m^2\phi\phi^*$

(ϕ, ϕ^* unabh. Variable)

$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \partial_\mu\partial^\mu\phi^* \qquad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi^*$

und $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) = \partial^\mu\partial_\mu\phi \qquad \frac{\partial\mathcal{L}}{\partial\phi^*} = -m^2\phi$

$\Rightarrow \boxed{(\Box + m^2)\phi^* = 0}$ und $\boxed{(\Box + m^2)\phi = 0}$

b) $\mathcal{L}' = i\bar{\Psi}e^{-i\alpha}\gamma^\mu\partial_\mu(e^{i\alpha}\Psi) - m\bar{\Psi}e^{-i\alpha}e^{i\alpha}\Psi = 0$

$\left(\bar{\Psi}' = \Psi^{+\prime}\gamma^0 = (e^{i\alpha}\Psi)^+\gamma^0 = e^{-i\alpha}\bar{\Psi}\right)$

c) trivial $\left(U(\alpha)U(\beta) - U(\beta)U(\alpha) = e^{i\alpha}e^{i\beta} - e^{i\beta}e^{i\alpha} = 0\right)$

d) $e^{i\alpha} = 1 + i\alpha + \frac{(i\alpha)^2}{2} + \ldots \approx 1 + i\alpha$

e) $\mathcal{L}' = i\bar{\Psi}e^{-i\alpha(x)}\gamma^\mu\partial_\mu(e^{i\alpha(x)}\Psi) - m\bar{\Psi}\underbrace{e^{-i\alpha(x)}e^{i\alpha(x)}}_{1}\Psi = 0$

$\left[\partial_\mu(e^{i\alpha(x)}\Psi) = (\partial_\mu\Psi)e^{i\alpha(x)} + i\partial_\mu\alpha(x)\,e^{i\alpha(x)}\Psi\right]$

$= i\bar{\Psi}\underbrace{e^{-i\alpha(x)}e^{i\alpha(x)}}_{1}\gamma^\mu\partial_\mu\Psi + i\bar{\Psi}e^{-i\alpha(x)}\gamma^\mu i\,\partial_\mu\alpha\,e^{i\alpha(x)}\Psi - m\Psi\bar{\Psi}$

$= \mathcal{L} - \bar{\Psi}\gamma^\mu\Psi\partial_\mu\alpha \neq \mathcal{L}$

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f) $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$ mit $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x)$

Dann ist das neue $\mathcal{L} = i\bar{\Psi}\gamma^\mu D_\mu \Psi - m\bar{\Psi}\Psi$

Gleichzeitige Eichtrf. $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$ und Phasentrf $\Psi \to \Psi e^{i\alpha(x)}$:

$$\mathcal{L}' = i\bar{\Psi}'\gamma^\mu D_\mu' \Psi - m\bar{\Psi}'\Psi'$$

$$= i\bar{\Psi}e^{-i\alpha(x)}\gamma^\mu(\underset{①}{\partial_\mu} - \underset{②}{ieA_\mu} - \underset{③}{i\partial_\mu\alpha(x)})e^{i\alpha(x)} - m\bar{\Psi}\Psi$$

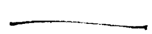
$$= i\bar{\Psi}e^{-i\alpha(x)}e^{i\alpha(x)}\gamma^\mu\partial_\mu\Psi + \bar{\Psi}\gamma^\mu\Psi\partial_\mu\alpha(x)$$

$$+ \underbrace{e\bar{\Psi}\gamma^\mu\Psi A_\mu}_{\text{aus } ②} - \underbrace{\bar{\Psi}e^{-i\alpha(x)}e^{i\alpha(x)}\gamma^\mu\Psi\partial_\mu\alpha}_{\text{aus } ③} - m\bar{\Psi}\Psi$$

$$= \mathcal{L} \quad \checkmark$$

g) es leicht zu zeigen daß $F_{\mu\nu}$ invariant ist. triv.

h) $\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + e\bar{\Psi}\gamma^\mu A_\mu\Psi + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$



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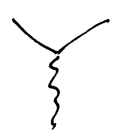

Wechsel-
wirkung

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