# Eichung: Bemerkungen.

## Coulomb-Eichung.

$$\left\{\begin{array}{l} \nabla^2 \vec{A} - \varepsilon_0 \mu_0 \dfrac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{j} + \varepsilon_0 \mu_0 \dfrac{\partial}{\partial t} \nabla \varphi \\ \nabla^2 \varphi = -\dfrac{1}{\varepsilon_0} \rho \\ \nabla \cdot \vec{A} = 0 \end{array}\right.$$

## Lorentz-Eichung

$$\left\{\begin{array}{l} \Box \vec{A} = -\mu_0 \vec{j} \\ \Box \varphi = -\dfrac{1}{\varepsilon_0} \rho \\ \nabla \cdot \vec{A} + \mu_0 \varepsilon_0 \dfrac{\partial}{\partial t} \varphi = 0 \end{array}\right.$$

# Energetische Beziehungen.

$$\begin{aligned} \bar{E}\cdot \quad & \nabla\times\bar{H} = \vec{J} + \frac{\partial\bar{D}}{\partial t} \\ -\ \bar{H}\cdot \quad & \nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} \end{aligned}$$

LHS: $\bar{E}\cdot(\nabla\times\bar{H}) - \bar{H}\cdot(\nabla\times\bar{E}) = -\nabla\cdot(\bar{E}\times\bar{H}) \equiv -\nabla\cdot\vec{S}$

$\vec{S} = (\bar{E}\times\bar{H})$ - der Poynting - Vektor

RHS: $\bar{E}\cdot\vec{J} + \bar{E}\cdot\frac{\partial\bar{D}}{\partial t} + \bar{H}\cdot\frac{\partial\bar{B}}{\partial t}$ $\quad (\bar{D} = \varepsilon_0\varepsilon_r\bar{E}, \quad \bar{B} = \mu_0\mu_r\bar{H})$

$$\bar{E}\cdot\frac{\partial\bar{D}}{\partial t} + \bar{H}\cdot\frac{\partial\bar{B}}{\partial t} = \varepsilon_0\varepsilon_r\,\bar{E}\cdot\frac{\partial\bar{E}}{\partial t} + \mu_0\mu_r\,\bar{H}\cdot\frac{\partial\bar{H}}{\partial t} =$$

$$= \varepsilon_0\varepsilon_r\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{E}^2\right) + \mu_0\mu_r\frac{\partial}{\partial t}\left(\frac{1}{2}\bar{H}^2\right) = \frac{\partial}{\partial t}\left\{\frac{1}{2}(\bar{E}\cdot\bar{D}) + \frac{1}{2}(\bar{H}\cdot\bar{B})\right\}.$$

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}(\bar{E}\cdot\bar{D}) + \frac{1}{2}(\bar{H}\cdot\bar{B})\right\} = -\nabla\cdot(\bar{E}\times\bar{H}) - \bar{j}\cdot\bar{E}$$

$$w = \frac{1}{2}(\bar{E}\cdot\bar{D}) + \frac{1}{2}(\bar{H}\cdot\bar{B})$$ - Energiedichte des EM-Feldes

$$\frac{\partial w}{\partial t} + \nabla\cdot\vec{S} = -\vec{j}\cdot\bar{E}$$ — Poynting-Satz.

$$\int_V \frac{\partial}{\partial t}\left\{\frac{1}{2}\bar{E}\cdot\bar{D} + \frac{1}{2}\bar{H}\cdot\bar{B}\right\} + \int_V \nabla\cdot(\bar{E}\times\bar{H}) = -\int_V \vec{j}\cdot\bar{E}\cdot dV$$

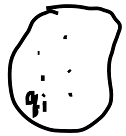
$$\int_V \nabla\cdot(\bar{E}\times\bar{H})\, dV = \oint_{\partial V} (\bar{E}\times\bar{H})\cdot\bar{n}\, dS = 0 \quad (S\to\infty)$$

$V \to \infty$

$$\int_V \frac{\partial}{\partial t} w\, dV = -\int_V \vec{j}\cdot\bar{E}\cdot dV$$

$$\int_V \left(\frac{\partial}{\partial t} w + \vec{j}\cdot\bar{E}\right) dV = 0$$

$$\int \vec{j}\cdot\vec{E}\, dV = \frac{d}{dt} E_k$$ - zu zeigen




$$m\dot{\vec v}_i = q_i(\vec E + \vec v_i \times \vec B)$$

$$m(\vec v_i \cdot \dot{\vec v}_i) = q_i(\vec v_i \cdot \vec E + \underbrace{\vec v_i \cdot (\vec v_i \times \vec B)}_{=0}).$$

$$m(\vec v_i \cdot \dot{\vec v}_i) = \frac{d}{dt}\left(\frac{m v_i^2}{2}\right) = \frac{d}{dt}E_{i,k}.$$

$$\frac{dE_k}{dt} = \sum_i \vec v_i \cdot \vec E(\vec r_i)\, q_i = \sum_i \vec j_i \cdot \vec E(\vec r_i) = \int_V \vec j \cdot \vec E \, dV$$

$$\vec v_i q_i = \vec j_i$$

$$\frac{d}{dt}(W_F + E_k) = 0 \iff \frac{\partial}{\partial t}\int_V \left\{\frac{1}{2}(\vec E \cdot \vec D) + \frac{1}{2}(\vec B \cdot \vec H)\right\} dV + \frac{dE_k}{dt}$$

---

ohne Ladungen, $\vec j = 0$

$$\frac{\partial W_F}{\partial t} = -\nabla \cdot \vec S$$ – Kontinuitätsgleichung.

# Kontinuitätsgleichung als Folge des Erhaltungssatzes.

$Q$ – eine erhaltende Größe (Skalar)

$\rho(\bar{r})$ – die Dichte von $Q$.

$$Q = \int_V \rho(\bar{r})\, dV = \text{const.} \quad \Rightarrow \quad \frac{dQ}{dt} = 0.$$

$$\frac{dQ}{dt} = \int_V \frac{d}{dt} \rho(\bar{r}, t)\, dV = \int_V \left\{ \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \vec{r}} \cdot \frac{d\bar{r}}{dt} \right\} =$$

$$= \int_V \left\{ \frac{\partial \rho}{\partial t} + (\nabla \rho) \cdot \bar{v} \right\} dV = 0$$

zu zeigen: $(\nabla\rho)\cdot\bar{v} = \nabla\cdot(\rho\bar{v}) = \nabla\cdot\bar{j}$

$\rho\vec{v} = \bar{j}$ – die Stromdichte

$$\nabla\cdot(\rho\bar{v}) = (\nabla\rho)\cdot\vec{v} + \underbrace{\rho(\nabla\cdot\vec{v})}_{=0} = (\nabla\rho)\cdot\bar{v}$$

$$\nabla\cdot\bar{v} = \nabla\cdot\left(\frac{d\bar{r}}{dt}\right) = \frac{\partial}{\partial t}(\nabla\cdot\bar{r}) = \frac{\partial}{\partial t} d = 0 \;; \quad d = 1, 2, \underline{\underline{3}}$$

– Dimension des Raumes.

$$\int_V \left\{ \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} \right\} dV = 0$$

$$\frac{\partial \rho}{\partial t} = - \nabla \cdot \vec{j}$$ – Kontinuitätsgleichung.

# Impulserhaltung. Energie - Impuls Tensor.

Gesamtimpuls: $\vec{P} = \vec{P}_M + \vec{P}_F$

$$\vec{P}_M = \int_V \vec{\Pi}_M \cdot dV$$

$\vec{\Pi}_M$ - mechanische Impulsdichte.

$$\vec{P} = \int_V \vec{\Pi}\, dV \,,$$

$\vec{\Pi}$ - Gesamtimpulsdichte.

Kontinuitätsgleichung für eine Vektorgröße.

$$\frac{\partial \vec{\Pi}}{\partial t} = -\nabla \cdot \overleftrightarrow{T} \,,$$

$\nabla \cdot \overleftrightarrow{T}$ - ein Vektor!

$$\left(\nabla \cdot \overleftrightarrow{T}\right)_j = \left(\sum_i \partial_i \left(\overleftrightarrow{T}\right)_i\right)_j = \sum_i \partial_i T_{ij}$$

$T_{ij}$ - <u>ein Tensor.</u>

$$\frac{d\vec{p}_M}{dt} = \vec{F} = \int_V \rho(\vec{r}) \left\{ \vec{E}(\vec{r},t) + \left( \vec{v}(\vec{r},t) \times \vec{B}(\vec{r},t) \right) \right\} dV$$

$$\frac{d\vec{\Pi}_M}{dt} = \rho \vec{E} + \rho [\vec{v} \times \vec{B}] = \rho \vec{E} + [\vec{J} \times \vec{B}] =$$

$$= (\mathrm{div}\, \vec{D}) \vec{E} + \left[ \left( \mathrm{rot}\, \vec{H} - \frac{\partial}{\partial t} \vec{D} \right) \times \vec{B} \right] =$$

$$= \vec{E}(\nabla \cdot \vec{D}) + [[\nabla \times \vec{H}] \times \vec{B}] - [\dot{\vec{D}} \times \vec{B}] =$$

$$= \vec{E}(\nabla \cdot \vec{D}) + (\nabla \times \vec{H}) \times \vec{B} - \frac{\partial}{\partial t}[\vec{D} \times \vec{B}] + \vec{D} \times \underbrace{\frac{\partial \vec{B}}{\partial t}}_{= -\nabla \times \vec{E}} =$$

$$= \vec{E}(\nabla \cdot \vec{D}) + (\nabla \times \vec{H}) \times \vec{B} - \vec{D} \times (\nabla \times \vec{E}) - \frac{\partial}{\partial t}[\vec{D} \times \vec{B}]$$

---

$$\frac{\partial}{\partial t} \left( \vec{\Pi}_M + \vec{D} \times \vec{B} \right) = \vec{E}(\nabla \cdot \vec{D}) - \vec{D} \times (\nabla \times \vec{E}) + \vec{H}(\nabla \cdot \vec{B}) - \vec{B} \times (\nabla \times \vec{H})$$

zu zeigen:

$$(\nabla\cdot\vec{D})\vec{E} - \vec{D}\times(\nabla\times\vec{E}) + (\nabla\cdot\vec{B})\vec{H} - \vec{B}\times(\nabla\times\vec{H}) = \nabla\cdot\overleftrightarrow{T}$$

$$(\nabla\cdot\vec{D})\vec{E} - \vec{D}\times(\nabla\times\vec{E}) = \varepsilon_0\varepsilon_r\{(\nabla\cdot\vec{E})\vec{E} - \vec{E}\times(\nabla\times\vec{E})\} =$$

$$= \varepsilon_0\varepsilon_r\{(\nabla\cdot\vec{E})\vec{E} - \nabla(\vec{E}\cdot\vec{E}) + (\vec{E}\cdot\nabla)\vec{E}\} =$$

$$= \varepsilon_0\varepsilon_r\{(\nabla\cdot\vec{E})\cdot\vec{E} + (\vec{E}\cdot\nabla)\vec{E} - \frac{1}{2}\nabla(\vec{E}^2)\}.$$