Energie - Impuls - Tensor

$$\frac{\partial \vec{\Pi}}{\partial t} = -\nabla \cdot \overleftrightarrow{T} \qquad \overleftrightarrow{T} \to T_{ij} \text{ - Tensor}$$

$$\frac{d\vec{p}_m}{dt} = \int_V \rho(\vec{r}) \left\{ \vec{E} + \vec{v} \times \vec{B} \right\} dV$$

$$\frac{d\vec{\Pi}_m}{dt} = -\frac{\partial}{\partial t}\left[\vec{D} \times \vec{B}\right] + (\nabla \cdot \vec{D}) \cdot \vec{E} - \vec{D} \times (\nabla \times \vec{E}) + $$
$$+ (\nabla \times \vec{H}) \times \vec{B} + (\nabla \cdot \vec{B}) \vec{H} .$$

$$(\nabla \cdot \vec{D}) \cdot \vec{E} - \vec{D} \times (\nabla \times \vec{E}) = \varepsilon_0 \varepsilon_r \left\{ (\nabla \cdot \vec{E}) \cdot \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla (\vec{E}^2) \right\}$$

$$(\nabla \cdot \vec{B}) \vec{H} - \vec{B} \times (\nabla \times \vec{H}) = \mu_0 \mu_r \left\{ (\nabla \cdot \vec{H}) \vec{H} + (\vec{H} \cdot \nabla) \vec{H} - \frac{1}{2} \nabla (\vec{H}^2) \right\}$$

$$\left\{ (\nabla \cdot \vec{E}) \cdot \vec{E} + (\vec{E} \cdot \nabla) \vec{E} - \frac{1}{2} \nabla (\vec{E}^2) \right\}_i = (\partial_j E_j) E_i + (E_j \partial_j) E_i - \frac{1}{2} \partial_i (E_j E_j) =$$

$$= \sum_j \left\{ \partial_j (E_j E_i) - \delta_{ij} \partial_j \left(\frac{1}{2} \vec{E}^2\right) \right\} = \sum_j \partial_j \left\{ E_j E_i - \frac{1}{2} \vec{E}^2 \delta_{ji} \right\} = \partial_j T^E_{ji} =$$

$$= (\nabla \cdot \overleftrightarrow{T}_E)_i$$

$$T_{ij} = \varepsilon_0 \varepsilon_r \left(E_i E_j - \frac{1}{2} \vec{E}^2 \delta_{ij} \right) + \frac{1}{\mu_0 \mu_r} \left(B_i B_j - \frac{1}{2} \vec{B}^2 \delta_{ij} \right).$$

Energie – Impuls – Tensor des EM-Feldes.

$$\frac{\partial}{\partial t} \left(\vec{\Pi}_M + \vec{D} \times \vec{B} \right) = \nabla \cdot \overleftrightarrow{T} \qquad \left((\nabla \cdot \overleftrightarrow{T})_i = \sum_j \partial_j T_{ji} \right)$$

$\vec{D} \times \vec{B} = \vec{\Pi}_F$ – Impulsdichte des EM-Feldes.

$$\vec{\Pi}_F = \vec{D} \times \vec{B} = \varepsilon_0 \mu_0 \left[\vec{E} \times \vec{H} \right] = \frac{1}{c^2} \vec{S}$$

$$\frac{\partial}{\partial t} \left(\vec{\Pi}^M + \vec{\Pi}^F \right) = \nabla \cdot \overleftrightarrow{T}$$ – Impulssatz der ED.

Integralform:

$$\frac{\partial}{\partial t} \left(\vec{P}_M + \vec{P}_F \right) = \oint_{\partial V} \overleftrightarrow{T} \cdot \vec{n} \, dS \qquad \left(\overleftrightarrow{T} \cdot \vec{n} \right)_i = \sum_j T_{ij} n_j$$

$$\left(S \to \infty: \ \overleftrightarrow{T} = 0 \right) \qquad \frac{\partial}{\partial t} \left(\vec{P}_M + \vec{P}_F \right) = 0.$$

EM-Wellen im Vakuum

$\rho = 0, \quad \bar{J} = 0, \quad \varepsilon_r = \mu_r = 1 \quad \Rightarrow \quad \begin{array}{l} \bar{B} = \mu_0 \bar{H} \\ \bar{D} = \varepsilon_0 \bar{E} \end{array}$

Maxwell-Gleichungen.

$$\nabla \cdot \bar{E} = 0, \qquad \nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}, \qquad \nabla \times \bar{B} = \varepsilon_0 \mu_0 \frac{\partial \bar{E}}{\partial t}$$

$$\nabla \times (\nabla \times \bar{E}) = \nabla \underbrace{(\nabla \cdot \bar{E})}_{0} - \nabla^2 \vec{E} = \nabla \times \left(-\frac{\partial \bar{B}}{\partial t}\right)$$

$$\nabla \times \frac{\partial \bar{B}}{\partial t} = \frac{\partial}{\partial t}(\nabla \times \bar{B}) = \varepsilon_0 \mu_0 \frac{\partial^2 \bar{E}}{\partial t^2}.$$

$$\nabla^2 \bar{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \bar{E}}{\partial t^2} = 0, \qquad \varepsilon_0 \mu_0 = \frac{1}{c^2}$$

Wellengl.

Analogerweise: $\nabla \times (\nabla \times \bar{B}) = \nabla \underbrace{(\nabla \cdot \bar{B})}_{=0} - \nabla^2 \bar{B} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \times \bar{E}) = -\varepsilon_0 \mu_0 \frac{\partial^2 \bar{B}}{\partial t^2}$

$$\nabla^2 \bar{B} - \varepsilon_0 \mu_0 \frac{\partial^2 \bar{B}}{\partial t^2} = 0.$$

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Box \quad \text{- D'Alembert-Operator}$$

$$\begin{cases} \Box \vec{E} = 0 \\ \Box \vec{B} = 0 \end{cases}$$

Allgemeine Lösung der Wellengleichung (homogen)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi(\vec{r}, t) = 0$$

$$\varphi(\vec{r}, t) = f_+(|\vec{r}| - ct) + f_-(|\vec{r}| + ct)$$

1D: $$\left(\partial^2_{xx} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi(x,t) = 0$$

$$\varphi(x,t) = f_+(x-ct) + f_-(x+ct)$$

$$\partial_x \varphi = f_+' + f_-' , \qquad \partial_t = -c f_+' + c f_-'$$

$$\partial^2_{xx}\varphi = f_+'' + f_-'' , \qquad \partial^2_{tt} = c^2 f_+'' + c^2 f_-''$$

$$\left(\partial^2_{xx} - \frac{1}{c^2}\partial^2_{tt}\right)\varphi = \left(f_+'' - \frac{1}{c^2}\cdot c^2 f_+''\right) + \left(f_-'' - \frac{1}{c^2}c^2 f_-''\right) = 0$$