Workshop on
Tracking In high Multiplicity
Environments

Zürich   3rd – 7th October 2005

A new fast track-fit algorithm based on broken lines

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Abstract

The determination of the particle momentum in HEP tracking chambers requires a fit of a parametrization to the measured points. Various effects can result in deviations to the ideal helix curve in the magnetic field of a solenoid, and the fit with a pure helix parametrization is not optimal. One effect is *multiple scattering*, which causes a ‘random walk” of the particle and affects especially low-momentum tracks and very accurate measurements.

A popular method of track reconstruction is the Kalman filter, an algorithm known from time-series analysis and signal processing, and with results mathematically equivalent to global least squares fits. It is recursive, includes measurements one after the other and has an execution time $\mathcal{O}(n)$, because large matrices are avoided.

The proposed method based on broken lines is non-recursive and allows to reconstruct the particle trajectory taking into account details of the multiple scattering. It provides optimal parameters and their covariance matrices at track start and end, and optimal values at each measured point along the trajectory including the variances. The method is constructed to allow the use of sparse-matrix techniques with a total execution time $\mathcal{O}(n)$, and is, under test conditions, a factor six faster than the Kalman filter.

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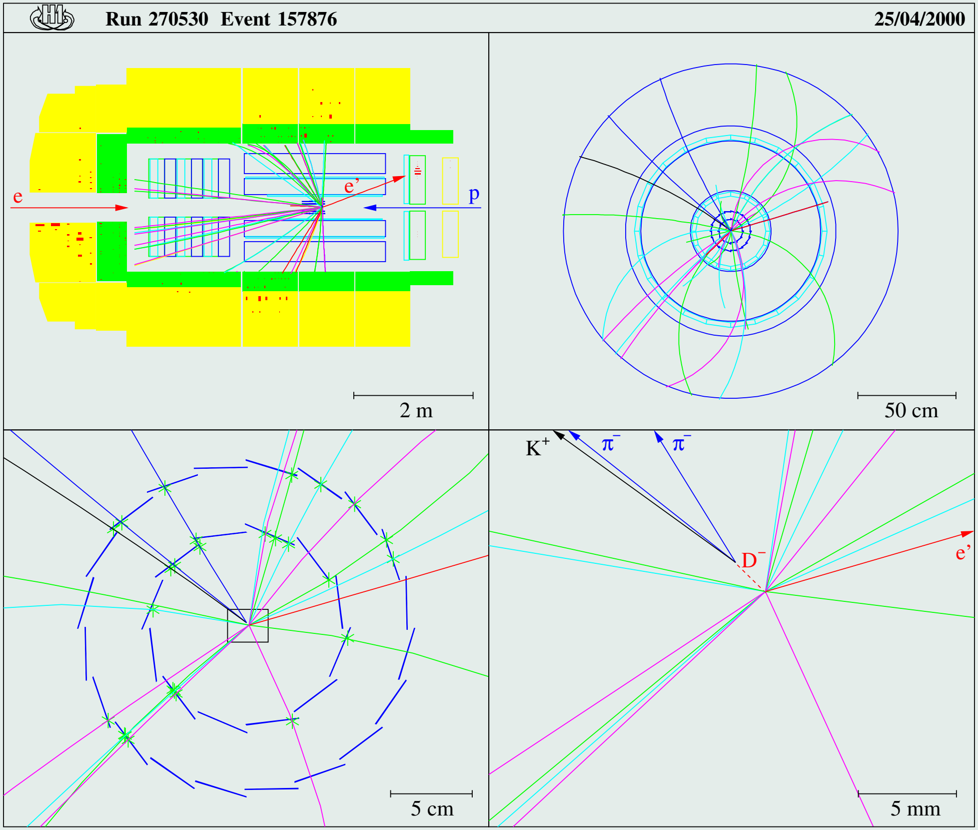
1. Track measurement in particle physics experiments



Charm event in the H1 detector at HERA with drift chamber and vertex detector.

LHC (2007 –): 1000 tracks per event, several events per bunch crossing (bunch spacing 25 ns)

Fast track reconstruction is essential.

Track parametrization

Ideal parametrization: a helix (κ, d_{ca}, ϕ, z_0, $\tan\lambda$) in a homogeneous magnetic field $\boldsymbol{B}_z$ with

xy- or $r\phi$-plane: circle, residual ε_i of a measured point (x_i, y_i) from a particle trajectory

$$\varepsilon_i = \frac{1}{2}\kappa\left(x_i^2 + y_i^2 + d_{\text{ca}}^2\right) - (1 + \kappa d_{\text{ca}})\left(x_i \sin\phi - y_i \cos\phi\right) + d_{\text{ca}} \ .$$

sz-plane: straight line, residual ε_i of a measured point (s_i, z_i)

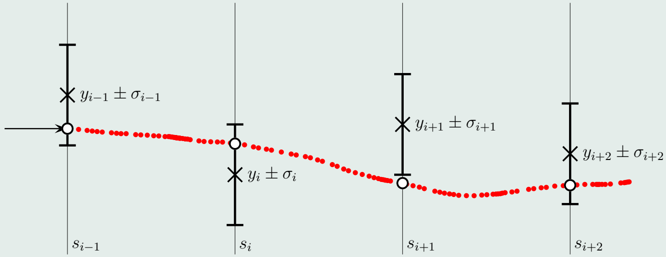
$$\varepsilon_i = z_0 + (\tan\lambda) \cdot s_i - z_i \qquad \text{with } s_i = \text{track length in } xy$$

. . . but there are random and non-random perturbations:

Multiple scattering (msc): Elastic scattering of charged particles in the Coulomb field of the nuclei in the detector material. The mean of the projected deflection angle θ after traversing a material layer is zero and the distribution has a variance of

$$\text{variance} \quad V[\theta] \propto \frac{t}{\beta^2 p^2} \qquad \text{where} \quad t = \Delta s / X_0$$

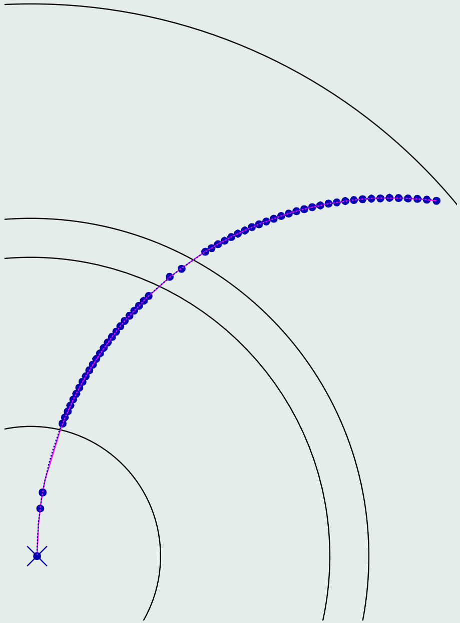
Multiple scattering deflections will influence all downstream measurement in a correlated way, and delimits the momentum measurement at low momenta.



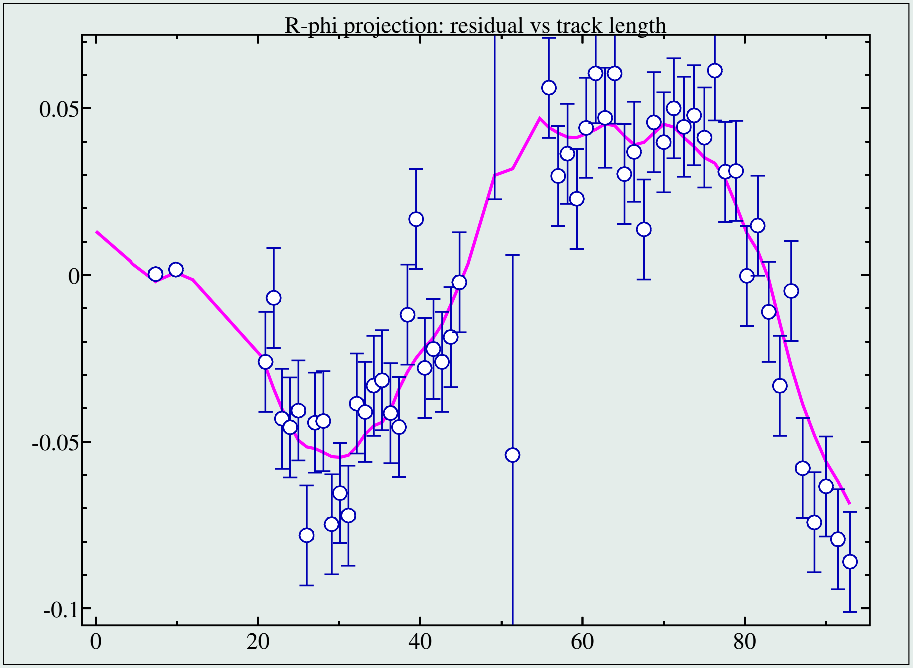
Energy loss, Radiation of electrons, Field inhomogeneity, . . .

Track example with msc *(magenta line is the true particle trajectory)*

Circle fit



Deviations from circle fit



→ broken-line fit

Large influence of multiple scattering (msc)

- for low momenta,
- for high position measurement accuracy, and
- for dense material (large value of $\sqrt{s/X_0}$).

2. Track fitting methods

Track parameters are obtained by the fit of a track parametrization to the n measured track hits:

0. Simple least square fit with ideal parametrization fast, with a computing time $\propto n$

1. Global track fitting methods computing time $\propto n^{2 \cdots 3}$
(either trajectory described by non-diagonal $n \times n$ covariance matrix (matrix method), or by including parameters for N scattering planes, large matrix (breakpoint method);

2. The *progressive* method (P. Billoir 1984) ($\approx$ Kalman filter) computing time $\propto n$
track is followed by incorporating measurement after measurement, starting from the outer detector, improving the parameter vector and covariance matrix;

2'. The Kalman filter (and smoothing) computing time $\propto n$
the state vector and its covariance matrix are propagated to the next measurement, additional error (msc …) introduced as process noise; smoothing in direction opposite to the filter.

3. Broken-line fit **("new")** computing time $\propto n$

Results on track parameters from 1 to 3 are (almost) identical, and are, at low momentum, (roughly) up to 30 % better than simple fit.

Track fitting is not only required to obtain the final track parameters for physics analysis, but also in the track-finding phase (pattern recognition). Very many fits of each track candidate are necessary to find the correct hits.

Requirements and computing times

Required results from a track fit algorithm:

- **A:** Optimal track parameters at **track-start** (vertex), for physics analysis;
- **B:** Correct covariance matrix for track parameters;
- **C:** Overall χ^2 of track, for test of quality of pattern recognition;
- **D:** Optimal track parameters at **track-end**, for extrapolation to other detectors;
- **E:** χ^2 of each single hit, for outlier test and improvement of hit selection.

Time for one track fit (no magnetic field), in **mikroseconds**, on standard PC:

Algorithm	$n = 25$	$n = 50$	$n = 100$	Remarks:
2-parameter least squares	0.270	0.420	0.720	bad fit in case of multiple scattering
Matrix method	150.000	943.000	6731.000	track-start parameters: **(A, B, C)**
Breakpoint method	117.000	556.000	2980.000	full reconstruction: **(A, B, C, D, E)**
Kalman backward	20.900	41.300	81.900	track-start parameters: **(A, B)**
Kalman back-/forward	approximately $\times 2$			full reconstruction: **(A, B, C, D, E)**
Broken-line fit	6.550	12.980	25.200	full reconstruction: **(A, B, C, D, E)**

Track-start parameters from methods in last 5 rows (almost) identical.

3. Track-fit algorithms based on broken lines

Is a method possible, which keeps the good properties of the Kalman filter/smoothing and allows to get the complete solution in one step?

- Treat the multiple scattering in all detail, which generates necessarily many parameters and matrices of large dimension;
- Use a mathematical model which results in equations with sparse matrices (many elements = 0), which can be **solved quickly**.

Simple and fast least squares track fits, with circle fit (Karimäkie) for κ, d_{ca}, ϕ and straight line fit (z_0, $\tan\lambda$), are done to prepare the data for a detailed fit:

- Momentum already known (from $\kappa, \tan\lambda$) with some precision, allows to calculate multiple scattering variances;
- Residuals in xy-plane and in sz-plane can be calculated;
- A detailed fit follows, which takes into account multiple scattering (and other perturbations), applied to the residuals:

  (a) fit of z-residual versus track length s (straight line, no magnetic field);

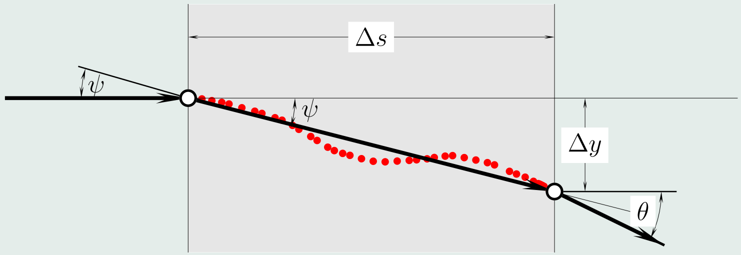
  (b) fit of circle residuals versus track length s including curvature correction $\Delta\kappa$.

A charged particle traversing a material layer is deflected by many small-angle scatters, mostly due to Coulomb scattering from nuclei, with variance

$$V[\theta] = \theta_0^2 = \left(\frac{13.6 \text{ MeV}}{\beta \boldsymbol{p} c}\right)^2 \boldsymbol{t} \left[1 + 0.038 \ln \boldsymbol{t}\right]^2$$

where $\boldsymbol{t} = \Delta s / X_0$

$$\boldsymbol{V}\begin{bmatrix}\theta \\ \psi\end{bmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} \theta_0^2$$



Multiple scattering in a layer of finite thickness

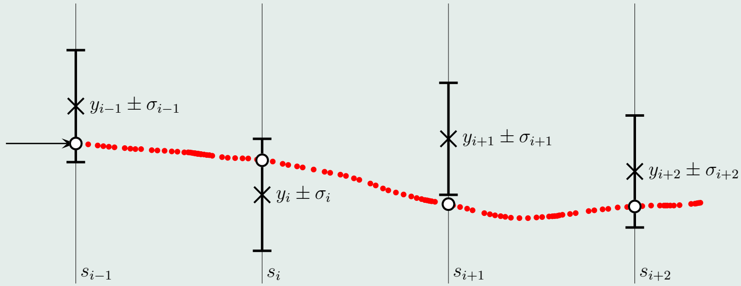
$\theta =$ projected angle of deflection, between direction before and behind layer

$\psi_{\text{left}} \equiv \psi$, $\psi_{\text{right}} \equiv \theta - \psi =$ angles between the **line, connecting the two intersection points**, and the direction before and behind layer

The **line, connecting the two intersection points**, can be determined by a measurement. Relevant for the reconstruction of the trajectory are the angles ψ_{left} and ψ_{right}, with covariance matrix

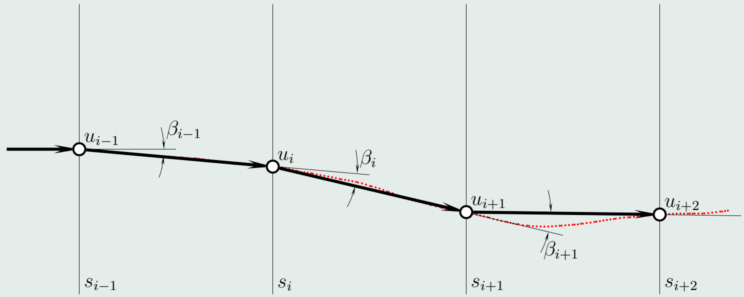
$$\boldsymbol{V}\begin{bmatrix}\psi_{\text{left}} \\ \psi_{\text{right}}\end{bmatrix}_i = \begin{pmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{pmatrix} \theta_{0,i}^2 \quad \text{for a homogeneous medium in layer } i$$

The particle track, given by the dotted curve, intersects the detector planes; the intersection points are drawn as circles. The result of the measurement are data points $y \pm \sigma$, given by crosses with error bar.



Instead of using a track parametrization with e.g. intercept and slopes as parameters, the proposed track-fit method uses **two phases in the track reconstruction**.

(1) Reconstruction of the trajectory:
The trajectory, represented by the intersection points of the trajectory with the detector planes, is determined in a least squares fit; the fitted estimates of the intersection points are denoted by u_i.



(2) Track parameter determination: From the fitted u_i-values the two track parameters intercept and slope, required for the physics analysis, are determined at both sides of the track.

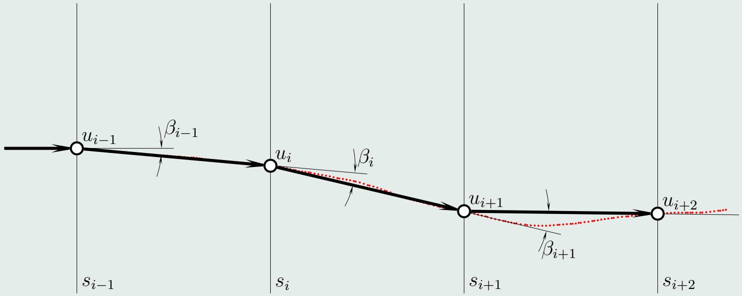
Kink angles

The intersection points u of the particle track with detector planes, drawn as circles, are connected by straight lines. The kink angles β are the angles between adjacent straight lines.

$$\beta_i = \psi_{\text{right},i-1} - \psi_{\text{left},i}$$

$$V[\beta_i] = \sigma^2_{\beta,i} = V[\psi_{\text{right},i-1}] + V[\psi_{\text{left},i}]$$

(multiple scattering)



There are $(n-2)$ kink angles β_i, which are linear functions of the values u_i (with $f_i \approx 1$):

$$\beta_i = f_i \cdot \left[u_{i-1} \frac{1}{s_i - s_{i-1}} - u_i \frac{s_{i+1} - s_{i-1}}{\left(s_{i+1} - s_i\right)\left(s_i - s_{i-1}\right)} + u_{i+1} \frac{1}{s_{i+1} - s_i} \right]$$

The values u_i are determined by minimization of the linear least squares expression (with weight $w_i = 1/\sigma_i^2$)

$$\boxed{S\left(\boldsymbol{u}\right) = \sum_{i=1}^{n} w_i \left(y_i - u_i\right)^2 + \sum_{i=2}^{n-1} \frac{\beta_i^2}{\sigma^2_{\beta,i}}}$$

with $n + (n-2)$ terms. Note: w_i may be zero or very small (vertex fit).

First phase: "Straight line" trajectory fit

The linear least squares expression $S(\boldsymbol{u})$ is minimized by the solution $\boldsymbol{u}$ of the matrix equation:

$$\boldsymbol{C}_u \boldsymbol{u} = \boldsymbol{r}_u \qquad \text{with the sparse } n\text{-by-}n \text{ matrix} \quad \boldsymbol{C}_u = \begin{pmatrix} d & x & x & & & & \\ x & d & x & x & & & \\ x & x & d & x & x & & \\ & x & x & d & x & x & \\ & & x & x & d & x & x \\ & & & & & \ddots & \\ & & & & & & \end{pmatrix}$$

i.e. $\boldsymbol{C}_u$ is a symmetric n-by-n band matrix of bandwidth $m = 2$. The elements of $\boldsymbol{C}_u$ and $\boldsymbol{r}_u$ are calculated by sums from the measured data.

After a decomposition $\boldsymbol{C}_u = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$ the matrix equation becomes $\boldsymbol{L}\left(\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}\boldsymbol{u}\right) = \boldsymbol{r}$. $\boldsymbol{L}$ is a left unit triangular matrix and $\boldsymbol{D}$ is diagonal; the band structure is kept! The matrix $\boldsymbol{C}_u$ can be stored in a 3-by-n array and the decomposition can be made *in-place*.

The solution can be obtained by the following calculations:

decompose	$\boldsymbol{C}_u = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$	decomposition	$(6n)$
solve	$\boldsymbol{L}\boldsymbol{v} = \boldsymbol{r}_u$	for $\boldsymbol{v}$ by forward substitution	$(2n)$
solve	$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{D}^{-1}\boldsymbol{v}$	for $\boldsymbol{u}$ by backward substitution	$(3n)$

The last column gives the number of dot-instructions (multiplication, division); the whole computation time is linear in n, i.e $\mathcal{O}(n)$ operations are necessary for the determination of $\boldsymbol{u}$.

Second phase: Track parameters ...and elements of the inverse matrix

Corrections Δz_0 and $\Delta(\tan\lambda)$ are calculated from the two first u-values u_1 and u_2 amd added to the initial approximations $\widehat{z_0}$ and $\widehat{(\tan\lambda)}$:

$$\begin{pmatrix} z_0 \\ \\ (\tan\lambda) \end{pmatrix} = \begin{pmatrix} \widehat{z_0} + u_1 \\ \\ \widehat{(\tan\lambda)} + \dfrac{u_2 - u_1}{s_2 - s_1} \end{pmatrix}$$

It is possible to calculate those **elements of the inverse matrix** $\boldsymbol{Z} = \boldsymbol{C}_u^{-1}$, which are in the band of the original matrix, in a computation **time linear in $\boldsymbol{n}$***), using the decomposition $\boldsymbol{LDL}^{\mathrm{T}}$. For a bandwidth of $m = 2$ these are $6n$ operations, with restrictions $|i-k| \leq m$ and $|j-k| \leq m$

$$\text{for } i = n \dots 1: \qquad Z_{ii} = D_{ii}^{-1} - \sum_{k=i+1}^{i+2} Z_{ik}\, L_{ki} \qquad\qquad Z_{ij} = -\sum_{k=j+1}^{j+2} Z_{ik}\, L_{kj} \qquad j = i+1,\ i+2$$

Starting with Z_{nn}, a sequence of computation can be performed by calculating elements of $\boldsymbol{Z}$ in reverse order; when calculating Z_{ij} all required elements of $\boldsymbol{Z}$ are already calculated.

By error propagation the covariance matrix of the track parameters is calculated from the elements of $\boldsymbol{V}_u \equiv \boldsymbol{Z}$.

*) K. Takahashi, J. Fagan and M. Chin, “Formation of a sparse bus impedance matrix and its applications to short circuit study”, Proceedings 8th PICA Conference (1973), Minneapolis, Minnesota

(b) Fit of circle residuals versus track length

In addition to the parameters in case (a) now a curvature correction $\Delta\kappa$ has to be determined. The mean value of the "kink angle" β_i, as defined before (a), is now different from zero, due to the magnetic deflection.

This magnetic deflection is taken into account by the **re-definition**

$$\beta_i \approx f_i \cdot \left[u_{i-1} \frac{1}{s_i - s_{i-1}} - u_i \frac{s_{i+1} - s_{i-1}}{\left(s_{i+1} - s_i\right)\left(s_i - s_{i-1}\right)} + u_{i+1} \frac{1}{s_{i+1} - s_i} \right] \quad + \frac{1}{2}\left(a_{i-1} + a_i\right) \cdot \boldsymbol{\Delta\kappa}$$

(a_i is the distance between the points i and $i+1$)
with $E\left[\beta_i\right] = 0$ and this has to be used in the function S

$$\boxed{S\left(\boldsymbol{u}, \Delta\kappa\right) = \sum_{i=1}^{n} \frac{\left(y_i - u_i\right)^2}{\sigma_i^2} + \sum_{i=2}^{n-1} \frac{\beta_i^2}{\sigma_{\beta,i}^2}}$$

which has to be minimized with respect to the values u_i and $\Delta\kappa$.

First phase: "Curved line" trajectory fit

The case of the additional parameter $\Delta\kappa$ is only slightly more complicated. The linear least squares expression $S(\boldsymbol{u}, \Delta\kappa)$ is minimized by the solution of the matrix equation:

$$\left(\begin{array}{c|c} C_\kappa & \boldsymbol{c}^{\mathrm{T}} \\ \hline \boldsymbol{c} & \boldsymbol{C}_u \end{array} \right) \left(\begin{array}{c} \Delta\kappa \\ \hline \boldsymbol{u} \end{array} \right) = \left(\begin{array}{c} \boldsymbol{r}_\kappa \\ \hline \boldsymbol{r}_u \end{array} \right)$$

with matrix

$$\left(\begin{array}{c|cccccccc} C_\kappa & c & c & c & c & c & c & c & \cdots \\ \hline c & d & x & x & & & & & \\ c & x & d & x & x & & & & \\ c & x & x & d & x & x & & & \\ c & & x & x & d & x & x & & \\ c & & & x & x & d & x & x & \\ c & & & & & & \ddots & & \\ \vdots & & & & & & & & \end{array} \right)$$

i.e. the matrix is a bordered band matrix.

Solution:

$\boldsymbol{C}_u = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{T}}$	decomposition	$(6n)$
$\boldsymbol{C}_u \boldsymbol{z} = \boldsymbol{c}$	solution for $\boldsymbol{z}$	$(5n)$
$\boldsymbol{B}_\kappa = \left(\boldsymbol{C}_\kappa - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{z}\right)^{-1}$	variance of curvature	$(n+1)$
$\boldsymbol{\Delta\kappa} = \boldsymbol{B}_\kappa \left(\boldsymbol{r}_\kappa - \boldsymbol{z}^{\mathrm{T}}\boldsymbol{r}_u\right)$	curvature	$(n+1)$
$\boldsymbol{C}_u \widetilde{\boldsymbol{u}} = \boldsymbol{r}_u$	solution for $\widetilde{\boldsymbol{u}}$	$(5n)$
$\boldsymbol{u} = \widetilde{\boldsymbol{u}} - \boldsymbol{z}\boldsymbol{\Delta\kappa}$	smoothed coordinates	(n)

Second phase: Track parameters

The curvature correction $\Delta\kappa$ is already calculated; the corrections for position and direction are calculated, as before, from u_1 and u_2, with covariance matrix by error propagation.

The complete inverse matrix of the bordered band matrix is written here,

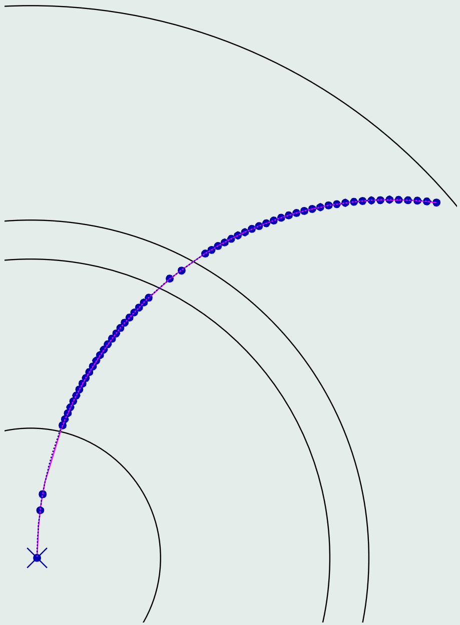
$$\left(\begin{array}{c|c} C_\kappa & \boldsymbol{c}^{\mathrm{T}} \\ \hline \boldsymbol{c} & \boldsymbol{C}_u \end{array}\right)^{-1} = \left(\begin{array}{c|c} \boldsymbol{B}_\kappa & -\boldsymbol{B}_\kappa \boldsymbol{z}^{\mathrm{T}} \\ \hline -\boldsymbol{z}\boldsymbol{B}_\kappa & \boldsymbol{C}_u^{-1} + \boldsymbol{z}\boldsymbol{B}_\kappa \boldsymbol{z}^{\mathrm{T}} \end{array}\right) ,$$

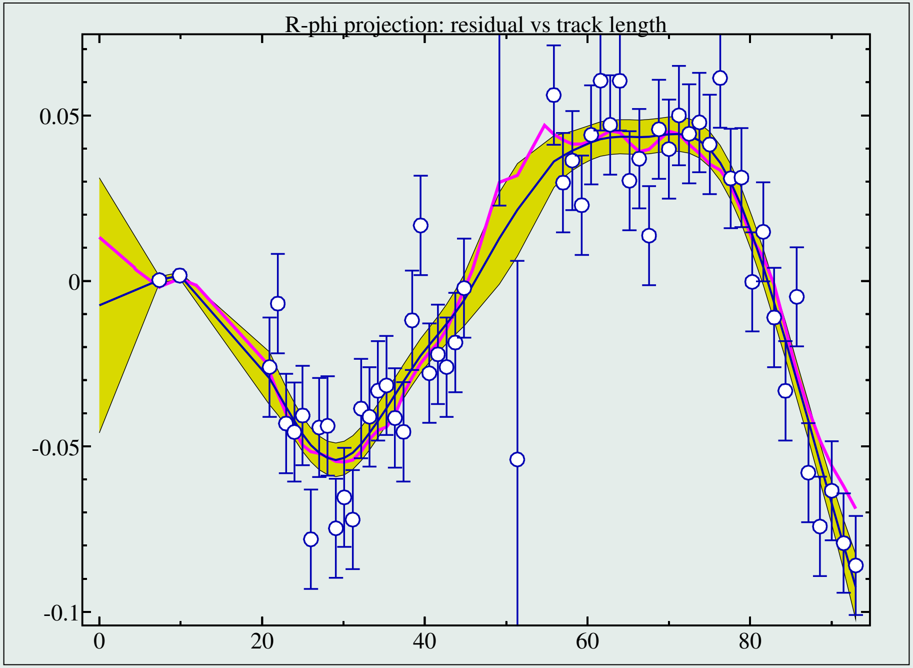
although only few elements are required, e.g. the element

$$V_{u,12} = \left(\boldsymbol{C}_u^{-1}\right)_{12} + \boldsymbol{B}_\kappa \cdot z_1 \, z_2$$

Note: The matrices C_κ and B_κ are in this application scalars (only one common parameter). They become 2-by-2 matrices, if two common parameters are determined, e.g. $\Delta\kappa$ and ΔT_0 (drift chamber time zero). The above formulas remain valid.

Blue line and yellow band is fit $\pm 1\sigma$

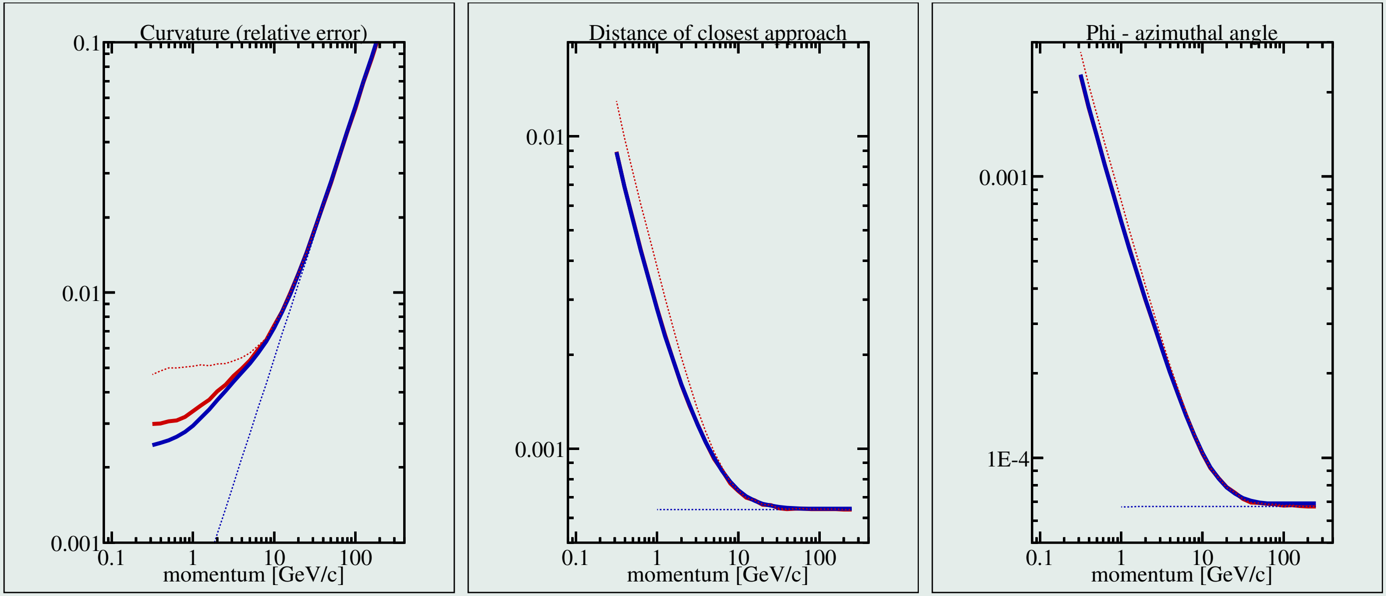




The broken-line fit result is given as blue broken-line with a ± 1 standard deviation yellow band, with extrapolation to the vertex at $s = 0$ (weight in fit was $w_1 = 0$). ← back

Thick red and thick blue lines: error from histogram fit (**red**) and calculated error (**blue**) for broken-line fit, with improvements by 37 % (momentum), by 31 % (d_{ca}) and 17 % (ϕ) at lowest momentum.

Thin dotted lines: error from histogram fit (red) and calculated error (blue) for simple LS fit



→ calculated errors are realistic for whole momentum range!

G. Lutz, Optimum track fitting in the presence of multiple scattering, NIMA 273 (1988) 349 – 361

Improved accuracy with correct covariance matrix: Improvement of the track reconstruction precision for tracks of low momentum in a dense detector, for high position measurement accuracy. Extension to include energy loss (deterministic) for heavy particles, and magnetic-field inhomogenities are possible.

Full information: The algorithm gives the full information on every measured point on a trajectory:

fitted value with propagated error, pull of position and kink angle

and this information can be used already during track finding, to remove bad hits or to cut the trajectory, and to adjust material assumption.

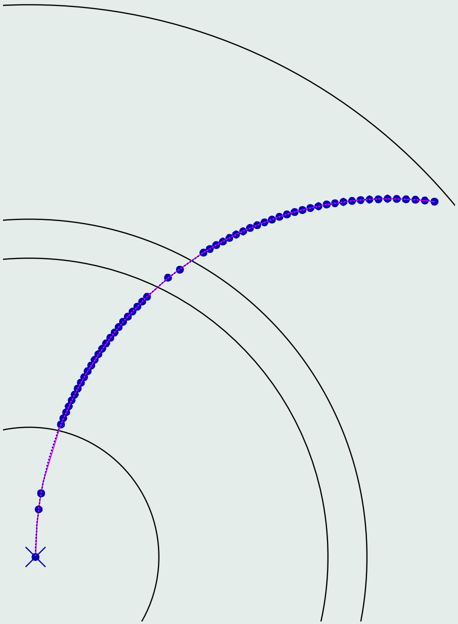
Outlier down-weighting: Outliers can be recognized; repeated fit with down-weighted hits done in reduced time.

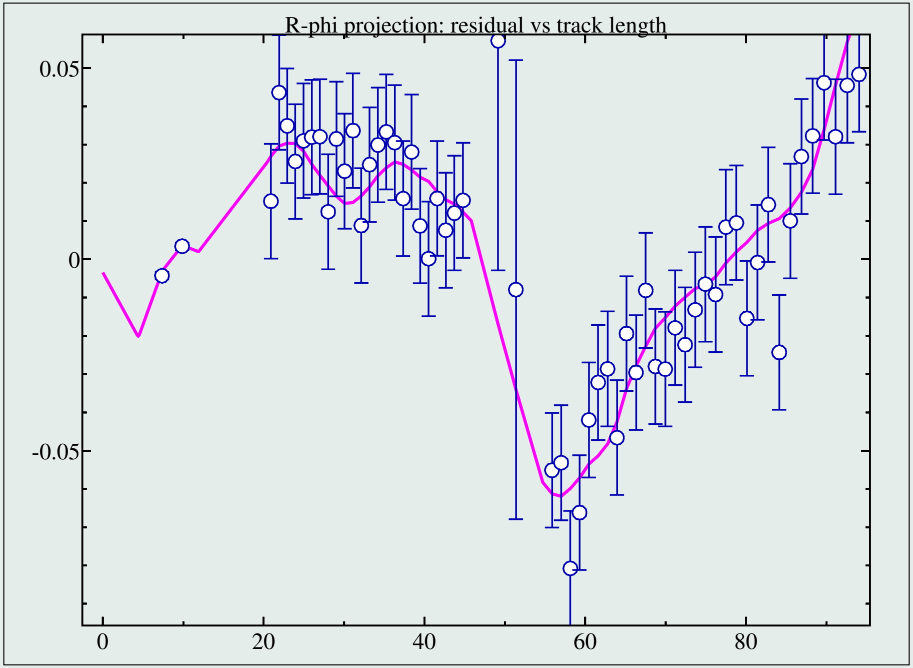
Broken-line fits: **New algorithm**, faster by factor ≈ 6 in comparison to Kalman filter (under test conditions), with fit in one step (no initial values, no iteration, no recursion, no direction backward or forward) in time $\mathcal{O}(n)$.

Backup slides

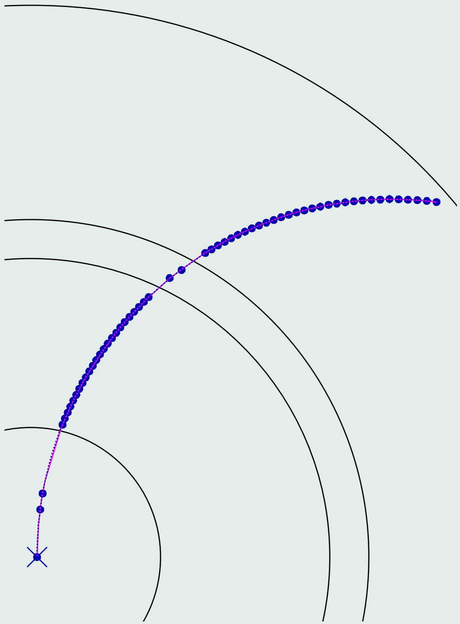
- 200 MeV/c track with multiple scattering
- . . . with broken-line fit
- χ^2- and P-values
- Pulls of position measurement
- Pulls of kink angle measurement
- Outlier examples
- A large-angle scatter

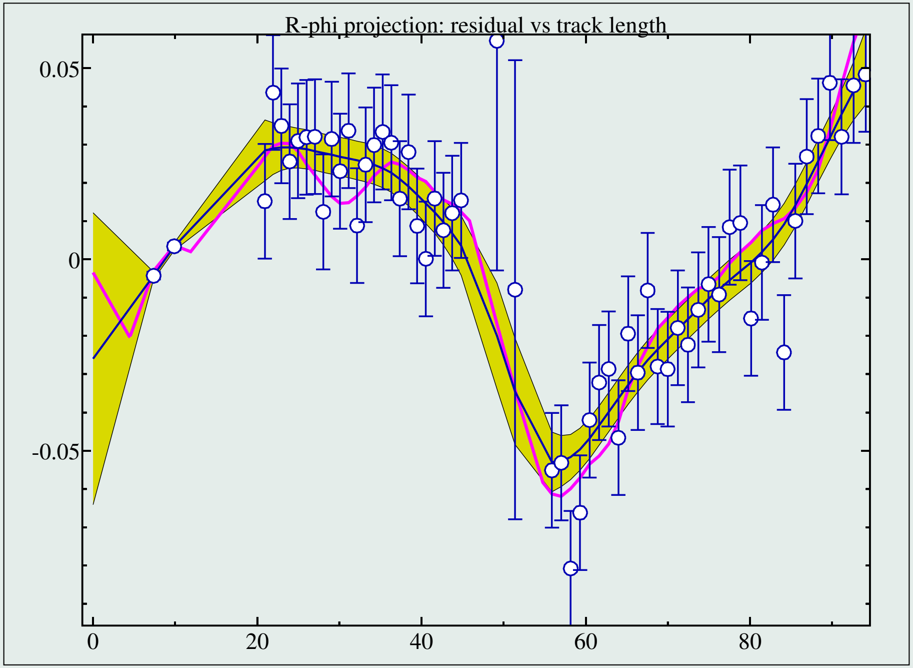
Track with multiple scattering 200 MeV/c





Magenta line is the true particle trajectory, with large kink angles in beam pipe, chamber walls etc.



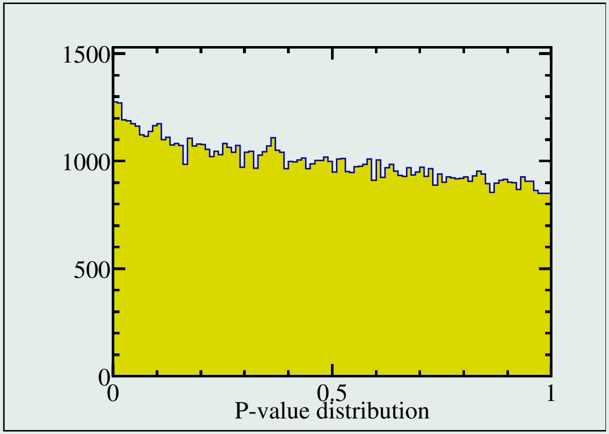


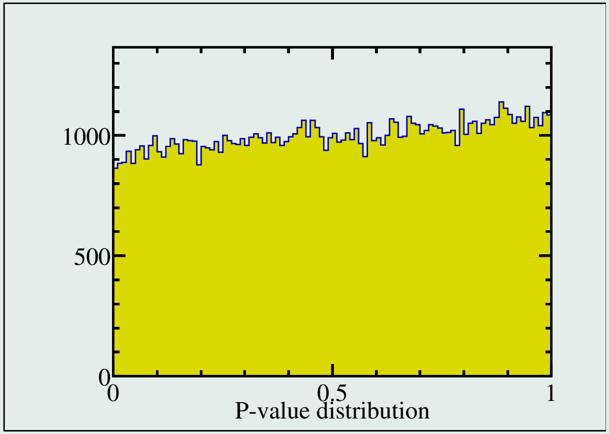
The broken-line fit result is given as a ± 1 standard deviation band, with extrapolation to the vertex at $s = 0$ (weight in fit was $w_1 = 0$).

$$\chi^2_{(n-2)} \equiv S(\boldsymbol{u})_{\min} = \chi^2\,(n \text{ measurements}) + \chi^2\,(n-2 \text{ kink angles})$$

with n fitted parameters.

$$\text{Number of degrees of freedom} = n - 2$$



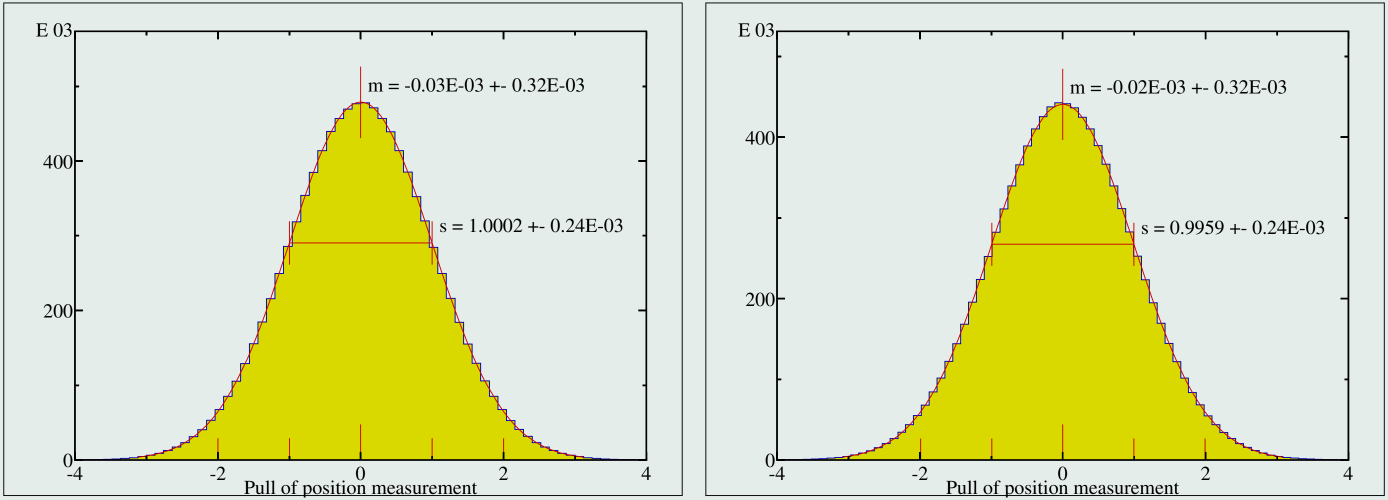


P-value distribution for the momenta of 0.5 GeV/c (left, mean value of $\chi^2 = 99.2$) and 10 GeV/c (right, mean value of $\chi^2 = 97.4$) are shown from the trajectory fit
for a modified H1 configuration (2 CST hits plus 98 drift chamber hits) in a vertex fit with a standard deviation of 30 μm.

Pulls of position measurement

The pull of the position measurement is defined as the difference of the measured and fitted position, divided by the standard deviation of the difference:

$$p_{y,i} = \frac{y_i - u_i}{\sqrt{\sigma_i^2 - (\boldsymbol{V}_u)_{ii}}}$$



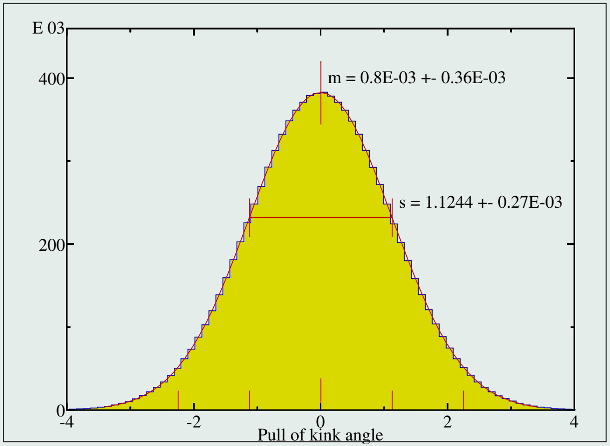
The pulls follow the expected $N(0,1)$ distribution for a momentum of 0.5 GeV/c (left) and for a momentum of 10 GeV/c (right) (H1 detector configuration with 2 CST and 56 CJC hits).

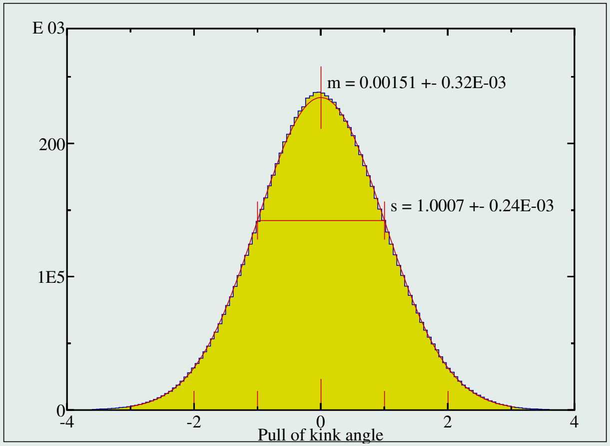
Pulls of kink angle measurement

The pull of the kink angle is defined as the difference of the expected (zero) and fitted angle, divided by the standard deviation of the difference:

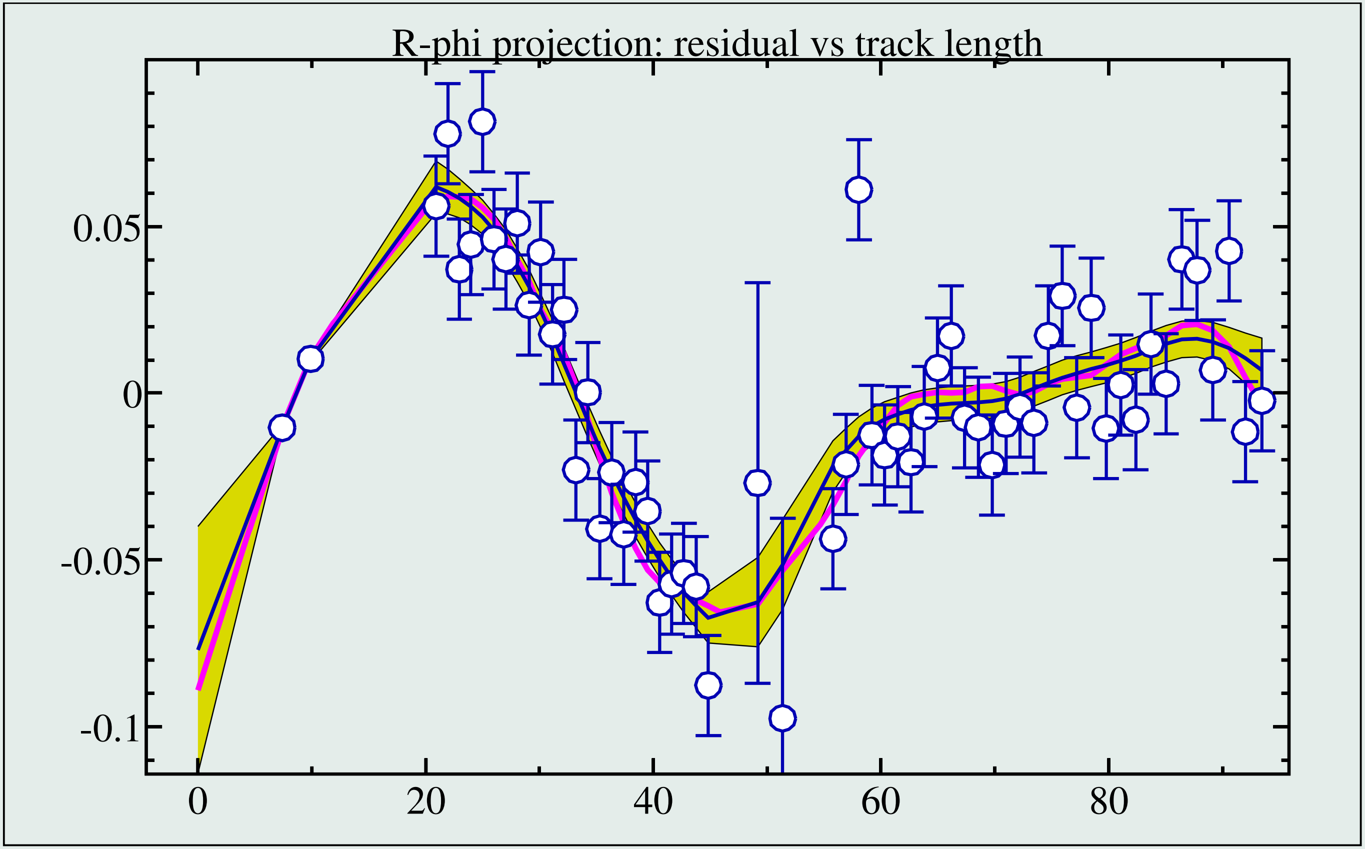
$$p_{\beta,i} = \frac{\beta_i}{\sqrt{\sigma^2_{\beta,i} - (\boldsymbol{V}_\beta)_{ii}}} \,,$$

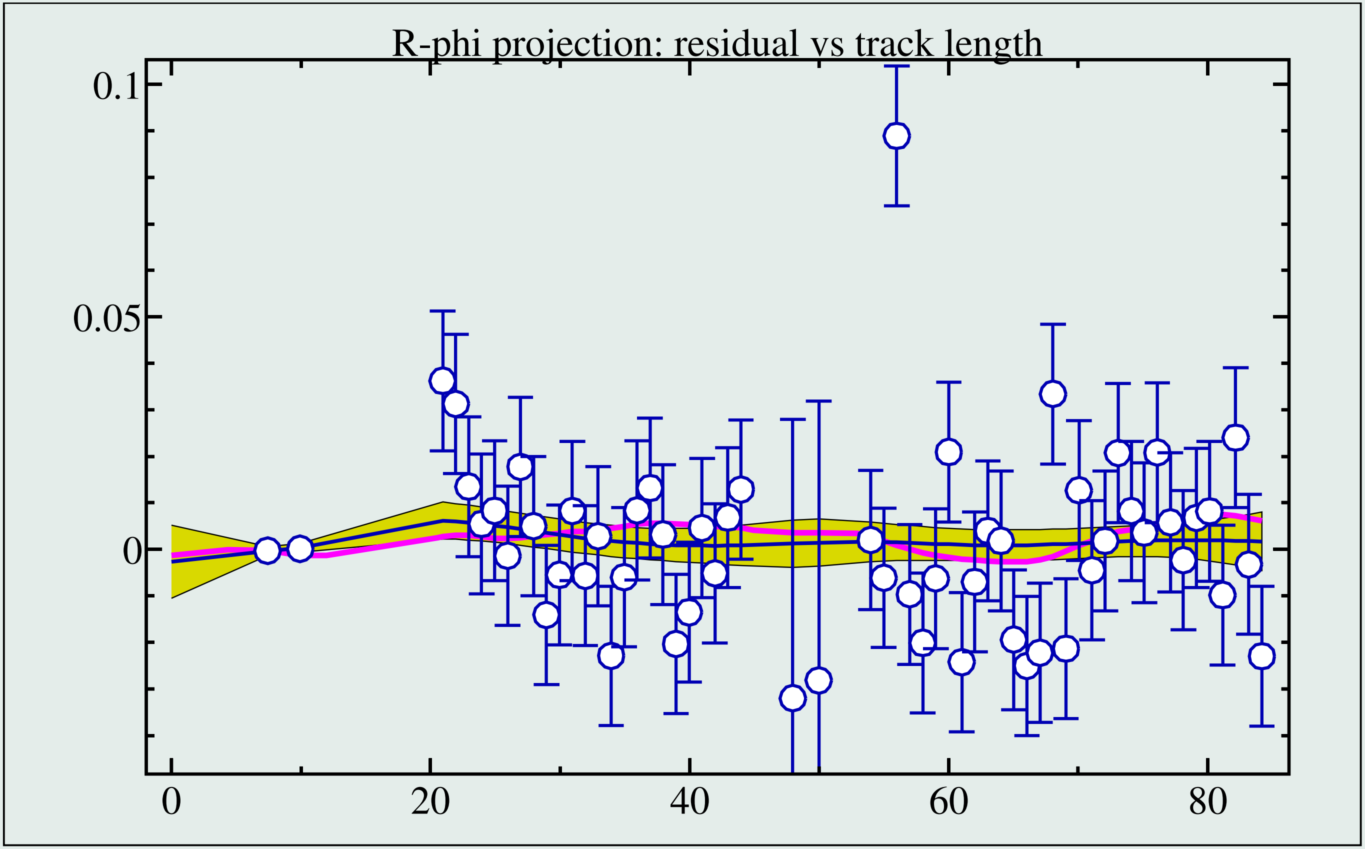
where $\boldsymbol{V}_\beta$ is calculated by error propagation from the covariance matrix $\boldsymbol{V}_u$.



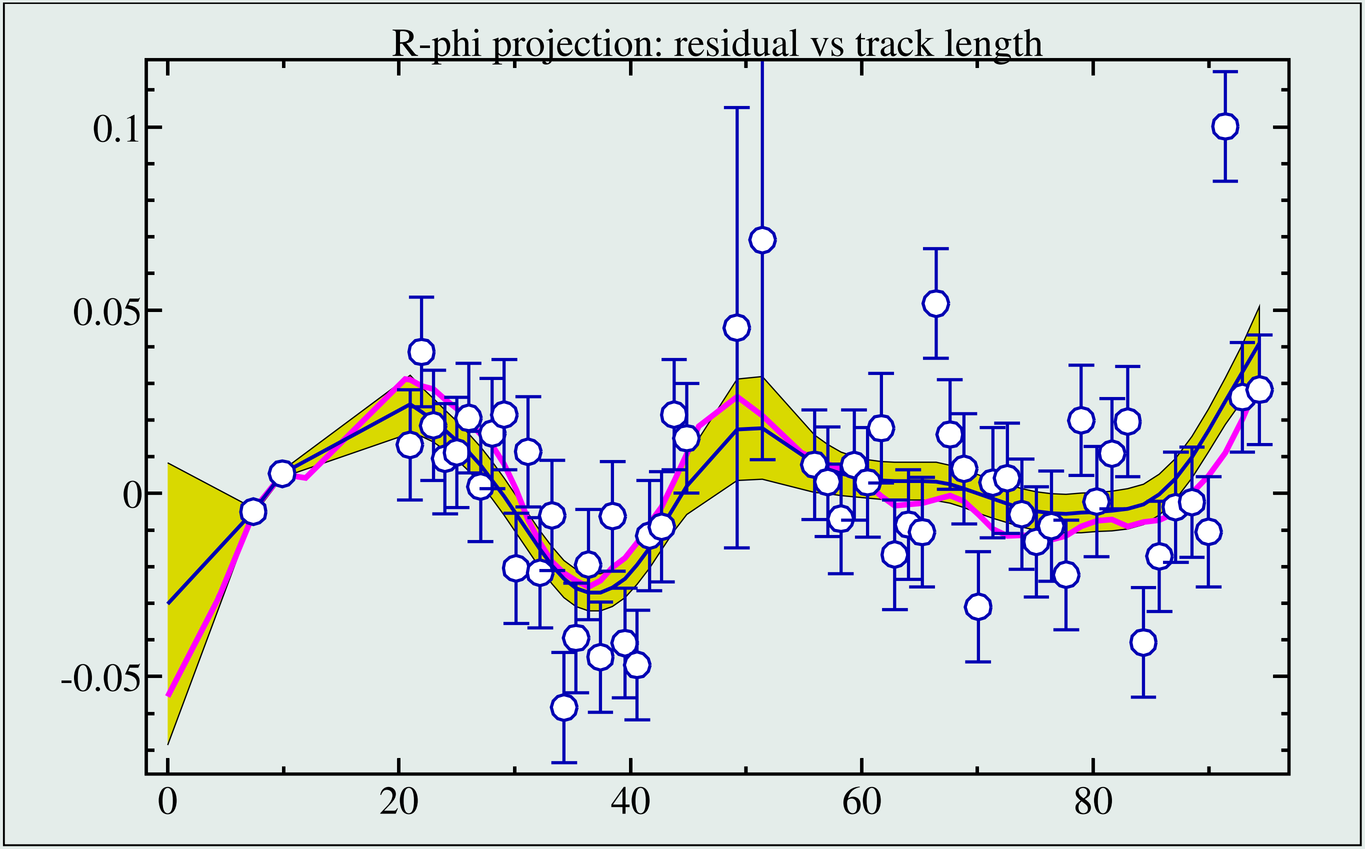


The pulls follow almost the expected $N(0,1)$ distribution for a momentum of 0.5 GeV/c (left) and for a momentum of 10 GeV/c (right).

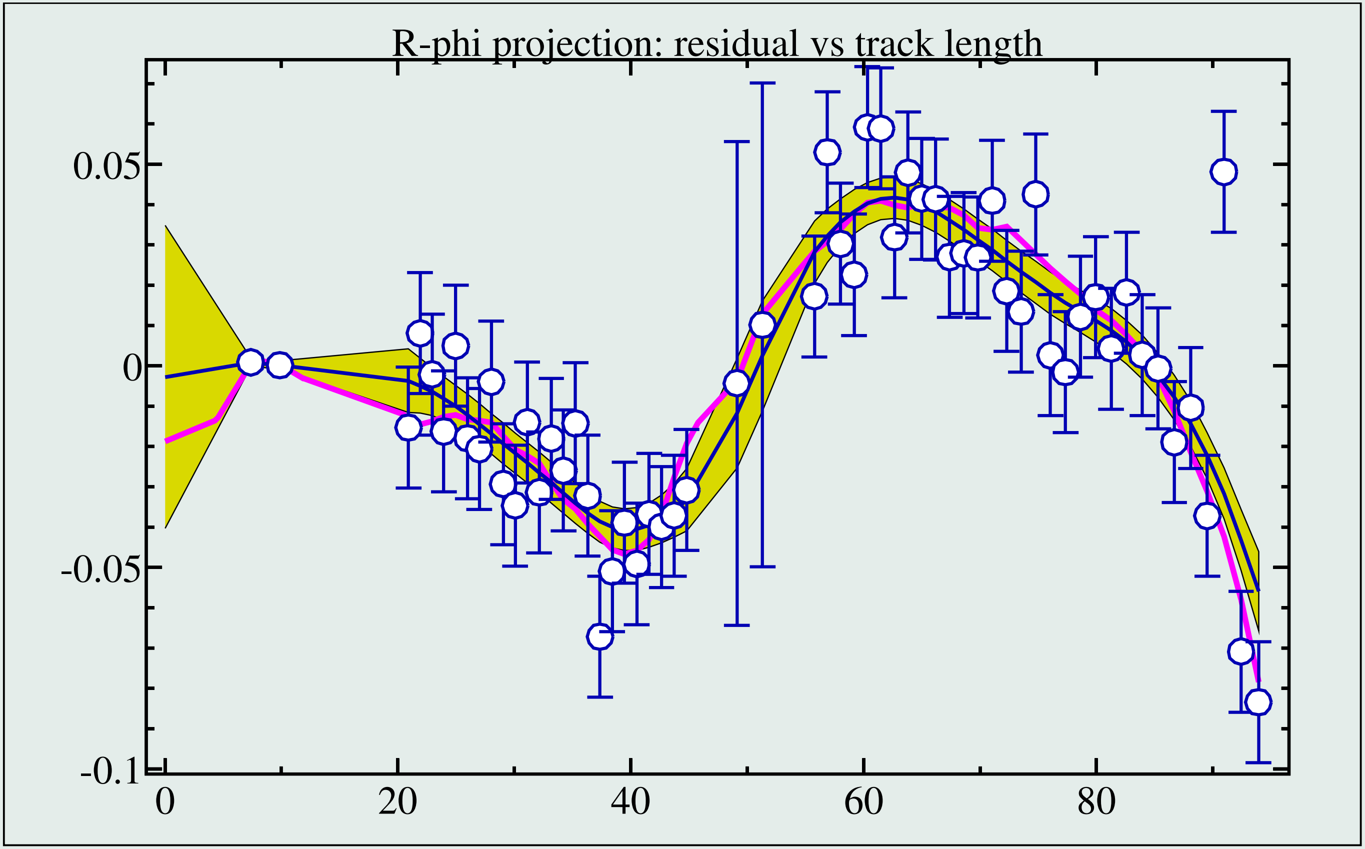

R-phi projection: residual vs track length
0.05
0
-0.05
-0.1
0
20
40
60
80


R-phi projection: residual vs track length
0.1
0.05
0
0
20
40
60
80

Outlier example 3 — 200 MeV/c, 6σ outlier

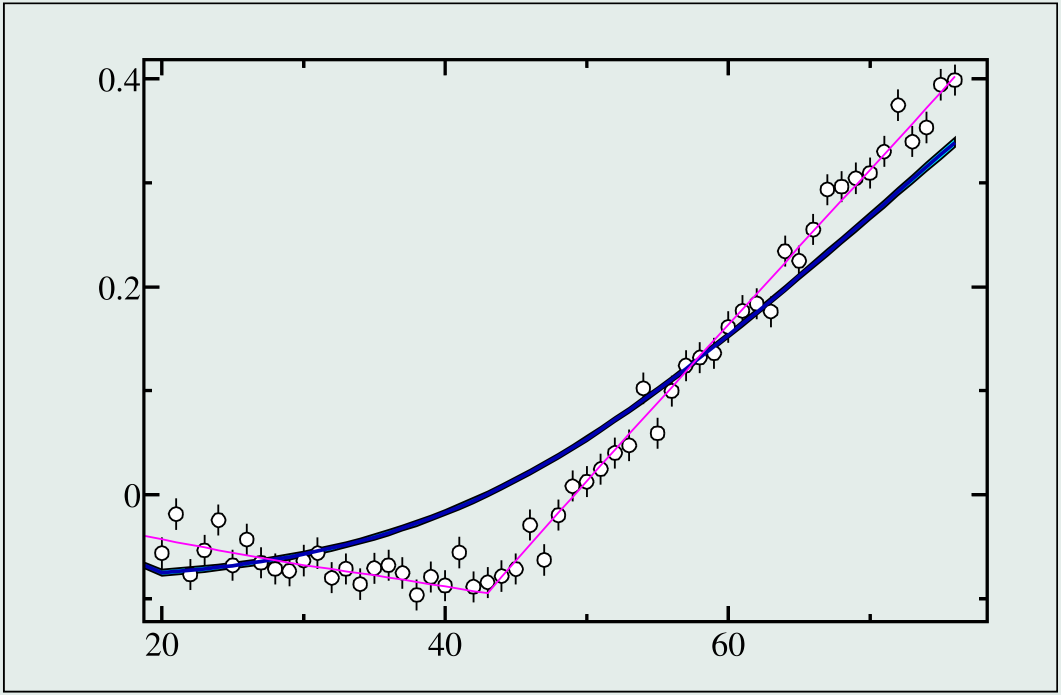


Outlier example 4 — 200 MeV/c, 6σ outlier



A large-angle scatter

A 1 GeV/c track with an artificial 1° kink around $s = 42$ cm:



The kink angle is much larger than the *rms* multiple scattering angle, expected for 1 GeV/c. The overall χ^2 of the trajectory fit (blue curve) is large.

Contents