Extracting azimuthal Fourier moments from sparse data

- Hermes data is always statistics-limited, because we always want to extract kinematic distributions in finer bins or more dimensions

⇨ Azimuthal Fourier moments must be extracted from sparse distributions containing bins with non-Gaussian statistics

- The acceptance can (and does!) cause a substantial systematic bias on observables extracted while integrating over kinematic variables on which that observable strongly depends. Examples:
  - P_t-weighted transverse target spin asymmetries
  - DVCS beam charge asymmetries (see next page)

⇨ Quantify ignorance of full kinematic dependence, propagate to result

⇨ fit the full kinematic dependence on (x,y,z,P_t) using some standard set of 4D orthogonal functions, then fold with known $\sigma_{UU}(x,y,z,P_t)$

Covariance matrix of fitted coefficients is propagated through folding

Maximum-Likelihood fit: unpol. example

For ML fit of unpol. azimuthal moments, the event dist'n and PDF are

$$C\,N(x,y,z,P_t,\phi,\phi_S) \;=\; \varepsilon(x,y,z,P_t,\phi,\phi_S)\,\underline{\sigma}_{UU}(x,y,z,P_t)\;\times$$

$$\left[1+A_{UU}^{\cos\phi}(\lambda_1,x,y,z,P_t)\,\cos\phi\;+\;A_{UU}^{\cos 2\phi}(\lambda_2,x,y,z,P_t)\,\cos(2\phi)\right]$$

$$\equiv\; F_{UU}(\lambda_1,\lambda_2,x,y,z,P_t,\phi,\phi_S)$$ (Probability Density Fun.)

Maximize Likelihood with respect to parameter sets λ_1, λ_2:

$$\mathcal{L}(\lambda_1,\lambda_2) \;=\; \frac{\prod_{i=1}^{N_x} F_{UU}(\lambda_1,\lambda_2,x_i,y_i,z_i,P_{ti},\phi_i,\phi_{Si})}{\mathcal{N}_{UU}^{N_x}(\lambda_1,\lambda_2)}$$

The denominator fixes the normalization of the PDF as the parameter sets λ_1 and λ_2 are stepped in the fit search:

$$\mathcal{N}_{UU}(\lambda_1,\lambda_2) \;=\; \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}P_t\,\mathrm{d}\phi\,\mathrm{d}\phi_S\;F_{UU}(\lambda_1,\lambda_2,x,y,z,P_t,\phi,\phi_S)$$

Acceptance ε and azimuthally averaged cross section $\underline{\sigma}_{UU}$ do not depend on the fitting parameter sets λ_1 and λ_2

⇨ **they can be omitted in calculation of the numerator!!**

How can we conveniently evaluate the normalization integral?

PDF Normalization: unpolarized case

Probability Density Function normalization:

$$\mathcal{N}_{UU}(\lambda_1,\lambda_2) = \int dx\,dy\,dz\,dP_t\,d\phi\,d\phi_S\,\varepsilon(x,y,z,P_t,\phi,\phi_S)\,\underline{\sigma}_{UU}(x,y,z,P_t)\times \left[1+A_{UU}^{\cos\phi}(\lambda_1,x,y,z,P_t)\cos\phi + A_{UU}^{\cos 2\phi}(\lambda_2,x,y,z,P_t)\cos(2\phi)\right]$$

Solution:

Use Monte Carlo integration method with azimuthal event weights.

As Pythia MC events are distributed according to $\varepsilon\underline{\sigma}_{UU}$, PDF integral is

$$\mathcal{N}_{UU}(\lambda_1,\lambda_2) = \sum_{j=1}^{N_{MC}} W_j^{MC}\left[1 + A_{UU}^{\cos\phi}(\lambda_1,x_j,y_j,z_j,P_{tj})\cos\phi_j + A_{UU}^{\cos 2\phi}(\lambda_2,x_j,y_j,z_j,P_{tj})\cos(2\phi_j)\right]$$

For efficiency:

All factors in both likelihood product (expt'l events) and integral sum (MC events) can be tabulated for all events before starting the fit search

Result of the fit

$$\sigma_{UU}(x,y,z,P_t,\phi) = \underline{\sigma}_{UU}(x,y,z,P_t) \times \left[1 + A_{UU}^{\cos\phi}(\lambda_1,x,y,z,P_t)\cos\phi + A_{UU}^{\cos 2\phi}(\lambda_2,x,y,z,P_t)\cos(2\phi)\right]$$

The parameter sets λ_1 and λ_2 could be archived in the Durham data base, but we compare models to asymmetries in yields integrated over some variables:

$$\langle \cos\phi \rangle_{UU}^{h}(x) = \frac{\int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}P_t\ \underline{\sigma}_{UU}^{Born}(x,y,z,P_t)\ A_{UU}^{\cos\phi}(\lambda_1,x,y,z,P_t)}{\int \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}P_t\ \underline{\sigma}_{UU}^{Born}(x,y,z,P_t)}$$

This integral can be evaluated using $\underline{\sigma}_{UU}(x,y,z,P_t)$ from parton dist. funs and measured hadron multiplicities, or a Pythia MC event set generated in 4π:

$$\langle \cos\phi \rangle_{UU}^{h}(x) = \sum_{j=1}^{N_{MC}} W_j^{MC} A_{UU}^{\cos\phi}(\lambda_1,x_j,y_j,z_j,P_{tj}) / \sum_{j=1}^{N_{MC}} W_j^{MC}$$

If the parameterization is linear in the fitted parameters, it is easy to propagate their covariance matrices through this sum

Exchange unknown systematic error for well-defined statistical uncertainty

Maximum-likelihood fit: transverse-polarized case

Using the predetermined full kinematic dependence of the cos(nϕ) moments, the event distribution and PDF for target polarization dist'n $\rho(P)$, $-1<P<1$, is:

$$
\begin{aligned}
C\,N(P,x,y,z,P_t,\phi,\phi_S) \;=\;& \rho(P)\;\varepsilon(x,y,z,P_t,\phi,\phi_S)\;\underline{\sigma}_{UU}(x,y,z,P_t)\times \\
& \Big\{ 1 + A_{UU}^{\cos\phi}(x,y,z,P_t)\cos\phi + A_{UU}^{\cos 2\phi}(x,y,z,P_t)\cos(2\phi) \\
& \quad + P\left[A_C(\lambda_1,x,y,z,P_t)\sin(\phi+\phi_S) + A_S(\lambda_2,x,y,z,P_t)\sin(\phi-\phi_S)\right]\Big\} \\
\equiv\;& F(\lambda_1,\lambda_2,P,x,y,z,P_t,\phi,\phi_S)
\end{aligned}
$$

ML treats the target polarization P like any other (e.g., kinematic) variable.

Again the parameter-independent factor $\varepsilon\underline{\sigma}_{UU}$ can be omitted in the numerator of the Likelihood:

$$
\mathcal{L}(\lambda_1,\lambda_2) \;=\; \prod_{i=1}^{N} \frac{F(\lambda_1,\lambda_2,P_i,x_i,y_i,z_i,P_{ti},\phi_i,\phi_{Si})^{W_i}}{\mathcal{N}(\lambda_1,\lambda_2)^{W_i}}
$$

Here, W_i are event weights (from e.g. RICH PId)

The product in the denominator is independent of λ_1 and λ_2 and can be ignored in the likelihood maximization, if the whole data set has no net polarization:

$$
\int \mathrm{d}P\; P\,\rho(P) = 0
$$

PDF normalization: transverse-polarized case

In the PDF normalization integral, the integration over P factorizes:

$$\mathcal{N}(\lambda_1,\lambda_2) = \int \mathrm{d}P\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}P_t\,\mathrm{d}\phi\,\mathrm{d}\phi_S\ \rho(P)\ \varepsilon(x,y,z,P_t,\phi,\phi_S)\ \underline{\sigma}_{UU}(x,y,z,P_t) \times$$
$$\Big\{1 + A_{UU}^{\cos\phi}(x,y,z,P_t)\cos\phi + A_{UU}^{\cos 2\phi}(x,y,z,P_t)\cos(2\phi)$$
$$+ P\left[A_C(\lambda_1,x,y,z,P_t)\sin(\phi+\phi_S) + A_S(\lambda_2,x,y,z,P_t)\sin(\phi-\phi_S)\right]\Big\}$$
$$= \int \mathrm{d}P\rho(P) \cdot \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}P_t\,\mathrm{d}\phi\,\mathrm{d}\phi_S\ \varepsilon(x,y,z,P_t,\phi,\phi_S)\ \underline{\sigma}_{UU}(x,y,z,P_t) \times$$
$$\Big\{1 + A_{UU}^{\cos\phi}(x,y,z,P_t)\cos\phi + A_{UU}^{\cos 2\phi}(x,y,z,P_t)\cos(2\phi)$$
$$+ \frac{\int \mathrm{d}P\ P\rho(P)}{\int \mathrm{d}P\ \rho(P)}\left[A_C(\lambda_1,x,y,z,P_t)\sin(\phi+\phi_S) + A_S(\lambda_2,x,y,z,P_t)\sin(\phi-\phi_S)\right]\Big\}$$

This normalization integral is obviously independent of λ_1 and λ_2 if...

$$\int \mathrm{d}P\ P\rho(P) = 0$$

If necessary, this might be arranged by scaling the weights of events recorded with one polarization sign.

ML fit: DVCS beam-helicity asymmetry

The event distribution and PDF for beam polarization dist'n $\rho(P)$, $-1<P<1$, is:

$$\begin{aligned} C\,N(P,x,y,t,\phi) &= \rho(P)\,\varepsilon(x,y,t,\phi)\,\underline{\sigma}_{UU}(x,y,t)\times \\ &\quad \{1 + P[A_1(\lambda_1,x,y,t)\sin(\phi) + A_2(\lambda_2,x,y,t)\sin(2\phi)]\} \\ &\equiv F(\lambda_1,\lambda_2,P,x,y,t,\phi) \end{aligned}$$

Again the parameter-independent factor $\varepsilon\underline{\sigma}_{UU}$ can be omitted in the numerator of the Likelihood:

$$\mathcal{L}(\lambda_1,\lambda_2) = \prod_{i=1}^{N} \frac{F(\lambda_1,\lambda_2,P_i,x_i,y_i,t_i,\phi_i)^{W_i}}{\mathcal{N}(\lambda_1,\lambda_2)^{W_i}}$$

Here, W_i are event weights (from e.g. RICH PId)

We will show the normalization in the denominator is independent of λ_1 and λ_2 and can therefore be ignored in the likelihood maximization if either:

-- the net target polarization for the whole data set is zero:

$$\int \mathrm{d}P\,P\,\rho(P) = 0$$

-- or if the acceptance has no odd harmonics in ϕ:

$$\varepsilon(x,y,t,\phi) = \varepsilon(x,y,t,-\phi)$$

PDF normalization: DVCS beam-helicity case

In the PDF normalization integral, the integration over P factorizes:

$$\begin{aligned}\mathcal{N}(\lambda_1,\lambda_2) &= \int dP\, dx\, dy\, dt\, d\phi\; \rho(P)\; \varepsilon(x,y,t,\phi)\; \underline{\sigma}_{UU}(x,y,t) \times \\ &\qquad \{1 + P\,[\,A_1(\lambda_1,x,y,t)\sin(\phi) + A_2(\lambda_2,x,y,t)\sin(2\phi)]\} \\ &= \int dP \rho(P) \cdot \int dx\, dy\, dt\, d\phi\; \varepsilon(x,y,t,\phi)\; \underline{\sigma}_{UU}(x,y,t) \times \\ &\qquad \left\{1 + \frac{\int dP\, P\, \rho(P)}{\int dP\, \rho(P)}\,[\,A_1(\lambda_1,x,y,t)\sin(\phi) + A_2(\lambda_2,x,y,t)\sin(2\phi)]\right\}\end{aligned}$$

This normalization integral is independent of λ_1 and λ_2 if either...

$$\int dP\, P\, \rho(P) = 0$$

(If necessary, this can be arranged by scaling the weights of events recorded with one polarization sign)

or if...

$$\varepsilon(x,y,t,\phi) = \varepsilon(x,y,t,-\phi)$$

because its convolution with sin(nϕ) again yields zero for the second term.

Summary

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- An unknown systematic error from multi-dimensional correlations between acceptance and asymmetry is exchanged for a slightly larger but well-known statistical uncertainty
- All available information about the correlated kinematic dependence of the asymmetries can be extracted and available for formation of any projection of the result