Transverse Target Moments of Dihadron Production in Semi-inclusive Deep Inelastic Scattering at HERMES

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Dihadron Fragmentation Function
Mini-Workshop
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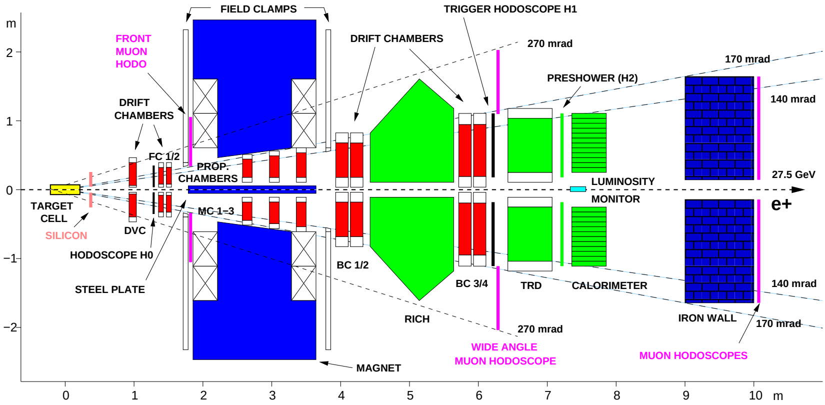
Outline

I. Background & Motivation
II. Theory
III. The TMDGen Generator
IV. Analysis
V. Results & Conclusions

Motivation & Background

The HERMES Spectrometer



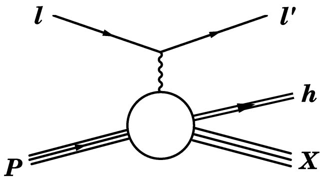
Beam Long. pol. $e^{\pm}$ at 27.6 GeV

Target Trans. pol. H ($\approx$ 75%)
Log. pol. H ($\approx$ 85%)
Unpol. H,D,Ne,Kr,...

Lep.-Had. Sep. High efficiency $\approx$ 98%
Low contamination (<2%)

Hadron PID Separates $\pi^{\pm}$, $K^{\pm}$, p, $\bar{p}$
with momenta in 2-15 GeV

SIDIS Production of Hadrons



- The SIDIS hadron & dihadron processes

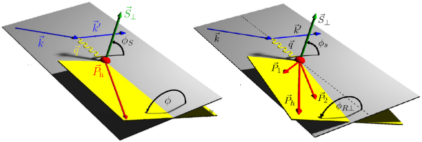
$$\begin{aligned} e + p &\rightarrow e' + h + X, \\ e + p &\rightarrow e' + h_1 + h_2 + X. \end{aligned}$$

- Dihadron production includes all sub-processes leading to hadron pair final states

- Factorization theorem implies $\sigma^{ep \to ehX} = \sum_q DF \otimes \sigma^{eq \to eq} \otimes FF$

- Access integrals of DFs and FFs through Fourier moments of ϕ_h, ϕ_S, ϕ_R & Legendre polynomials in $\cos\vartheta$.

$$\begin{aligned} \phi_h &= \mathrm{signum}\left[(\boldsymbol{k} \times \boldsymbol{P}_h) \cdot \boldsymbol{q} \right] \arccos \frac{(\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{q} \times \boldsymbol{P}_h)}{|\boldsymbol{q} \times \boldsymbol{k}|\, |\boldsymbol{q} \times \boldsymbol{P}_h|}, \\ \phi_S &= \mathrm{signum}\left[(\boldsymbol{k} \times \boldsymbol{S}) \cdot \boldsymbol{q} \right] \arccos \frac{(\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{q} \times \boldsymbol{S})}{|\boldsymbol{q} \times \boldsymbol{k}|\, |\boldsymbol{q} \times \boldsymbol{S}|}, \\ \phi_R &= \mathrm{signum}\left[(\boldsymbol{R} \times \boldsymbol{P}_h) \cdot \boldsymbol{n} \right] \arccos \frac{(\boldsymbol{q} \times \boldsymbol{k}) \cdot (\boldsymbol{P}_h \times \boldsymbol{R})}{|\boldsymbol{q} \times \boldsymbol{k}|\, |\boldsymbol{P}_h \times \boldsymbol{R}|}. \end{aligned}$$

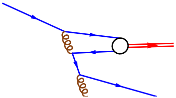


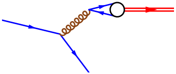
Motivation

- Collinear SIDIS Dihadron cross section
  - Collinear access to transversity through two transverse target moments.
  - Transversity is coupled with "Collins-like" fragmentation functions $H_{1,OT}^{\sphericalangle\, sp}$ and $H_{1,LT}^{\sphericalangle\, pp}$, associated with *sp* and *pp* interference.
- TMD SIDIS Dihadron cross section
  - The Lund/Artru string fragmentation model predicts Collins function for pseudo-scalar and vector meson final states have opposite signs.
- Two types of fragmentation are usually defined

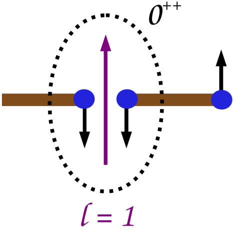
  Favored: struck quark present in the observed particles.

  Disfavored: struck quark not present in the observed particles.





Lund/Artru String Fragmentation Model



- Favored fragmentation modeled as the breaking of a gluon flux tube between the struck quark and the remnant.
- Assume that the flux tube breaks into a $q\bar{q}$ pair with quantum numbers equal to the vacuum.

- Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ states to prefer "quark left".
  - $|0, 0\rangle =$ pseudo-scalar mesons.
  - $|1, 0\rangle =$ longitudinally polarized vector mesons.
- Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$ states to prefer "quark right".
  - $|1, \pm 1\rangle =$ transversely polarized vector mesons.
- For the two ρ_T's, "the Collins function" should have opposite sign to that for π
- For ρ_L, "the Collins function" is zero.

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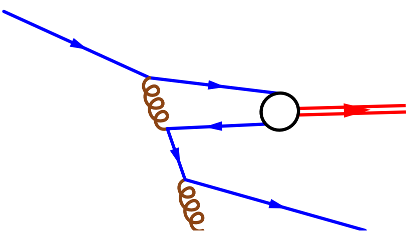

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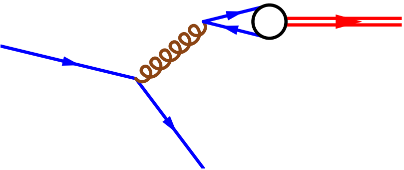
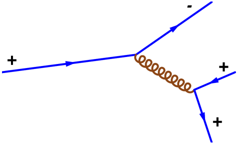
Gluon Radiation Fragmentation Model



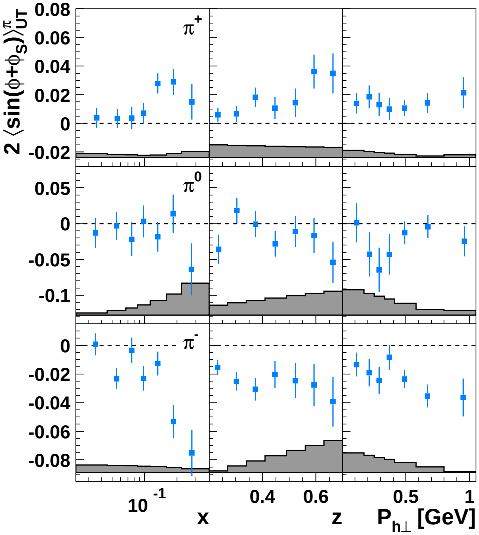
- Disfavored frag. model: assume produced diquark forms the observed meson
- Assume additional final state interaction to set pseudo-scalar quantum numbers
- Assume no additional interactions in dihadron production.

- Exists common sub-diagram between this model and the Lund/Artru model.
- Keeping track of quark polarization states, sub-diagram for disfavored $|1, 1\rangle$ diquark production identical to sub-diagram for favored $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ diquark production.



- Implies that the disfavored Collins function for transverse vector mesons also has opposite sign as the favored pseudo-scalar Collins function
  - Thus fav. = disfav. for Vector Mesons
  - Data suggests fav. ≈ -disfav. for pseudo-scalar mesons.

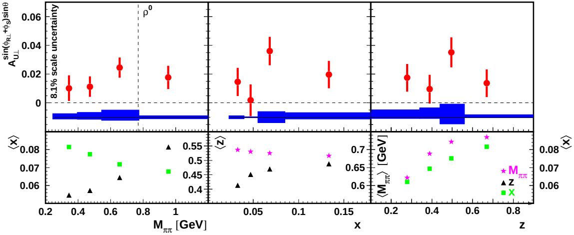
HERMES Collins Moments for Pions



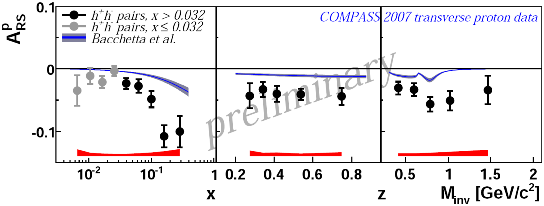
- Final result published in January A. Airapetian et al, Phys. Lett. B 693 (2010) 11-16. arXiv:1006.4221 (hep-ex)
- Significant π^- asymmetry implies $H_1^{\perp,disf} \approx -H_1^{\perp,fav}$
- Pions have small, but non-zero asymmetry
- Expect Collins moments negative for $\rho^{\pm}$.
- Would like uncertainties on dihadron moments on the order of 0.02.

Collinear Dihadron Results

HERMES



COMPASS



- Measure asymmetry $2\langle\sin(\phi_{R\perp}+\phi_S)\sin\theta\rangle$ in $\pi^+\pi^-$ pair production.
- Related to h_1 DF (transversity) and *sp* interference FF $H_{1,UT}^{\sphericalangle sp}$.
- Model based on HERMES results by Bacchetta, *et al.* (PRD 74:114007, 2006)
- Prediction for COMPASS results yields too small of an asymmetry. (arXiv:0907.0961v1)
- Both experiments indicate non-zero h_1 and $H_{1,UT}^{\sphericalangle sp}$.

The Angles ϕ_R verses $\phi_{R\perp}$

- The angle ϕ_R is the fundamental quantity
- The angle $\phi_{R\perp}$ is supposed to be an experimentally "easier" quantity.
- The difference is suppressed by $(Q^2)^{-2}$
  - Doesn't matter for leading twist analysis (twist-2)
  - Might matter at twist-3 and twist-4
- Can compute one as easily as the other, so should really use ϕ_R
- Note, the equations for ϕ_R and $\phi_{R\perp}$ are similar

$$\begin{aligned} \phi_R &= \text{signum}\big[\,(\boldsymbol{R}\times\boldsymbol{P}_h)\cdot\boldsymbol{n}\big]\arccos\frac{(\boldsymbol{q}\times\boldsymbol{k})\cdot(\boldsymbol{P}_h\times\boldsymbol{R}_T)}{|\boldsymbol{q}\times\boldsymbol{k}|\,|\boldsymbol{P}_h\times\boldsymbol{R}_T|}. \\ \phi_{R\perp} &= \text{signum}\big[\,(\boldsymbol{q}\times\boldsymbol{k})\cdot\boldsymbol{R}_T\big]\arccos\frac{(\boldsymbol{q}\times\boldsymbol{k})\cdot(\boldsymbol{q}\times\boldsymbol{R}_T)}{|\boldsymbol{q}\times\boldsymbol{k}|\,|\boldsymbol{q}\times\boldsymbol{R}_T|}, \end{aligned}$$

with

$$\boldsymbol{n} = (\boldsymbol{q}\cdot\boldsymbol{P}_h)\,\boldsymbol{k} - (\boldsymbol{k}\cdot\boldsymbol{P}_h)\,\boldsymbol{q}. \tag{1}$$

Second SIDIS Dihadron Program at HERMES

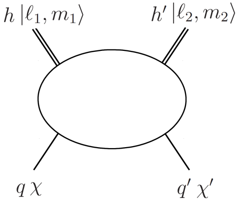
- Uses ϕ_R not $\phi_{R\perp}$ and also use $\cos\vartheta$.
- Analyzes full TMD (i.e. non-collinear), sub-leading twist cross section.
  - Number of unpol. moments: 15 (24 at Tw. 3), compared with pseudo-scalar mesons 2 (3 at Tw. 3).
  - Number of transverse target moments: 27 (54 at Tw. 3), compared with pseudo-scalars 3 (6 at Tw. 3).
  - Must determine which moments are suitable for release.
- Apply acceptance correction.
  - Note: RICH momentum cuts significantly effect $\cos\vartheta$ distribution.
- Attempt background subtraction to separate vector mesons from hadron pairs.
- Measure at least 4 vector mesons/hadron pairs (ρ-triplet and ϕ).
  - Have data for K^*s (less background than ρ)
  - Theory regarding mixed mass pairs (πK) not as well developed.

Items Which Required Additional Development

- Non-collinear SIDIS Monte Carlo generator at sub-leading twist.
  - Must simulate azimuthal dependence of cross section for systematic studies.
  - Cannot use polynomial fits to the data as was done for pseudo-scalar analysis.
- Generator requires
  - Non-collinear cross section at sub-leading twist.
  - Non-collinear fragmentation models.
- Would also like to understand “Which term in the cross section includes ‘the Collins function’ for ρ_L, ρ_T?”
  - Use alternate partial wave expansion
  - Note: perhaps possible to answer question without new expansion
  - However, pursuit of the answer in this manner has led to new theoretical results: the sub-leading twist, TMD cross section.

Theory

Fragmentation Functions and Spin/Polarization



- Leading twist Fragmentation functions are related to number densities
  - Amplitudes squared rather than amplitudes
- Difficult to relate Artru/Lund prediction with published notation and cross section.
- Propose new convention for fragmentation functions
  - Functions entirely identified by the polarization states of the quarks, χ and χ'
  - Any final-state polarization, i.e. $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$, contained within partial wave expansion of fragmentation functions
- Exists exactly two fragmentation functions
  - D_1, the unpolarized fragmentation function ($\chi = \chi'$)
  - $H_1^\perp$, the polarized (Collins) fragmentation function ($\chi \neq \chi'$)
- New partial waves analysis proposed, using direct sum basis $|\ell, m\rangle$ rather than the direct product basis $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$.

Rigorous Definitions

- Fragmentation Correlation Matrix

$$\Delta_{mn}(P_h, S_h; k) = \sum_X \int \frac{d^4x}{(2\pi)^4} e^{ip\cdot x} \langle 0 | \Psi_m(x) | P_h, S_h; X \rangle \langle P_h, S_h; X | \overline{\Psi}_n(0) | 0 \rangle$$

- Trace Notation

$$\Delta^{[\Gamma]}(z, M_h, |\boldsymbol{k}_T|, \cos\vartheta, \phi_R - \phi_k) = 4\pi \frac{z|\boldsymbol{R}|}{16M_h} \int dk^+ \, \mathrm{Tr}\left[\Gamma \Delta(k, P_h, R)\right]\Big|_{k^- = P_h^- / z}.$$

- Define fragmentation functions via trace relations

FF	Previous Definitions Pseudo-Scalar	 Dihadron	New Definition All Final States
D_1	$\Delta^{[\gamma^-]}$	$\Delta^{[\gamma^-]}$	$\Delta^{[\gamma^-(1+i\gamma^5)]}$
$G_1^\perp$	$--$	$\propto \Delta^{[\gamma^-\gamma^5]}$	$--$
$H_1^\perp$	$\Delta^{[(\sigma^{1-})\gamma^5]}$	$\Delta^{[(\sigma^{1-})\gamma^5]}$	$\Delta^{[(\sigma^{1-}+i\sigma^{2-})\gamma^5]}$
$\bar{H}_1^{\sphericalangle}$	$--$	$\propto \Delta^{[(\sigma^{2-})\gamma^5]}$	$--$

Relation with Previous Notation

- Real part of fragmentation function similar
- New definition of D_1 & $H_1^\perp$
  - Adds "imaginary" part to D_1 & $H_1^\perp$, instead of introducing new functions.
  - Functions are complex valued and depend on Q^2, z, $|k_T|$, M_h, $\cos\vartheta$, $(\phi_R - \phi_k)$.
- Comparing with similar trace definitions, e.g. PRD 67:094002, yields the relations

$$D_1\Big|_{Gliske} = \left[D_1 + i\frac{|\boldsymbol{R}||\boldsymbol{k}_T|}{M_h^2}\sin\vartheta\sin(\phi_R - \phi_k)\,G_1^\perp\right]_{other},$$

$$H_1^\perp\Big|_{Gliske} = \left[H_1^\perp + \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\sin\vartheta e^{i(\phi_R-\phi_k)}\,\bar{H}_1^{\sphericalangle}\right]_{other} = \frac{|\boldsymbol{R}|^2}{|\boldsymbol{k}_T|^2}H_1^{\sphericalangle}\Big|_{other},$$

- Note: there are inconsistencies in the literature between definitions of $H_1^{\sphericalangle}$, $\bar{H}_1^{\sphericalangle}$, and $H_1'^{\sphericalangle}$.

Partial Wave Expansion

- Fragmentation functions expanded into partial waves in the direct sum basis according to

$$D_1 = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos\vartheta) e^{im(\phi_R - \phi_k)} D_1^{|\ell,m\rangle}(z, M_h, |\boldsymbol{k}_T|),$$

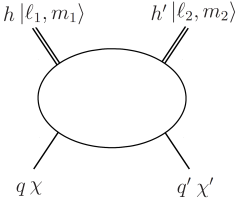
$$H_1^{\perp} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos\vartheta) e^{im(\phi_R - \phi_k)} H_1^{\perp|\ell,m\rangle}(z, M_h, |\boldsymbol{k}_T|),$$

- Each term in pseudo-scalar and dihadron cross section uniquely related to a specific partial wave $|\ell, m\rangle$.
- Cross section looks the same for all final states, excepting certain partial waves may or may not be present
  - Pseudo-scalar production is $\ell = 0$ sector
  - Dihadron production is $\ell = 0, 1, 2$ sector
  - Given the pseudo-scalar cross section (at any twist) can extrapolate cross section for other final states

Where is "the Collins function?"

- Consider direct sum vs. direct product basis

$$\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} = \left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\left(\frac{1}{2}\otimes\frac{1}{2}\right), \\ = (1\oplus 0)\otimes(1\oplus 0), \\ = 2\oplus 1\oplus 1\oplus 1\oplus 0\oplus 0.$$



- Three $\ell = 1$ and two $\ell = 0$ cannot be separated experimentally
  - Theoretically distinguishable via Generalized Casimir Operators
- Longitudinal vector meson state $|1,0\rangle|1,0\rangle$ is a mixture of $|2,0\rangle$ and $|0,0\rangle$
  - Cannot access, due to $\ell = 0$ multiplicity
  - Model predictions for longitudinal vector mesons not testable
- Transverse vector meson states $|1,\pm1\rangle|1,\pm1\rangle$ are exactly $|2,\pm2\rangle$
  - Models predict dihadron $H_1^{\perp|2,\pm2\rangle}$ has opposite sign as pseudo-scalar $H_1^{\perp}$.
  - Cross section has direct access to $H_1^{\perp|2,\pm2\rangle}$
- Note: the usual IFF, related to $H_1^{\perp|1,1\rangle}$ is not pure *sp*, but also includes *pp* interference.

Dihadron Twist-3 Cross Section

$$
\begin{aligned}
d\sigma_{UU} &= \frac{\alpha^2 M_h P_{h\perp}}{2\pi x y Q^2}\left(1+\frac{\gamma^2}{2x}\right) \\
&\times \sum_{\ell=0}^{2}\Bigg\{ A(x,y)\sum_{m=0}^{\ell}\Big[P_{\ell,m}\cos(m(\phi_h-\phi_R))\Big(F_{UU,T}^{P_{\ell,m}\cos(m(\phi_h-\phi_R))}+\epsilon F_{UU,L}^{P_{\ell,m}\cos(m(\phi_h-\phi_R))}\Big)\Big] \\
&\quad + B(x,y)\sum_{m=-\ell}^{\ell}P_{\ell,m}\cos((2-m)\phi_h+m\phi_R)F_{UU}^{P_{\ell,m}\cos((2-m)\phi_h+m\phi_R)} \\
&\quad + V(x,y)\sum_{m=-\ell}^{\ell}P_{\ell,m}\cos((1-m)\phi_h+m\phi_R)F_{UU}^{P_{\ell,m}\cos((1-m)\phi_h+m\phi_R)}\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
d\sigma_{UT} &= \frac{\alpha^2 M_h P_{h\perp}}{2\pi x y Q^2}\left(1+\frac{\gamma^2}{2x}\right)|S_\perp|\sum_{\ell=0}^{2}\sum_{m=-\ell}^{\ell}\Bigg\{A(x,y)\Big[P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)) \\
&\quad\times\Big(F_{UT,T}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)}+\epsilon F_{UT,L}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)}\Big)\Big] \\
&\quad + B(x,y)\Big[P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)F_{UT}^{P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)} \\
&\qquad + P_{\ell,m}\sin((3-m)\phi_h+m\phi_R-\phi_S)F_{UT}^{P_{\ell,m}\sin((3-m)\phi_h+m\phi_R-\phi_S)}\Big] \\
&\quad + V(x,y)\Big[P_{\ell,m}\sin(-m\phi_h+m\phi_R+\phi_S)F_{UT}^{P_{\ell,m}\sin(-m\phi_h+m\phi_R+\phi_S)} \\
&\qquad + P_{\ell,m}\sin((2-m)\phi_h+m\phi_R-\phi_S)F_{UT}^{P_{\ell,m}\sin((2-m)\phi_h+m\phi_R-\phi_S)}\Big]\Bigg\}.
\end{aligned}
$$

Structure Functions, Unpolarized

$$F_{UU,L}^{P_{\ell,m}\cos(m\phi_h - m\phi_R)} = 0,$$

$$F_{UU,T}^{P_{\ell,m}\cos(m\phi_h - m\phi_R)} = \begin{cases} \mathcal{I}\left[f_1 D_1^{|\ell,0\rangle}\right] & m = 0, \\ \mathcal{I}\left[2\cos(m\phi_h - m\phi_k)\, f_1 \left(D_1^{|\ell,m\rangle} + D_1^{|\ell,-m\rangle}\right)\right] & m > 0, \end{cases}$$

$$F_{UU}^{P_{\ell,m}\cos((2-m)\phi_h + m\phi_R)} = -\mathcal{I}\left[\frac{|\boldsymbol{p}_T||\boldsymbol{k}_T|}{MM_h}\cos\left((m-2)\phi_h + \phi_p + (1-m)\phi_k\right) h_1^{\perp} H_1^{\perp|\ell,m\rangle}\right],$$

$$\begin{aligned} F_{UU}^{P_{\ell,m}\cos((1-m)\phi_h + m\phi_R)} = & -\frac{2M}{Q}\mathcal{I}\Bigg[\frac{|\boldsymbol{k}_T|}{M_h}\cos((m-1)\phi_h + (1-m)\phi_k) \\ & \times \left(xhH_1^{\perp|\ell,m\rangle} + \frac{M_h}{M} f_1 \frac{\tilde{D}^{\perp|\ell,m\rangle}}{z}\right) \\ & + \frac{|\boldsymbol{p}_T|}{M}\cos((m-1)\phi_h + \phi_p - m\phi_k) \\ & \times \left(xf^{\perp} D_1^{|\ell,m\rangle} + \frac{M}{M_h} h_1^{\perp} \frac{\tilde{H}^{|\ell,m\rangle}}{z}\right)\Bigg]. \end{aligned}$$

- Can test Lund/Artru model with $F_{UU}^{\sin^2\vartheta\cos(2\phi_R)}$, $F_{UU}^{\sin^2\vartheta\cos(4\phi_h - 2\phi_R)}$ via Boer-Mulder's function

Twist-2 Structure Functions, Transverse Target

$$F_{UT,L}^{P_{\ell,m}\sin((m+1)\phi_h - m\phi_R - \phi_S)} = 0$$

$$F_{UT,T}^{P_{\ell,m}\sin((m+1)\phi_h - m\phi_R - \phi_S)} = -\mathfrak{I}\left[\frac{|\boldsymbol{p}_T|}{M}\cos\left((m+1)\phi_h - \phi_p - m\phi_k\right) \times \left(f_{1T}^{\perp}\left(D_1^{|\ell,m\rangle} + D_1^{|\ell,-m\rangle}\right) + \chi(m) g_{1T}\left(D_1^{|\ell,m\rangle} - D_1^{|\ell,-m\rangle}\right)\right)\right],$$

$$F_{UT}^{P_{\ell,m}\sin((1-m)\phi_h + m\phi_R + \phi_S)} = -\mathfrak{I}\left[\frac{|\boldsymbol{k}_T|}{M_h}\cos\left((m-1)\phi_h - \phi_p - m\phi_k\right) h_1 H_1^{\perp|\ell,m\rangle}\right],$$

$$F_{UT}^{P_{\ell,m}\sin((3-m)\phi_h + m\phi_R - \phi_S)} = \mathfrak{I}\left[\frac{|\boldsymbol{p}_T|^2|\boldsymbol{k}_T|}{M^2 M_h}\cos\left((m-3)\phi_h + 2\phi_p - (m-1)\phi_k\right) h_{1T}^{\perp} H_1^{\perp|\ell,m\rangle}\right].$$

- Can test Lund/Artru model with $F_{UT}^{\sin^2\vartheta\sin(-\phi_h+2\phi_R+\phi_S)}$ and $F_{UT}^{\sin^2\vartheta\sin(3\phi_h-2\phi_R+\phi_S)}$ via transversity
- In theory, could also test Lund/Artru and gluon radiation models with $F_{UT}^{\sin^2\vartheta\sin(\phi_h+2\phi_R-\phi_S)}$ and $F_{UT}^{\sin^2\vartheta\sin(5\phi_h-2\phi_R-\phi_S)}$ via pretzelocity
- Data from SIDIS pseudo-scalar production indicate pretzelocity very small or possibly zero

Collinear versus TMD Moments

- It is not the particulars of the DF or FF that make a moment survive in the collinear case, but rather the $\sum m = 0$ (necessary condition).
  - Moments with $h_1 H_1^{\perp|\ell,m\rangle}$ (Collins moments)
    - h_1 has $\Delta m = 0$; $H_1^{\perp}$ has $\chi \neq \chi'$, and thus $\Delta m = -1$.
    - Fragmentation functions surviving in collinear case must have $m = 1$ so $\sum m = 0$.
    - Collinear moments are $|1,1\rangle$, $|2,1\rangle$.
  - Moments with $h_1^{\perp} H_1^{\perp|\ell,m\rangle}$ (Boer-Mulders moments)
    - $h_1^{\perp}$ has $\Delta m = -1$.
    - $H_1^{\perp}$ again has $\Delta m = -1$.
    - Moments surviving in collinear case have $m = 2$, i.e. $|2,2\rangle$.
- TMD Structure function for the $|1,1\rangle$ A_{UT} moment

$$F_{UT}^{\sin\vartheta\sin(\phi_R+\phi_S)}(x,y,z,P_{h\perp},\boldsymbol{p}_T,\boldsymbol{k}_T) = -\mathfrak{I}\left[\frac{|\boldsymbol{k}_T|}{M_h}\cos\left(\phi_p-\phi_k\right) h_1(x,p_T)\, H_1^{\perp|1,1\rangle}(z,zk_T)\right]$$

- Collinear assumption implies

$$\int d\phi_h\, dP_{h\perp}\, F_{UT}^{\sin\vartheta\sin(\phi_R+\phi_S)}(x,y,z,P_{h\perp},\boldsymbol{p}_T,\boldsymbol{k}_T) \approx h_1(x)\, H_1^{\perp|1,1\rangle\,(1)}(z),$$

with $h_1(x) = \int dp_T\, h_1(x,p_T)$, $\quad H_1^{\perp|1,1\rangle\,(1)}(z) = \int dk_T \frac{|\boldsymbol{k}_T|}{M_h} H_1^{\perp|1,1\rangle}(z,zk_T)$.

The TMDGen Generator

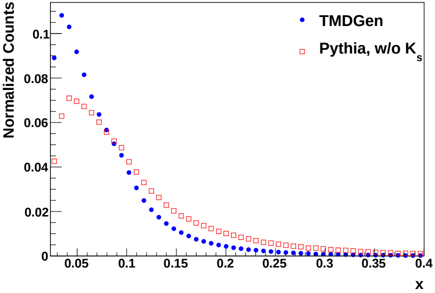
Collinear Dihadron Spectator Model

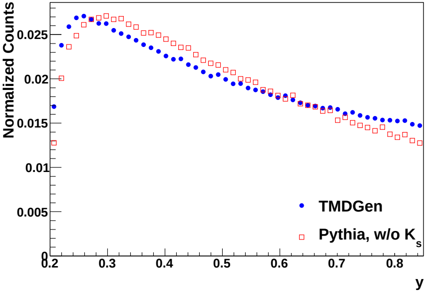
- Based on Bacchetta/Radici spectator model for collinear dihadron production *Phys. Rev.* D74 (2006)
  - The SIDIS X is replaced with a single, on-shell, particle of mass $M_s \propto M_h$.
  - Assume one spectator for hadron pairs and vector mesons.
  - Integration over transverse momenta is performed before extracting fragmentation functions.
- One can use the same correlator to extract TMD fragmentation functions
  - One just needs to not integrate and follow the Dirac-matrix algebra and partial wave expansion.
  - Numeric studies show need for additional $\boldsymbol{k}_T$ cut-off.
- Original model intended for $\pi^+\pi^-$ pairs
  - Adding flavor dependence allows generalization to $\pi^+\pi^0$, $\pi^-\pi^0$ pairs.
  - Slight change to vertex function allows generalization to K^+K^- pairs.
  - Slight change to vertex function and allows generalization to K^+K^- pairs.
- Unfortunately, the model only includes partial waves of the Collins function for $\ell < 2$.
  - Instead, one can set $|2, \pm 2\rangle$ partial waves proportional to partial waves of either $H_1^\perp$ or D_1.

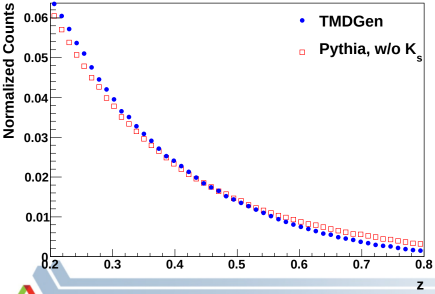
New TMDGEN Generator

- No previous Monte Carlo generator has TMD dihadron production with full angular dependence
- Method
  - Integrates cross section per flavor to determine "quark branching ratios"
  - Throw a flavor type according to ratios
  - Throw kinematic/angular variables by evaluating cross section
    - Can use weights or acceptance rejection
  - Full TMD simulation: each event has specific $|\boldsymbol{p}_T|$, ϕ_p, $|\boldsymbol{k}_T|$, ϕ_k values
  - Includes both pseudo-scalar and dihadron SIDIS cross sections
- Guiding plans
  - Extreme flexibility
    - Allow many models for fragmentation and distribution functions
    - Various final states: pseudo-scalars, vector mesons, hadron pairs, etc.
    - Output options & connecting to analysis chains of various experiments
    - Minimize dependencies on other libraries
  - Full flavor and transverse momentum dependence.
- Current C++ package considered stable and allows further expansion
- Can be useful for both experimentalists and theorists.

$\pi^+\pi^0$ Kinematic Distributions, p.1

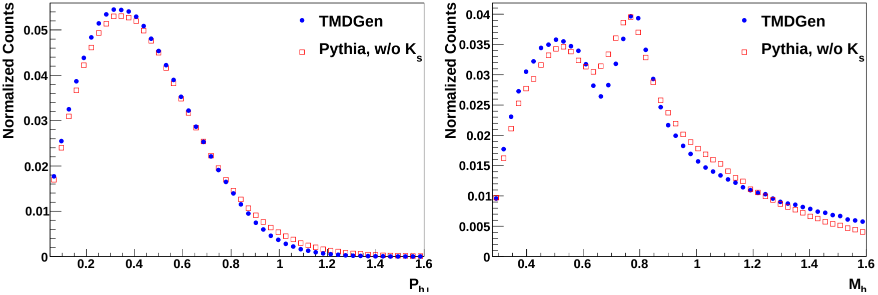






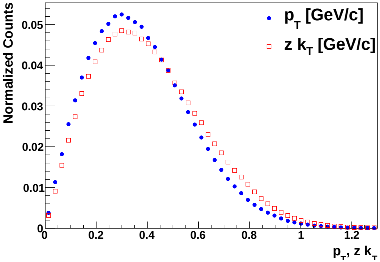
- Close agreement for x, y, z distributions.
- Main discrepancy in x—may be due to imbalance in the flavor contributions, or Q^2 effects.
- Similar results for other $\pi\pi$ and KK dihadrons.

$\pi^+\pi^0$ Kinematic Distributions, p.2



- Fairly good agreement in both $P_{h\perp}$ and M_h distributions.
- Note: some discrepancies in full 5D kinematic, but PYTHIA also doesn't match data in full 5D

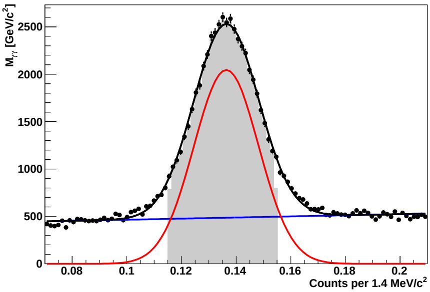
$\pi^+\pi^0$ Kinematic Distributions, Intrinsic Transverse Momentum

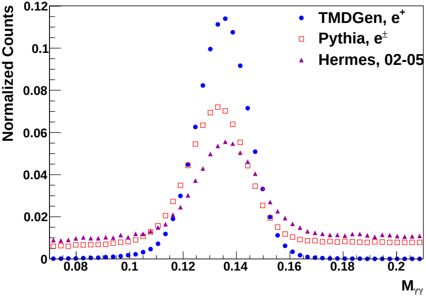


- Partonic transverse momentum denoted p_T
- The fragmenting quark's transverse momentum is zk_T
- Model requires $p_T \approx zk_T$ in order to get narrow $P_{h\perp}$ peak
- Model does not require any flavor dependence to k^2, k_T^2 cut-offs
- However, model poorly constrains RMS values $\langle p_T^2 \rangle$, $\langle k_T^2 \rangle$
- No other generator can show p_T, k_T distributions

Analysis

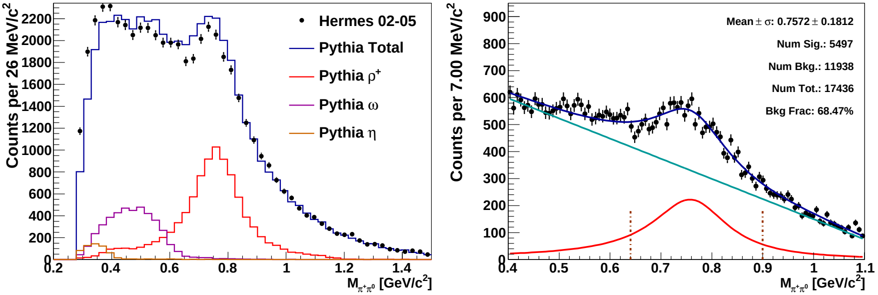
Neutral Pion Reconstruction





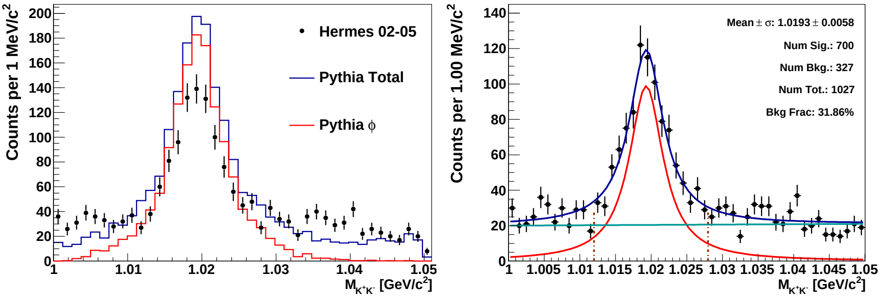
- Invariant mass spectrum of $\gamma\gamma$-system for $\pi^+\gamma\gamma$ events.
- $E_{\text{clus.}} = \alpha E_\gamma$, with α equal to 0.97, 0.9255 and 0.95 for HERMES, PYTHIA, and TMDGEN data, respectively.
- Central value of the peak is sufficiently close to the accepted value.
- Width of the peak is reflection of the resolution of the spectrometer for the π^0 mass.

Mass Distribution: $\pi^+\pi^0$



- Left panel: comparison with PYTHIA, highlighting various process decaying into $\pi^+\pi^-$ pair.
- Right panel: Hermes 02-05 data, fit to Breit-Wigner plus linear background to estimate background fraction.
- High background fraction, but hope only VMs in *pp*-wave.
- Distributions for other $\pi\pi$ dihadron effectively the same.

Mass Distribution: K^+K^-



- Lower signal, but much lower background fraction.
- No other mesons decaying into K^+K^- within mass window.
- Clean access to strange quark distribution and fragmentation functions.

Fitting Functions

- Perform angular fit in each kinematic bin
- Main focus is on transverse target Collins and Sivers moments
- Fit function includes 41 angular moments plus constant term
  - Unpolarized moments, twist-2 and twist-3 (24 moments)
  - The transverse target Collins and Sivers moments (18 moments)

$$
\begin{aligned}
f(\cos\vartheta, \phi_h, \phi_R, \phi_S) = & \sum_{\ell=0}^{2}\Bigg[\sum_{m=0}^{\ell} a_1^{|\ell,m\rangle} P_{\ell,m}\cos(m\phi_h - m\phi_R) \\
& + \sum_{m=-\ell}^{\ell}\left(a_2^{|\ell,m\rangle} P_{\ell,m}\cos((2-m)\phi_h + m\phi_R) + a_3^{|\ell,m\rangle} P_{\ell,m}\cos((1-m)\phi_h + m\phi_R)\right) \\
& + \sum_{m=-\ell}^{\ell}\left(b_1^{|\ell,m\rangle} P_{\ell,m}\sin((m+1)\phi_h - m\phi_R - \phi_S) + b_2^{|\ell,m\rangle} P_{\ell,m}\sin((1-m)\phi_h + m\phi_R + \phi_S)\right)
\end{aligned}
$$

- Constrain $a_1^{|0,0\rangle} = 1$.
- Fit parameters are integrals of structure functions, which are integrals of distribution and fragmentation functions

$$
\begin{aligned}
a_1^{|\ell,m\rangle} &\propto f_1 D_1^{|\ell,m\rangle} \\
a_2^{|\ell,m\rangle} &\propto h_1^{\perp} H_1^{\perp|\ell,m\rangle}
\end{aligned}
\qquad
a_3^{|\ell,m\rangle} \propto f_1 D_1^{|\ell,m\rangle}, h_1^{\perp} H_1^{\perp|\ell,m\rangle}
\qquad
\begin{aligned}
b_1^{|\ell,m\rangle} &\propto f_{1T}^{\perp} D_1^{|\ell,m\rangle} \\
b_2^{|\ell,m\rangle} &\propto h_1 H_1^{\perp|\ell,m\rangle}
\end{aligned}
$$

Summary of Further Analysis Details

- The angular acceptance per kinematic bin was correct using a least squares method and a basis expansion.
- A naive test of the acceptance correction method using TMDGen data for both training and "HERMES" data.
- The the non-resonant photon pair background was estimated and subtracted from the results.
- The charge symmetric background was studied and found to be negligible.
- Exclusive background fraction determined to be less than 3.5% with negligible effects
- The overall vector meson fraction was determined for each final state.
- Using a simple MLE fit (no acceptance correction) the results were also compared with the published results, using the same data productions, binning, cuts, etc.

Systematic Uncertainty

- Three non-negligible sources of systematic uncertainty were found:
  - Acceptance and the Acceptance Correction
    - PYTHIA+RADGEN is used to simulate data
    - Moments are induced in PYTHIA+RADGEN data using weights computed from the angular part of cross section using TMDGEN
    - Angular integrated TMDGEN is used as training data for the acceptance correction.
    - Uncertainty set to half the difference between 4π weighted PYTHIA moments and the corrected PYTHIAmoments.
  - Year dependence
    - 2002-2004 is with e^+ beam, 2005 is with e^- beam—almost equal statistics (about 40/60 split)
    - Systematic uncertainty is estimated as half the uncertainty needed to reduce the χ^2 per moment per bin to 1.
  - RICH Unfolding vs. No Unfolding
    - Two methods exist: either assign a track the most likely PID or assigning weights according to some unfolding.
    - Half the difference is taken as the systematic uncertainty.

Results and Conclusions

Conclusions

- Non-collinear SIDIS Dihadron production provides unique access to
  - Strange quark distribution and fragmentation functions
  - Testing the Lund/Artru model
  - The TMD spin structure of fragmentation
- Theoretical developments include
  - Clarifying the prediction of the Lund/Artru Model
  - Developing the gluon radiation model
  - Defining a new partial wave expansion
  - Computing the twist-3 dihadron cross section
- Numerical Methods and Software
  - Smearing and acceptance correction method
  - TMDGEN Monte Carlo generator
- Analysis and systematic studies completed
- Results are in agreement with Lund/Artru model and the gluon radiation model, assuming u-quark dominance
- Much more detailed information now provided regarding $H_1^{\perp|1,1\rangle}$
- Just need release of preliminary results by the HERMES Collaboration

Backup Slides

Relations with Previous Notation, Partial Waves

$$
\begin{aligned}
D_1^{|0,0\rangle} &= D_{1,OO} = \left(\frac{1}{4}D^{s}_{1,OO} + \frac{3}{4}D^{p}_{1,OO}\right), \\
D_1^{|1,0\rangle} &= D_{1,OL}, \\
D_1^{|1,\pm1\rangle} &= D_{1,OT} \mp \frac{|\boldsymbol{k}_T|\,|\boldsymbol{R}|}{M_h^2} G^{\perp}_{1,OT}, \\
D_1^{|2,0\rangle} &= \frac{1}{2}D_{1,LL}, \\
D_1^{|2,\pm1\rangle} &= \frac{1}{2}\left(D_{1,LT} \mp \frac{|\boldsymbol{k}_T|\,|\boldsymbol{R}|}{M_h^2} G^{\perp}_{1,LT}\right), \\
D_1^{|2,\pm2\rangle} &= D_{1,TT} \mp \frac{1}{2}\frac{|\boldsymbol{k}_T|\,|\boldsymbol{R}|}{M_h^2} G^{\perp}_{1,TT},
\end{aligned}
$$

$$
\begin{aligned}
H_1^{\perp\,|0,0\rangle} &= H^{\perp}_{1,OO} = \frac{1}{4}H^{\perp s}_{1,OO} + \frac{3}{4}H^{\perp p}_{1,OO}, \\
H_1^{\perp\,|1,1\rangle} &= H^{\perp}_{1,OT} + \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\bar{H}^{\sphericalangle}_{1,OT} = \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}H^{\sphericalangle}_{1,OT} \\
H_1^{\perp\,|1,0\rangle} &= H^{\perp}_{1,OL}, \\
H_1^{\perp\,|1,-1\rangle} &= H^{\perp}_{1,OT}, \\
H_1^{\perp\,|2,2\rangle} &= H^{\perp}_{1,TT} + \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\bar{H}^{\sphericalangle}_{1,TT} = \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}H^{\sphericalangle}_{1,TT} \\
H_1^{\perp\,|2,1\rangle} &= \frac{1}{2}H^{\perp}_{1,LT} + \frac{1}{2}\frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\bar{H}^{\sphericalangle}_{1,LT} = \frac{1}{2}\frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}H^{\sphericalangle}_{1,LT} \\
H_1^{\perp\,|2,0\rangle} &= \frac{1}{2}H^{\perp}_{1,LL}, \\
H_1^{\perp\,|2,-1\rangle} &= \frac{1}{2}H^{\perp}_{1,LT}, \\
H_1^{\perp\,|2,-2\rangle} &= H^{\perp}_{1,TT}.
\end{aligned}
$$

Fragmentation Correlation Function

- Described spectator model uses the following fragmentation correlation function

$$
\begin{aligned}
\Delta^q(k, P_h, R) \; = \; & \Big\{ |F^s|^2 \, e^{-2\frac{k^2}{\Lambda_s^2}} \not{k} \left(\not{k} - \not{P}_h + M_s\right) \not{k} \\
& + |F^p|^2 \, e^{-2\frac{k^2}{\Lambda_p^2}} \not{k}\not{R} \left(\not{k} - \not{P}_h + M_s\right) \not{R}\not{k} \\
& + F^{s*} F^p e^{-2\frac{k^2}{\Lambda_{sp}^2}} \not{k} \left(\not{k} - \not{P}_h + M_s\right) \not{R}\not{k} \\
& + F^s F^{p*} e^{-2\frac{k^2}{\Lambda_{sp}^2}} \not{k}\not{R} \left(\not{k} - \not{P}_h + M_s\right) \not{k} \Big\} \\
& \times \frac{1}{(2\pi)^3} \frac{1}{k^4} \delta\left((k - P_h)^2 - M_s^2\right) e^{-2\frac{k_T^2}{\Lambda_b^2}} .
\end{aligned}
$$

Model Calculation for Fragmentation Functions

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|0,0\rangle} = \left(\frac{z^2|\boldsymbol{k}_T|^2 + M_s^2}{1-z}\right)\left[\left|F^s\right|^2 e^{-2\frac{k^2}{\Lambda_s^2}} - R^2\left|F^p\right|^2 e^{-2\frac{k^2}{\Lambda_p^2}}\right]$$

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|1,1\rangle} = -2M_s|\boldsymbol{R}||\boldsymbol{k}_T|\left[\mathrm{Re}\left(F^{s*}F^p\right) e^{-2\frac{k^2}{\Lambda_{sp}^2}}\right]$$

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|1,0\rangle} = -2\frac{M_s|\boldsymbol{R}|}{zM_h}\left(M_h^2 + z^2|\boldsymbol{k}_T|^2\right)\left[\mathrm{Re}\left(F^{s*}F^p\right) e^{-2\frac{k^2}{\Lambda_{sp}^2}}\right]$$

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|2,2\rangle} = |\boldsymbol{k}_T|^2|\boldsymbol{R}|^2\left[\left|F^p\right|^2 e^{-2\frac{k^2}{\Lambda_p^2}}\right],$$

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|2,1\rangle} = \frac{|\boldsymbol{k}_T||\boldsymbol{R}|^2}{zM_h}\left(M_h^2 + z^2|\boldsymbol{k}_T|^2 + \frac{1}{2}z^2k^2\right)\left[\left|F^p\right|^2 e^{-2\frac{k^2}{\Lambda_p^2}}\right],$$

$$\frac{16\pi^2 M_h k^4}{|\boldsymbol{R}|} D_1^{|2,0\rangle} = \left(\frac{|\boldsymbol{R}|^2}{z^2M_h^2}\left(M_h^2 + z^2|\boldsymbol{k}_T|^2\right)\left(M_h^2 + z^2|\boldsymbol{k}_T|^2 + z^2k^2\right) - 2|\boldsymbol{k}_T|^2|\boldsymbol{R}|^2\right)\left[\left|F^p\right|^2 e^{-2\frac{k^2}{\Lambda_p^2}}\right],$$

$$D_1^{|\ell,-m\rangle} = D_1^{|\ell,m\rangle}.$$

Model Calculation for Fragmentation Functions

$$
\begin{aligned}
\frac{8\pi^2 k^4}{|\boldsymbol{R}|} H_1^{\perp|1,1\rangle} &= -\frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\left(k^2+|\boldsymbol{k}_T|^2\right)\left(\left(1-z^2\right)k^2 - z^2|\boldsymbol{k}_T|^2\right) \\
&\quad \times \left[\mathrm{Im}\left(F^{s*}F^{p}\right) e^{-2\frac{k^2}{\Lambda_{sp}^2}}\right], \\
\frac{8\pi^2 k^4}{|\boldsymbol{R}|} H_1^{\perp|1,0\rangle} &= \frac{1}{z} M_h |\boldsymbol{R}| \left(zk^2 - 2\left(M_h^2 + z^2(k^2+|\boldsymbol{k}_T|^2)\right)\right) \\
&\quad \times \left[\mathrm{Im}\left(F^{s*}F^{p}\right) e^{-2\frac{k^2}{\Lambda_{sp}^2}}\right], \\
\frac{8\pi^2 k^4}{|\boldsymbol{R}|} H_1^{\perp|1,-1\rangle} &= -M_h^2 |\boldsymbol{R}||\boldsymbol{k}_T| \left[\mathrm{Im}\left(F^{s*}F^{p}\right) e^{-2\frac{k^2}{\Lambda_{sp}^2}}\right].
\end{aligned}
$$

Smearing/Acceptance Effects

- Let $\boldsymbol{x}^{(T)}$ be true value of variables, $\boldsymbol{x}^{(R)}$ the reconstructed values
- A conditional probability $p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right)$ relates the true PDF $p\left(\boldsymbol{x}^{(T)}\right)$ with the PDF of the reconstructed variables, $p\left(\boldsymbol{x}^{(R)}\right)$.
- Specific relation given by Fredholm integral equation

$$\begin{aligned} p\left(\boldsymbol{x}^{(R)}\right) &= \eta \int d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) p\left(\boldsymbol{x}^{(T)}\right), \\ \frac{1}{\eta} &= \int d^D\boldsymbol{x}^{(R)} d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) p\left(\boldsymbol{x}^{(T)}\right). \end{aligned}$$

- Can rewrite in terms of a smearing operator

$$\begin{aligned} \tilde{g}\left(\boldsymbol{x}^{(R)}\right) &= S\left[g(\boldsymbol{x}^{(T)})\right], \\ &= \int d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) g\left(\boldsymbol{x}^{(T)}\right). \end{aligned}$$

- Fredholm equation is simply

$$p\left(\boldsymbol{x}^{(R)}\right) = S\left[\eta p\left(\boldsymbol{x}^{(T)}\right)\right].$$

Solution with Finite Basis and Integrated Squared Error

- Restrict to finite basis

$$\begin{aligned} \eta p\left(\boldsymbol{x}^{(T)}\right) &= \sum_i \alpha_i f_i\left(\boldsymbol{x}^{(T)}\right), \\ p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) &= \sum_{i,j} \Gamma_{i,j} f_i\left(\boldsymbol{x}^{(R)}\right) f_j\left(\boldsymbol{x}^{(T)}\right). \end{aligned}$$

- Determine parameters by minimizing the integrated squared error (ISE)

$$\begin{aligned} ISE_1 &= \int d^D\boldsymbol{x}^{(R)} d^D\boldsymbol{x}^{(T)} \left[p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) - \sum_{i,j} \Gamma_{i,j} f_i(\boldsymbol{x}^{(R)}) f_j(\boldsymbol{x}^{(T)})\right]^2, \\ ISE_2 &= \int d^D\boldsymbol{x}^{(R)} \left\{p\left(\boldsymbol{x}^{(R)}\right) - S\left[\eta p\left(\boldsymbol{x}^{(T)}\right)\right]\right\}^2. \end{aligned}$$

Numerical Solution

- Define/compute

$$
\begin{aligned}
F_{i,j} &= \int d^D\boldsymbol{x}^{(\boldsymbol{T})} f_i\left(\boldsymbol{x}^{(\boldsymbol{T})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
B_{i,j} &= \int d^D\boldsymbol{x}^{(\boldsymbol{R})} d^D\boldsymbol{x}^{(\boldsymbol{T})}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) f_i\left(\boldsymbol{x}^{(\boldsymbol{R})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
&= V \int d^D\boldsymbol{x}^{(\boldsymbol{R})} d^D\boldsymbol{x}^{(\boldsymbol{T})}\, p_{MC}\left(\boldsymbol{x}^{(T)}, \boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(\boldsymbol{R})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
b_i &= \int d^D\boldsymbol{x}^{(R)} p\left(\boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(R)}\right), \\
&= \frac{V}{N_R} \sum_{k=1}^{N_R} f_i\left(\boldsymbol{x}^{(R,k)}\right),
\end{aligned}
$$

- ISEs reduce to the matrix equation

$$B^T F^{-1} B \boldsymbol{\alpha} = B^T F^{-1} \boldsymbol{b}.$$

- Assuming $(B^T F^{-1} B)$ and B are invertible, the solution for the given ISEs is

$$\boldsymbol{\alpha} = \left(B^T F^{-1} B\right)^{-1} B^T F^{-1} \boldsymbol{b} = B^{-1} \boldsymbol{b}.$$

Uncertainty Calculation

- Define

$$\left(C^{\boldsymbol{b}}\right)_{j,j'} = \frac{\delta_{j,j'}}{N_R - 1}\left[\frac{V^2}{N_R}\sum_{k=1}^{N_R} f_i^2\left(\boldsymbol{x}^{(R,k)}\right) - (b_i)^2\right],$$

$$\left(C^{B}\right)_{j,k;j',k'} = \frac{\delta_{j,j'}\delta_{k,k'}}{N_\epsilon - 1}\left[\frac{V^4}{N_\epsilon}\sum_{k=1}^{N_\epsilon} f_j^2\left(\boldsymbol{x}^{(M,k)}\right) f_k^2\left(\boldsymbol{x}^{(T,k)}\right) - (B_{j,k})^2\right],$$

$$C'^{(B)}_{i,i'} = \sum_{j,j'} C^{(B)}_{i,j;i',j'}\alpha_j\alpha_{j'}.$$

- The uncertainty on $\boldsymbol{\alpha}$ is then

$$C^{(\boldsymbol{\alpha})} = B^{-1}C^{(\boldsymbol{b})}B^{-T} + B^{-1}C'^{(B)}B^{-T}.$$

- One could consider a third term $\left(B^T F^{-1} B\right)^{-1}$, the Hessian of the matrix eq.
  - Numeric studies show this term is not a meaningful estimate of the uncertainty, and that it can be neglected.