Dihadron production in semi-inclusive DIS from transversely polarized protons

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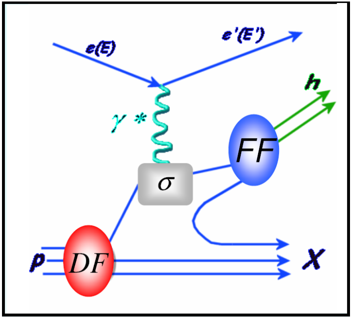
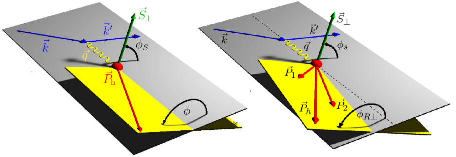
SIDIS Meson Production



- SIDIS cross section can be written $\sigma^{ep\to ehX} = \sum_q DF \otimes \sigma^{eq\to eq} \otimes FF$
- Access integrals of DFs and FFs through azimuthal asymmetries in ϕ_h, ϕ_S, ϕ_R



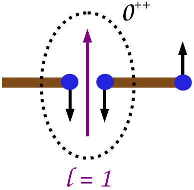
Distribution Functions (DF)

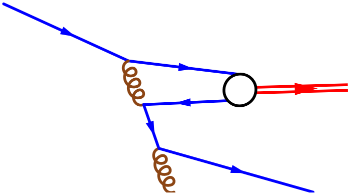
		quark		
		U	L	T
nucleon	U	f_1		$h_1^\perp$
	L		g_1	$h_{1L}^\perp$
	T	$f_{1T}^\perp$	$g_{1T}^\perp$	h_1 $h_{1T}^\perp$

Fragmentation Functions (FF)

quark	
Unpol.	**Pol.**
D_1	$H_1^\perp$

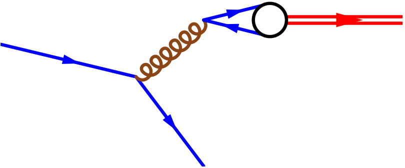
Lund/Artru String Fragmentation Model





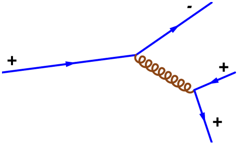
- Favored fragmentation modeled as the breaking of a gluon flux tube.
- Assume flux tube breaks into $q\bar{q}$ pair with vacuum quantum numbers.
- Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ states to prefer "quark left".
  - $|0, 0\rangle =$ pseudo-scalar mesons; $|1, 0\rangle =$ long. pol. vector mesons.
- Expect mesons overlapping with $|\frac{1}{2}, \frac{1}{2}\rangle|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle|\frac{1}{2}, -\frac{1}{2}\rangle$ states to prefer "quark right".
  - $|1, \pm 1\rangle =$ transversely polarized vector mesons.
- For the two ρ_T's, "the Collins function" should have opposite sign to that for π

Gluon Radiation Fragmentation Model



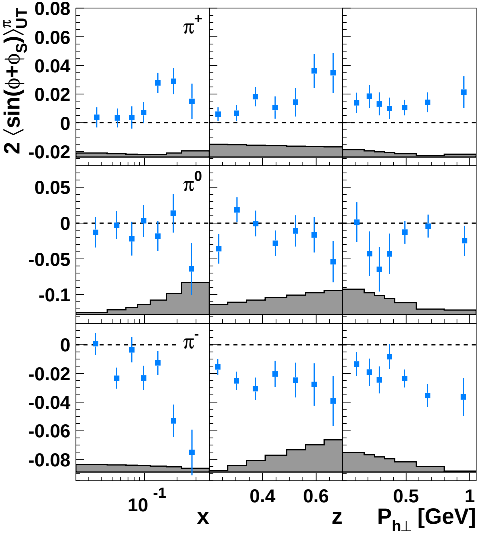
- Disfavored frag. model: assume produced diquark forms the observed meson
- Assume additional final state interaction to set pseudo-scalar quantum numbers
- Assume no additional interactions in dihadron production.

- Exists common sub-diagram between this model and the Lund/Artru model.
- Keeping track of quark polarization states, sub-diagram for disfavored $|1,1\rangle$ diquark production identical to sub-diagram for favored $|\frac{1}{2},-\frac{1}{2}\rangle|\frac{1}{2},\frac{1}{2}\rangle$ diquark production.



- Implies that the disfavored Collins function for transverse vector mesons also has opposite sign as the favored pseudo-scalar Collins function
  - Thus fav. = disfav. for Vector Mesons
  - Data suggests fav. $\approx$ -disfav. for pseudo-scalar mesons.

HERMES Collins Moments for Pions

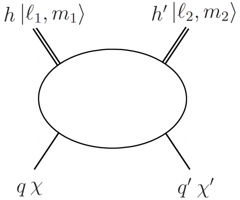


- Results published in Jan 2010
  A. Airapetian et al, Phys. Lett. B 693 (2010) 11-16. arXiv:1006.4221 (hep-ex)
- Significant π^- asymmetry implies $H_1^{\perp,disf} \approx -H_1^{\perp,fav}$
- Pions have small, but non-zero asymmetry

Vector Meson Expectation

Species	Type	Sign
ρ^+	fav.	-
ρ^0	mix	$\approx$ 0 or -
ρ^-	disfav.	-

Fragmentation Functions and Spin/Polarization

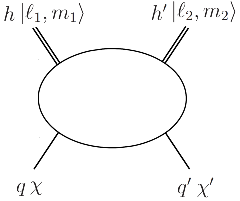


- Leading twist Fragmentation functions are related to number densities
  - Amplitudes squared rather than amplitudes
- Difficult to relate Artru/Lund prediction with published notation and cross section.
- Propose new convention for fragmentation functions
  - Functions entirely identified by the polarization states of the quarks, χ and χ'
  - Any final-state polarization, i.e. $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$, contained within partial wave expansion of fragmentation functions
- Exists exactly two fragmentation functions
  - D_1, the unpolarized fragmentation function ($\chi = \chi'$)
  - $H_1^{\perp}$, the polarized (Collins) fragmentation function ($\chi \neq \chi'$)
- New partial waves analysis proposed, using direct sum basis $|\ell, m\rangle$ rather than the direct product basis $|\ell_1, m_1\rangle|\ell_2, m_2\rangle$.

$$H_1^{\perp} = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos\vartheta) e^{im(\phi_R - \phi_k)} H_1^{\perp|\ell,m\rangle}(z, M_h, |\boldsymbol{k}_T|),$$

Where is "the Collins function?"

- Consider direct sum vs. direct product basis



$$\begin{aligned}\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2} &= \left(\frac{1}{2}\otimes\frac{1}{2}\right)\otimes\left(\frac{1}{2}\otimes\frac{1}{2}\right),\\ &= (1\oplus 0)\otimes(1\oplus 0)\,,\\ &= 2\oplus 1\oplus 1\oplus 1\oplus 0\oplus 0.\end{aligned}$$

- The three $\ell = 1$ cannot be separated experimentally
- Longitudinal vector meson state $|1,0\rangle|1,0\rangle$ is a mixture of $|2,0\rangle$ and $|0,0\rangle$
  - Cannot access, due to $\ell = 0$ multiplicity
- Transverse vector meson states $|1,\pm1\rangle|1,\pm1\rangle$ are exactly $|2,\pm2\rangle$
  - Models predict dihadron $H_1^{\perp|2,\pm2\rangle}$ has opposite sign as pseudo-scalar $H_1^{\perp}$.
  - Cross section has direct access to $H_1^{\perp|2,\pm2\rangle}$
- Note: the usual IFF, related to $H_1^{\perp|1,1\rangle}$ is not pure *sp*, but also includes *pp* interference.
- Using symmetry, can calculate cross section for any polarized final state from the scalar final state cross section

Dihadron Twist-2 and Twist-3 Cross Section

$$
\begin{aligned}
d\sigma_{UU} &= \frac{\alpha^2 M_h P_{h\perp}}{2\pi x y Q^2}\left(1+\frac{\gamma^2}{2x}\right) \\
&\times \sum_{\ell=0}^{2}\Bigg\{ A(x,y)\sum_{m=0}^{\ell}\Big[P_{\ell,m}\cos(m(\phi_h-\phi_R))\Big(F_{UU,T}^{P_{\ell,m}\cos(m(\phi_h-\phi_R))}+\epsilon F_{UU,L}^{P_{\ell,m}\cos(m(\phi_h-\phi_R))}\Big)\Big] \\
&\quad + B(x,y)\sum_{m=-\ell}^{\ell}P_{\ell,m}\cos((2-m)\phi_h+m\phi_R)F_{UU}^{P_{\ell,m}\cos((2-m)\phi_h+m\phi_R)} \\
&\quad + V(x,y)\sum_{m=-\ell}^{\ell}P_{\ell,m}\cos((1-m)\phi_h+m\phi_R)F_{UU}^{P_{\ell,m}\cos((1-m)\phi_h+m\phi_R)}\Bigg\}, \\
d\sigma_{UT} &= \frac{\alpha^2 M_h P_{h\perp}}{2\pi x y Q^2}\left(1+\frac{\gamma^2}{2x}\right)|\boldsymbol{S}_\perp|\sum_{\ell=0}^{2}\sum_{m=-\ell}^{\ell}\Bigg\{A(x,y)\Big[P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)) \\
&\qquad \times\Big(F_{UT,T}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)}+\epsilon F_{UT,L}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)}\Big)\Big] \\
&\quad + B(x,y)\Big[P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)F_{UT}^{P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)} \\
&\qquad + P_{\ell,m}\sin((3-m)\phi_h+m\phi_R-\phi_S)F_{UT}^{P_{\ell,m}\sin((3-m)\phi_h+m\phi_R-\phi_S)}\Big] \\
&\quad + V(x,y)\Big[P_{\ell,m}\sin(-m\phi_h+m\phi_R+\phi_S)F_{UT}^{P_{\ell,m}\sin(-m\phi_h+m\phi_R+\phi_S)} \\
&\qquad + P_{\ell,m}\sin((2-m)\phi_h+m\phi_R-\phi_S)F_{UT}^{P_{\ell,m}\sin((2-m)\phi_h+m\phi_R-\phi_S)}\Big]\Bigg\}.
\end{aligned}
$$

Structure Functions, Unpolarized

$$F_{UU,L}^{P_{\ell,m}\cos(m\phi_h - m\phi_R)} = 0,$$

$$F_{UU,T}^{P_{\ell,m}\cos(m\phi_h - m\phi_R)} = \begin{cases} \mathcal{I}\left[f_1 D_1^{|\ell,0\rangle}\right] & m = 0, \\ \mathcal{I}\left[2\cos(m\phi_h - m\phi_k)\, f_1 \left(D_1^{|\ell,m\rangle} + D_1^{|\ell,-m\rangle}\right)\right] & m > 0, \end{cases}$$

$$F_{UU}^{P_{\ell,m}\cos((2-m)\phi_h + m\phi_R)} = -\mathcal{I}\left[\frac{|\boldsymbol{p}_T||\boldsymbol{k}_T|}{MM_h}\cos\left((m-2)\phi_h + \phi_p + (1-m)\phi_k\right) h_1^{\perp} H_1^{\perp|\ell,m\rangle}\right],$$

$$F_{UU}^{P_{\ell,m}\cos((1-m)\phi_h + m\phi_R)} = -\frac{2M}{Q}\mathcal{I}\left[\frac{|\boldsymbol{k}_T|}{M_h}\cos((m-1)\phi_h + (1-m)\phi_k) \times \left(xhH_1^{\perp|\ell,m\rangle} + \frac{M_h}{M} f_1 \frac{\tilde{D}^{\perp|\ell,m\rangle}}{z}\right) + \frac{|\boldsymbol{p}_T|}{M}\cos((m-1)\phi_h + \phi_p - m\phi_k) \times \left(xf^{\perp} D_1^{|\ell,m\rangle} + \frac{M}{M_h} h_1^{\perp} \frac{\tilde{H}^{|\ell,m\rangle}}{z}\right)\right].$$

- Can test Lund/Artru model with $F_{UU}^{\sin^2\vartheta\cos(2\phi_R)}$, $F_{UU}^{\sin^2\vartheta\cos(4\phi_h - 2\phi_R)}$ via Boer-Mulder's function

Twist-2 Structure Functions, Transverse Target

$$F_{UT,L}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)} = 0$$

$$F_{UT,T}^{P_{\ell,m}\sin((m+1)\phi_h-m\phi_R-\phi_S)} = -\mathcal{I}\left[\frac{|\boldsymbol{p}_T|}{M}\cos\left((m+1)\phi_h-\phi_p-m\phi_k\right)\times\left(f_{1T}^{\perp}\left(D_1^{|\ell,m\rangle}+D_1^{|\ell,-m\rangle}\right)+\chi(m)g_{1T}\left(D_1^{|\ell,m\rangle}-D_1^{|\ell,-m\rangle}\right)\right)\right],$$

$$F_{UT}^{P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)} = -\mathcal{I}\left[\frac{|\boldsymbol{k}_T|}{M_h}\cos\left((m-1)\phi_h-\phi_p-m\phi_k\right)h_1H_1^{\perp|\ell,m\rangle}\right],$$

$$F_{UT}^{P_{\ell,m}\sin((3-m)\phi_h+m\phi_R-\phi_S)} = \mathcal{I}\left[\frac{|\boldsymbol{p}_T|^2|\boldsymbol{k}_T|}{M^2M_h}\cos\left((m-3)\phi_h+2\phi_p-(m-1)\phi_k\right)h_{1T}^{\perp}H_1^{\perp|\ell,m\rangle}\right].$$

- Can test Lund/Artru model with $F_{UT}^{\sin^2\vartheta\sin(-\phi_h+2\phi_R+\phi_S)}$ and $F_{UT}^{\sin^2\vartheta\sin(3\phi_h-2\phi_R+\phi_S)}$ via transversity
- In theory, could also test Lund/Artru and gluon radiation models with $F_{UT}^{\sin^2\vartheta\sin(\phi_h+2\phi_R-\phi_S)}$ and $F_{UT}^{\sin^2\vartheta\sin(5\phi_h-2\phi_R-\phi_S)}$ via pretzelocity
- Data from SIDIS pseudo-scalar production indicate pretzelocity very small or possibly zero

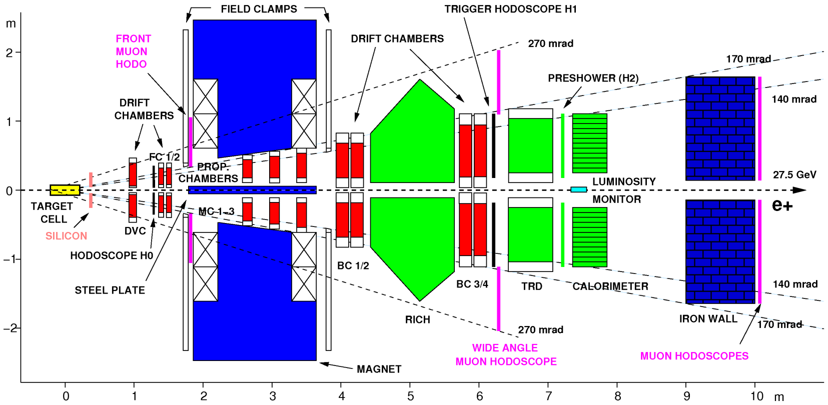
Collinear versus TMD Moments

- It is not the particulars of the DF or FF that make a moment survive in the collinear case, but rather the $\sum m = 0$ (necessary condition).
  - Moments with $h_1^\perp H_1^{\perp|\ell,m\rangle}$ (Boer-Mulders moments)
    - $h_1^\perp$ has $\chi \neq \chi'$, and thus $\Delta m = -1$
    - $H_1^\perp$ similarly has $\Delta m = -1$.
    - Final state polarization must have $m = 2$ in order that $\sum m = 0$.
    - Only surviving moment in collinear dihadron production is $|2, 2\rangle$.
  - Moments with $h_1 H_1^{\perp|\ell,m\rangle}$ (Collins moments)
    - h_1 has $\Delta m = 0$.
    - $H_1^\perp$ again has $\Delta m = -1$.
    - Collinear moments are $|1, 1\rangle$, $|2, 1\rangle$.
- Can also look for the m which cancels the ϕ_h dependence

$$F_{UU}^{P_{\ell,m}\cos((2-m)\phi_h+m\phi_R)} = -\mathcal{I}\left[\frac{|\boldsymbol{p}_T||\boldsymbol{k}_T|}{MM_h}\cos\left((m-2)\phi_h+\phi_p+(1-m)\phi_k\right) h_1^\perp H_1^{\perp|\ell,m\rangle}\right],$$

$$F_{UT}^{P_{\ell,m}\sin((1-m)\phi_h+m\phi_R+\phi_S)} = -\mathcal{I}\left[\frac{|\boldsymbol{k}_T|}{M_h}\cos\left((m-1)\phi_h-\phi_p-m\phi_k\right) h_1 H_1^{\perp|\ell,m\rangle}\right],$$

The HERMES Experiment



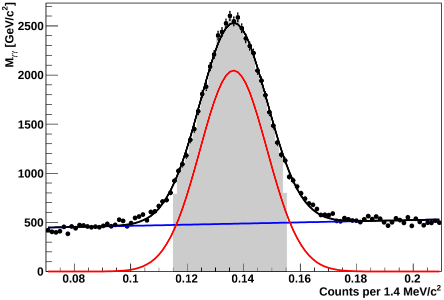
Beam Long. pol. $e^{\pm}$ at 27.6 GeV

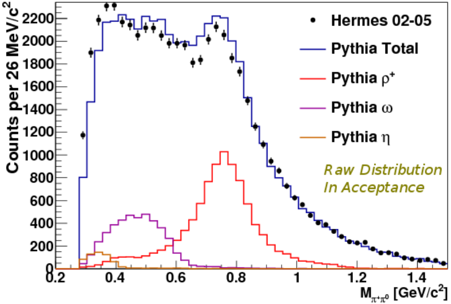
Target Trans. pol. H ($\approx$ 75%)
Long. pol. H ($\approx$ 85%)
Unpol. H,D,Ne,Kr,...

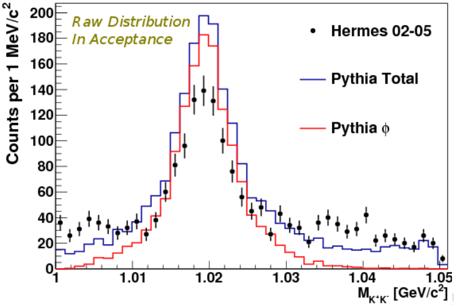
Lep.-Had. Sep. High efficiency $\approx$ 98%
Low contamination (<2%)

Hadron PID Separates $\pi^{\pm}$, $K^{\pm}$, p, $\bar{p}$
with momenta in 2-15 GeV

Particle Reconstruction







- Central value of π^0 peak is close to PDG value—observed width due to detector resolution
- Pythia vs data plots indicate the many subprocesses in $\pi\pi$-dihadron production
- K^+K^- much cleaner–only processes are one resonant (ϕ) and one non-resonant production

Analysis Details

- Considering final states of $\pi^+\pi^-$, $\pi^+\gamma\gamma$, $\pi^-\gamma\gamma$, K^+K^-
  - Need a model for TMDGen, and then could likewise analyze $K^+\pi^-$, $K^-\pi^+$, $K^+\gamma\gamma$, $K^-\gamma\gamma$.
- Need to correct for acceptance, which requires a new Monte Carlo generator and new TMD fragmentation functions.
- Correction applied for non-resonant $\gamma\gamma$ pairs.
- Integrated charge symmetric background $\lesssim 5\%$ and exclusive background $\lesssim 3.5\%$.
  - Effects determined to be negligible.
- Systematics include
  - Acceptance, smearing, and radiative effects
  - Dependence on the beam charge
  - Particle identification procedures

New TMDGEN Generator

- No previous Monte Carlo generator has TMD dihadron production with full angular dependence
- Method
  - Integrates cross section per flavor to determine "quark branching ratios"
  - Throw a flavor type according to ratios
  - Throw kinematic/angular variables by evaluating cross section
    - Can use weights or acceptance rejection
  - Full TMD simulation: each event has specific $|\boldsymbol{p}_T|$, ϕ_p, $|\boldsymbol{k}_T|$, ϕ_k values
  - Includes both pseudo-scalar and dihadron SIDIS cross sections
- Guiding plans
  - Extreme flexibility
    - Allow many models for fragmentation and distribution functions
    - Various final states: pseudo-scalars, vector mesons, hadron pairs, etc.
    - Output options & connecting to analysis chains of various experiments
    - Minimize dependencies on other libraries
  - Full flavor and transverse momentum dependence.
- Current C++ package considered stable and allows further expansion
- Can be useful for both experimentalists and theorists.

Acceptance/Smearing

- One could do two step process
  1. Unfold the yield $\boldsymbol{y} = S\boldsymbol{x}$
  2. Solve for moments $\boldsymbol{x} = X\boldsymbol{\alpha}$
- Or do all at once by solving $\boldsymbol{y} = SX\boldsymbol{\alpha}$
- Or unfold in parameter space via $X^{-1}\boldsymbol{y} = X^{-1}SX\boldsymbol{\alpha} \Leftrightarrow \boldsymbol{\beta} = S'\boldsymbol{\alpha}$
- In practice, we solve $\boldsymbol{b} = B\boldsymbol{\alpha}$ with

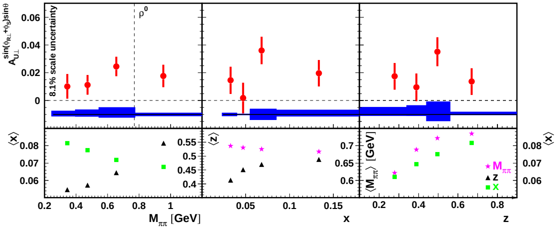
$$b_i = \frac{V}{N_R}\sum_{k=1}^{N_R} f_i\left(\boldsymbol{x}^{(R,k)}\right), \qquad \left(C^{\boldsymbol{b}}\right)_{j,j'} = \frac{\delta_{j,j'}}{N_R - 1}\left[\frac{V^2}{N_R}\sum_{k=1}^{N_R} f_i^2\left(\boldsymbol{x}^{(R,k)}\right) - (b_i)^2\right],$$

$$B_{i,j} = \frac{V^3}{N_{MC}}\sum_{k=1}^{N_{MC}} f_i\left(\boldsymbol{x}^{(R,k)}\right) f_j\left(\boldsymbol{x}^{(T,k)}\right), \qquad \left(C^{B}\right)_{j,k;j',k'} = \frac{\delta_{j,j'}\delta_{k,k'}}{N_\epsilon - 1}\left[\frac{V^4}{N_\epsilon}\sum_{k=1}^{N_\epsilon} f_j^2\left(\boldsymbol{x}^{(M,k)}\right) f_k^2\left(\boldsymbol{x}^{(T,k)}\right) - (B_{j,k})^2\right].$$

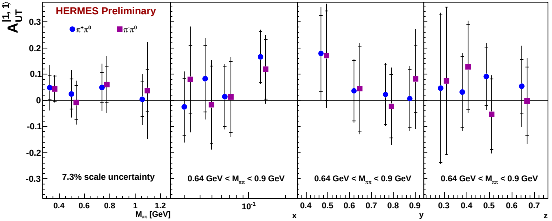
- The fit is applied over the angular variables in several different binning options:
  - 1D M_{hh} bins or various 2D bins: M_{hh} and one of $\{x, y, z, P_{h\perp}\}$
- We "unfold" acceptance only using TMDGen, thus $\boldsymbol{x}^{(R)} = \boldsymbol{x}^{(T)}$ and $B = B^T$.
- Basis consists of 24 unpolarized and 18 polarized moments

$|1,1\rangle$ Moment for $\pi\pi$ Dihadrons

Published $\pi^+\pi^-$ Results

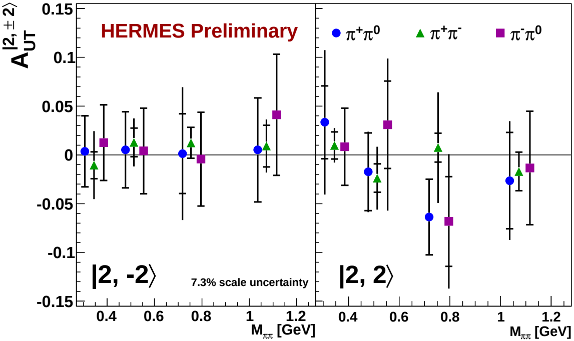


New $\pi^\pm\pi^0$ Results



- Signs of moments are consistent for all $\pi\pi$ dihadron species.
- Statistics are much more limited for $\pi^\pm\pi^0$ dihadrons.
- Despite uncertainties, may still help constrain global fits.

$|2, \pm 2\rangle$ Moments for $\pi\pi$ Dihadrons



- $|2, -2\rangle$ moment very consistent with zero for all flavors
- Results for $|2, 2\rangle$ are consistent with expectations
  - No indication of any signal outside the ρ-mass bin
  - Negative moments for $\rho^{\pm}$, very small ρ^0 moments
  - Results are sufficiently suggestive to merit measurements at current experiments.

Conclusions and Outlook

- First preliminary results for transverse target moments of dihadron production
- Current work continues on the finalization and publication of these preliminary results
- Transverse momentum dependent $|2, \pm 2\rangle$ moments related to string models of fragmentation
  - Measurements are consistent with models
  - Results point towards needing a higher statistic data set
- Measured $|1, 1\rangle$ moments allow collinear access to transversity
  - These additional $\pi^{\pm}\pi^{0}$ species will assist in the u-d flavor separation
- Future work with $K^{+}K^{-}$
  - Little data near ϕ-mass, but much more for $M_{KK} > 1.05$ GeV
  - Can again measure $|1, 1\rangle$ to access to strange flavor of transversity
  - Sivers moments related to strange flavor of Sivers function.
- Also have data for πK-dihadrons
  - However, we are lacking a fragmentation function model.

Backup Slides

Partial Wave Expansion

- Fragmentation functions expanded into partial waves in the direct sum basis according to

$$
\begin{aligned}
D_1 &= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos\vartheta) e^{im(\phi_R-\phi_k)} D_1^{|\ell,m\rangle}(z, M_h, |\boldsymbol{k}_T|), \\
H_1^{\perp} &= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} P_{\ell,m}(\cos\vartheta) e^{im(\phi_R-\phi_k)} H_1^{\perp|\ell,m\rangle}(z, M_h, |\boldsymbol{k}_T|),
\end{aligned}
$$

- Each term in pseudo-scalar and dihadron cross section uniquely related to a specific partial wave $|\ell, m\rangle$.
- Cross section looks the same for all final states, excepting certain partial waves may or may not be present
  - Pseudo-scalar production is $\ell = 0$ sector
  - Dihadron production is $\ell = 0, 1, 2$ sector
  - Given the pseudo-scalar cross section (at any twist) can extrapolate cross section for other final states

Rigorous Definitions

- Fragmentation Correlation Matrix

$$\Delta_{mn}(P_h, S_h; k) = \sum_X \int \frac{d^4x}{(2\pi)^4} e^{ik\cdot x} \langle 0 | \Psi_m(x) | P_h, S_h; X \rangle \langle P_h, S_h; X | \overline{\Psi}_n(0) | 0 \rangle$$

- Trace Notation

$$\Delta^{[\Gamma]}(z, M_h, |\boldsymbol{k}_T|, \cos\vartheta, \phi_R - \phi_k) = 4\pi \frac{z|\boldsymbol{R}|}{16 M_h} \int dk^+ \, \mathrm{Tr}\left[\Gamma \Delta(k, P_h, R)\right]\Big|_{k^- = P_h^-/z}.$$

- Define fragmentation functions via trace relations

	Previous Definitions		New Definition
FF	Pseudo-Scalar	Dihadron	All Final States
D_1	$\Delta^{[\gamma^-]}$	$\Delta^{[\gamma^-]}$	$\Delta^{[\gamma^-(1+i\gamma^5)]}$
$G_1^\perp$	$--$	$\propto \Delta^{[\gamma^-\gamma^5]}$	$--$
$H_1^\perp$	$\Delta^{[(\sigma^{1-})\gamma^5]}$	$\Delta^{[(\sigma^{1-})\gamma^5]}$	$\Delta^{[(\sigma^{1-}+i\sigma^{2-})\gamma^5]}$
$\bar{H}_1^{\sphericalangle}$	$--$	$\propto \Delta^{[(\sigma^{2-})\gamma^5]}$	$--$

Relation with Previous Notation

- Real part of fragmentation function similar
- New definition of D_1 & $H_1^\perp$
  - Adds "imaginary" part to D_1 & $H_1^\perp$, instead of introducing new functions.
  - Functions are complex valued and depend on Q^2, z, $|k_T|$, M_h, $\cos\vartheta$, $(\phi_R - \phi_k)$.
- Comparing with similar trace definitions, e.g. PRD 67:094002, yields the relations

$$D_1\Big|_{Gliske} = \left[D_1 + i\frac{|\boldsymbol{R}||\boldsymbol{k}_T|}{M_h^2}\sin\vartheta\sin(\phi_R - \phi_k)\, G_1^\perp\right]_{other},$$

$$H_1^\perp\Big|_{Gliske} = \left[H_1^\perp + \frac{|\boldsymbol{R}|}{|\boldsymbol{k}_T|}\sin\vartheta e^{i(\phi_R-\phi_k)}\,\bar{H}_1^{\sphericalangle}\right]_{other} = \frac{|\boldsymbol{R}|^2}{|\boldsymbol{k}_T|^2}H_1^{\sphericalangle}\Big|_{other},$$

- Note: there are inconsistencies in the literature between definitions of $H_1^{\sphericalangle}$, $\bar{H}_1^{\sphericalangle}$, and $H_1'^{\sphericalangle}$.

Collinear Dihadron Spectator Model

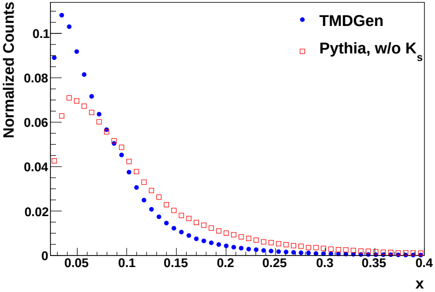
- Based on Bacchetta/Radici spectator model for collinear dihadron production *Phys. Rev.* D74 11 (2006) 114007
  - The SIDIS X is replaced with a single, on-shell, particle of mass $M_s \propto M_h$.
  - Assume one spectator for hadron pairs and vector mesons.
  - Integration over transverse momenta is performed before extracting fragmentation functions.
- One can use the same correlator to extract TMD fragmentation functions
  - One just needs to not integrate and follow the Dirac-matrix algebra and partial wave expansion.
  - Numeric studies show need for additional $\boldsymbol{k}_T$ cut-off.
- Original model intended for $\pi^+\pi^-$ pairs
  - Adding flavor dependence allows generalization to $\pi^+\pi^0$, $\pi^-\pi^0$ pairs.
  - Slight change to vertex function allows generalization to K^+K^- pairs.
  - Slight change to vertex function and allows generalization to K^+K^- pairs.
- The model only includes partial waves of the Collins function for $\ell < 2$.
- Model cannot easily be extended to mixed mass pairs $(K\pi)$

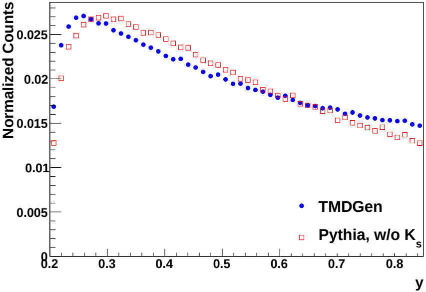
Available Models in TMDGen

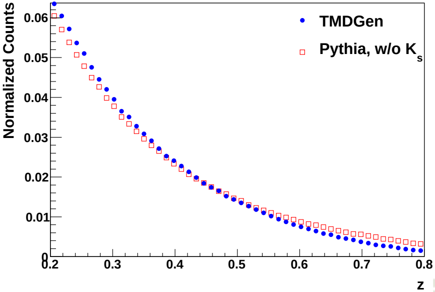
Distribution Functions	Model Identifier
f_1	CTEQ
f_1	LHAPDF
f_1	BCR08
f_1	GRV98
g_1	GRSV2000
$f_{1T}, h_{1T}^{\perp}, h_1$	Torino Group
$f_1, g_1, g_{1L}, g_{1T}, f_{1T}, h_1, h_1^{\perp}, h_{1T}^{\perp}$	Pavia Spectator Model

Frag. Functions	Final State	Model Identifier
D_1	pseudo-scalar	fDSS
D_1	pseudo-scalar	Kretzer
$D_1, H_1^{\perp}$	dihadron	Spectator Model
$D_1, H_1^{\perp}$	dihadron	Set given partial wave proportional to any other partial wave

$\pi^+\pi^0$ Kinematic Distributions, p.1

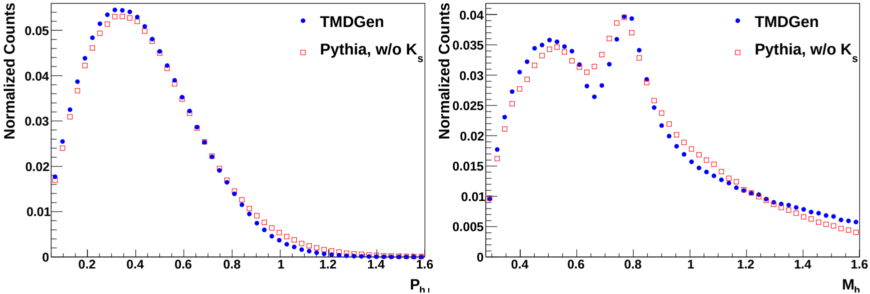






- Close agreement for x, y, z distributions.
- Main discrepancy in x—may be due to imbalance in the flavor contributions, or Q^2 effects.
- Similar results for other $\pi\pi$ and KK dihadrons.

$\pi^+\pi^0$ Kinematic Distributions, p.2



- Fairly good agreement in both $P_{h\perp}$ and M_h distributions.
- Note: some discrepancies in full $5D$ kinematic, but PYTHIA also doesn't match data in full $5D$

Smearing/Acceptance Effects

- Let $\boldsymbol{x}^{(T)}$ be true value of variables, $\boldsymbol{x}^{(R)}$ the reconstructed values
- A conditional probability $p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right)$ relates the true PDF $p\left(\boldsymbol{x}^{(T)}\right)$ with the PDF of the reconstructed variables, $p\left(\boldsymbol{x}^{(R)}\right)$.
- Specific relation given by Fredholm integral equation

$$
\begin{aligned}
p\left(\boldsymbol{x}^{(R)}\right) &= \eta \int d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) p\left(\boldsymbol{x}^{(T)}\right), \\
\frac{1}{\eta} &= \int d^D\boldsymbol{x}^{(R)} d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) p\left(\boldsymbol{x}^{(T)}\right).
\end{aligned}
$$

- Can rewrite in terms of a smearing operator

$$
S\left[g(\boldsymbol{x}^{(T)})\right] = \int d^D\boldsymbol{x}^{(T)}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) g\left(\boldsymbol{x}^{(T)}\right).
$$

- Fredholm equation is simply

$$
p\left(\boldsymbol{x}^{(R)}\right) = S\left[\eta p\left(\boldsymbol{x}^{(T)}\right)\right].
$$

Solution with Finite Basis and Integrated Squared Error

- Restrict to finite basis

$$
\begin{aligned}
\eta p\left(\boldsymbol{x}^{(T)}\right) &= \sum_i \alpha_i f_i\left(\boldsymbol{x}^{(T)}\right), \\
p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) &= \sum_{i,j} \Gamma_{i,j} f_i\left(\boldsymbol{x}^{(R)}\right) f_j\left(\boldsymbol{x}^{(T)}\right).
\end{aligned}
$$

- Determine parameters by minimizing the integrated squared error (ISE)

$$
\begin{aligned}
ISE_1 &= \int d^D\boldsymbol{x}^{(R)} d^D\boldsymbol{x}^{(T)} \left[p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) - \sum_{i,j} \Gamma_{i,j} f_i(\boldsymbol{x}^{(R)}) f_j(\boldsymbol{x}^{(T)})\right]^2, \\
ISE_2 &= \int d^D\boldsymbol{x}^{(R)} \left\{p\left(\boldsymbol{x}^{(R)}\right) - \sum_i \alpha_i S\left[f_i\left(\boldsymbol{x}^{(T)}\right)\right]\right\}^2.
\end{aligned}
$$

Analytic Solution

- Define/compute

$$
\begin{aligned}
F_{i,j} &= \int d^D \boldsymbol{x}^{(\boldsymbol{T})} f_i\left(\boldsymbol{x}^{(\boldsymbol{T})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
B_{i,j} &= \int d^D \boldsymbol{x}^{(\boldsymbol{R})} d^D \boldsymbol{x}^{(\boldsymbol{T})}\, p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) f_i\left(\boldsymbol{x}^{(\boldsymbol{R})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
&= V \int d^D \boldsymbol{x}^{(\boldsymbol{R})} d^D \boldsymbol{x}^{(\boldsymbol{T})}\, p_{MC}\left(\boldsymbol{x}^{(T)}, \boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(\boldsymbol{R})}\right) f_j\left(\boldsymbol{x}^{(\boldsymbol{T})}\right), \\
b_i &= \int d^D \boldsymbol{x}^{(R)} p\left(\boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(R)}\right).
\end{aligned}
$$

- ISEs reduce to the matrix equation

$$B^T F^{-1} B \boldsymbol{\alpha} = B^T F^{-1} \boldsymbol{b}.$$

- Assuming B is invertible, this reduces to $B\boldsymbol{\alpha} = \boldsymbol{b}$.
- Note: the least squares solution, ignoring smearing, is $F\boldsymbol{\alpha} = \boldsymbol{b}$.

Numeric Solution

- The quantities can be computed as

$$
\begin{aligned}
b_i &= \frac{V}{N_R} \sum_{k=1}^{N_R} f_i\left(\boldsymbol{x}^{(R,k)}\right), \\
B_{i,j} &= \frac{V^3}{N_{MC}} \sum_{k=1}^{N_{MC}} f_i\left(\boldsymbol{x}^{(R,k)}\right) f_j\left(\boldsymbol{x}^{(T,k)}\right).
\end{aligned}
$$

- Use standard methods to solve $B\boldsymbol{\alpha} = \boldsymbol{b}$.
- One is simply unfolding in the parameter space.

Uncertainty Calculation

- Define

$$\left(C^{\boldsymbol{b}}\right)_{j,j'} = \frac{\delta_{j,j'}}{N_R - 1}\left[\frac{V^2}{N_R}\sum_{k=1}^{N_R} f_i^2\left(\boldsymbol{x}^{(R,k)}\right) - (b_i)^2\right],$$

$$\left(C^{B}\right)_{j,k;j',k'} = \frac{\delta_{j,j'}\delta_{k,k'}}{N_\epsilon - 1}\left[\frac{V^6}{N_\epsilon}\sum_{k=1}^{N_\epsilon} f_j^2\left(\boldsymbol{x}^{(M,k)}\right) f_k^2\left(\boldsymbol{x}^{(T,k)}\right) - (B_{j,k})^2\right],$$

$$C'^{(B)}_{i,i'} = \sum_{j,j'} C^{(B)}_{i,j;i',j'}\alpha_j\alpha_{j'}.$$

- The uncertainty on $\boldsymbol{\alpha}$ is then

$$C^{(\boldsymbol{\alpha})} = B^{-1}C^{(\boldsymbol{b})}B^{-T} + B^{-1}C'^{(B)}B^{-T}.$$

- One could consider a third term $\left(B^T F^{-1} B\right)^{-1}$, the Hessian of the matrix eq.
  - Numeric studies show this term is not a meaningful estimate of the uncertainty, and that it can be neglected.

Alternate Derivation

- Again, assume that $p\left(\boldsymbol{x}^{(R)} \middle| \boldsymbol{x}^{(T)}\right) = Vp(\boldsymbol{x}^{(R)}, \boldsymbol{x}^{(T)})$.
- Substitute $\eta p\left(\boldsymbol{x}^{(T)}\right) = \sum_i \alpha_i f_i\left(\boldsymbol{x}^{(T)}\right)$ into the Fredholm integral equation:

$$p\left(\boldsymbol{x}^{(R)}\right) = V \sum_i \alpha_i \int d^D\boldsymbol{x}^{(T)}\, p_{MC}\left(\boldsymbol{x}^{(T)}, \boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(T)}\right).$$

- Applying the operator $\int d^D\boldsymbol{x}^{(R)} f_j\left(\boldsymbol{x}^{(R)}\right)$ to both sides yields

$$\int d^D\boldsymbol{x}^{(R)} f_j\left(\boldsymbol{x}^{(R)}\right) p\left(\boldsymbol{x}^{(R)}\right) = V \sum_i \alpha_i \int d^D\boldsymbol{x}^{(R)} d^D\boldsymbol{x}^{(T)}\, p_{MC}\left(\boldsymbol{x}^{(T)}, \boldsymbol{x}^{(R)}\right) f_i\left(\boldsymbol{x}^{(T)}\right),$$

- Using the definitions of $\boldsymbol{b}$ and B, this reduces to

$$\boldsymbol{b} = B\boldsymbol{\alpha}.$$