Some remarks on dipole showers and the DGLAP equation

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Introduction: **Event generators and perturbative calculations**

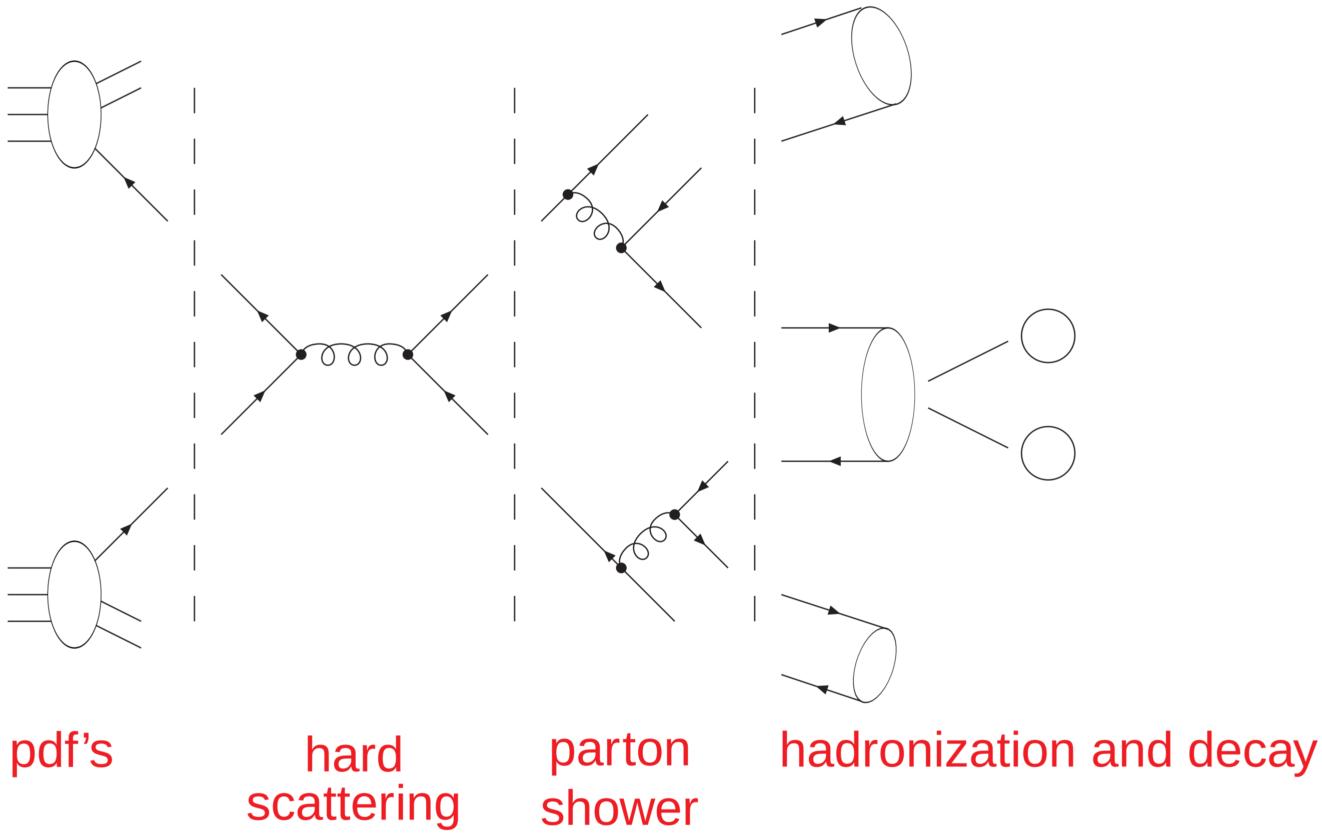
I.: **Parton showers from dipoles**

II: **The DGLAP equation**

Phys. Rev. D76, (2007), 094003, arxiv:0709.1026 [hep-ph], (with M. Dinsdale and M. Ternick)

Phys. Rev. D79, (2009), 074021, arxiv:0903.2150 [hep-ph], (with P. Skands)

Event generators



Underlying event: Interactions of the proton remnants.

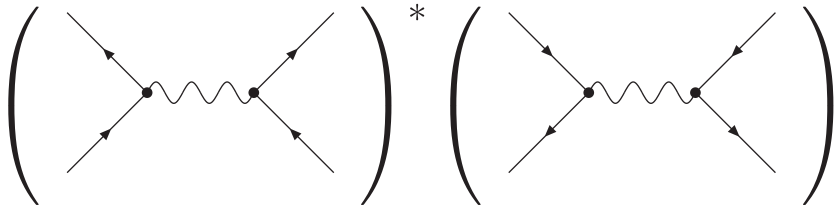
Multiple interactions: more than one pair of partons undergo hard scattering

Pile-up events: more than one hadron-hadron scattering within a bunch crossing

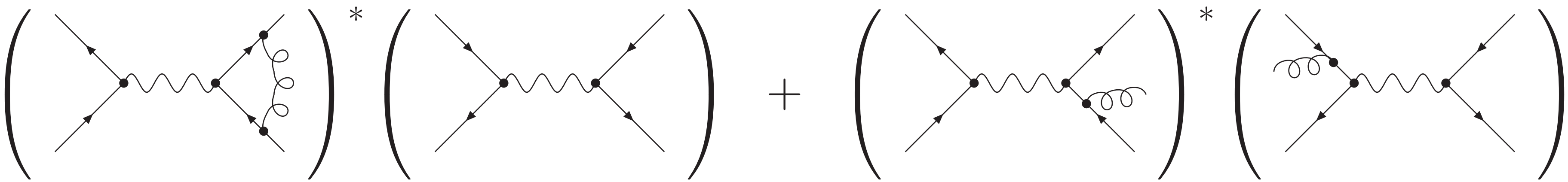
Exact perturbative calculations

Leading order (LO) and next-to-leading order (NLO):

At leading order only Born amplitudes contribute:



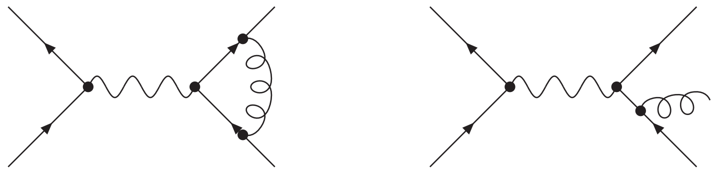
At next-to-leading order: One-loop amplitudes and Born amplitudes with an additional parton.

2 Re 

Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\sigma^{NLO} = \int_{n+1} d\sigma^R + \int_n d\sigma^V$$
$$= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

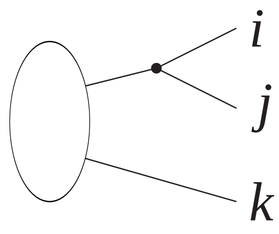
- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

The subtraction terms

The approximation term $d\sigma^A$ is given as a sum over dipoles:

$$d\sigma^A = \sum_{pairs\; i,j} \sum_{k \neq i,j} \mathcal{D}_{ij,k}.$$



Each dipole contribution has the following form:

$$\mathcal{D}_{ij,k} = -\frac{1}{2p_i \cdot p_j} \mathcal{A}_n^{(0)\,*}\left(p_1,\dots,\tilde{p}_{(ij)},\dots,\tilde{p}_k,\dots\right) \frac{\mathbf{T}_k \cdot \mathbf{T}_{ij}}{\mathbf{T}_{ij}^2} V_{ij,k} \mathcal{A}_n^{(0)}\left(p_1,\dots,\tilde{p}_{(ij)},\dots,\tilde{p}_k,\dots\right).$$

- Colour correlations through $\mathbf{T}_k \cdot \mathbf{T}_{ij}$.
- Spin correlations through $V_{ij,k}$.

The dipoles have the correct soft and collinear limit.

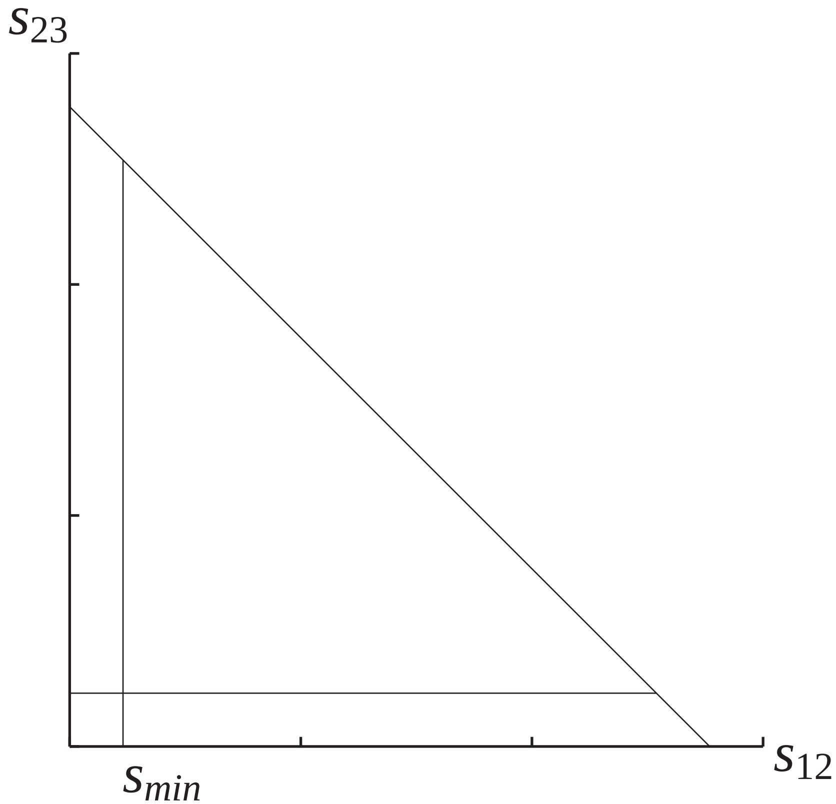
An example: $e^+e^- \longrightarrow$ 2 jets at NLO

The matrix element squared for $\gamma^* \longrightarrow qg\bar{q}$:

$$M_3 = 8(1-\varepsilon)\left[2\frac{s_{123}^2}{s_{12}s_{23}} - 2\frac{s_{123}}{s_{12}} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{23}}{s_{12}} + (1-\varepsilon)\frac{s_{12}}{s_{23}} - 2\varepsilon\right]$$

The dipole subtraction terms:

$$\mathcal{D}_{12,3} + \mathcal{D}_{32,1} = 8(1-\varepsilon) \left\{ \left[2\frac{s_{123}^2}{s_{12}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{12}} + (1-\varepsilon)\frac{s_{23}}{s_{12}}\right] + \left[2\frac{s_{123}^2}{s_{23}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{12}}{s_{23}}\right] \right\}$$

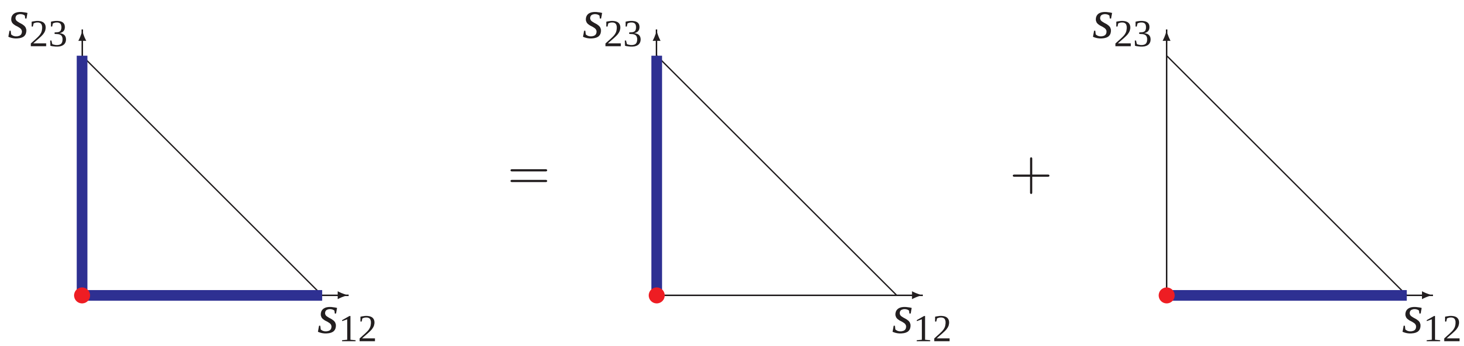


The antenna subtraction term:

$$\mathcal{A}_{123} = \mathcal{D}_{12,3} + \mathcal{D}_{32,1}$$

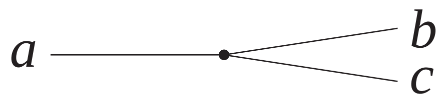
The Dalitz plot

$$\mathcal{A}_{123} = \mathcal{D}_{12,3} + \mathcal{D}_{32,1}$$



Basics of shower algorithm

Starting point: Collinear factorization.



Probability for particle a to split into particles b and c:

$$d\mathcal{P}_a = \sum_{b,c} \frac{\alpha_s}{2\pi} P_{a\to bc}(z)\,dt\,dz, \qquad t = \ln\left(\frac{Q^2}{\Lambda^2}\right)$$

Splitting kernels:

$$\begin{aligned} P_{q\to qg}(z) &= C_F \frac{1+z^2}{1-z}, \\ P_{g\to gg}(z) &= C_A \frac{(1-z(1-z))^2}{z(1-z)}, \\ P_{g\to q\bar{q}}(z) &= T_R\left(z^2+(1-z)^2\right). \end{aligned}$$

$P_{q\to qg}$ has a soft singularity for $z \longrightarrow 1$, $P_{g\to gg}$ has a soft singularity for $z \longrightarrow 1$ and $z \longrightarrow 0$.

The Sudakov factor

Probability that a branching occurs during a small range of t:

$$dI(t) \quad = \quad dt \int_{z_-(t)}^{z_+(t)} dz \sum_{b,c} \frac{\alpha_s}{2\pi} P_{a\to bc}(z),$$

Sudakov factor: Probability that no branching occurs between t_0 and t_1:

$$\Delta(t_1, t_0) \quad = \quad \exp\left(- \int_{t_0}^{t_1} dt \int_{z_-(t)}^{z_+(t)} dz \sum_{b,c} \frac{\alpha_s}{2\pi} P_{a\to bc}(z) \right)$$

A typical shower algorithm

- Choose the next scale t according to the Sudakov factor.
- Choose the momentum fraction z according to $P_{a\to bc}(z)$.
- Choose the azimuthal angle uniform or according to spin-dependent splitting functions.
- Insert the new particle.
- If $t > t_{min}$ goto first step, otherwise stop.

Angular ordering

Amplitude for the emission of a soft gluon from a q-$\bar{q}$-antenna:

$$d\sigma_g = d\sigma_0 \frac{\alpha_s C_F}{\pi} \frac{dk^0}{k^0} \frac{d\phi}{2\pi} d\cos\theta \frac{1-\cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{g\bar{q}})}$$

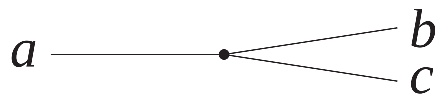
$$\frac{1-\cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{g\bar{q}})} = W_q + W_{\bar{q}}, \qquad W_q = \frac{1}{2}\left[\frac{\cos\theta_{g\bar{q}} - \cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{g\bar{q}})} + \frac{1}{(1-\cos\theta_{qg})}\right],$$

$$W_{\bar{q}} = \frac{1}{2}\left[\frac{\cos\theta_{qg} - \cos\theta_{q\bar{q}}}{(1-\cos\theta_{qg})(1-\cos\theta_{g\bar{q}})} + \frac{1}{(1-\cos\theta_{g\bar{q}})}\right].$$

$$\int \frac{d\phi}{2\pi} W_q = \begin{cases} \frac{1}{1-\cos\theta_{qg}}, & \text{if } \theta_{qg} < \theta_{q\bar{q}}, \\ 0 & \text{otherwise} \end{cases}$$

Angular ordering: No emission if $\theta_{qg} > \theta_{q\bar{q}}$!

Momentum conservation



1. Momentum conservation:

$$p_a \quad = \quad p_b + p_c$$

2. Momenta are on-shell, for massless particles:

$$p_a^2 = p_b^2 = p_c^2 = 0.$$

3. Momenta are real.

For $1 \longrightarrow 2$ splittings it is not possible to satisfy all three requirements.

Recent developments

- Rewriting and improvement of Pythia, Herwig and Ariadne, Sherpa as a new event generator

  Sjöstrand, Skands; Gieseke, Stephens, Webber; Lönnblad, Krauss, Kuhn, Schälicke, Soff;

- Uncertainties of parton showers

  Gieseke; Stephens, van Hameren; Bauer, Tackmann

- Parton showers in the framework of SCET

  Bauer, Schwartz, Tackmann, Thaler

- Matching of parton showers with fixed-order tree level matrix elements

  Catani, Krauss, Kuhn, Webber; Mangano, Moretti, Pittau; Mrenna and Richardson;

- Matching of parton showers with NLO: $\longrightarrow$ next page

Recent developments

- Matching of parton showers with NLO

  Frixione, Gieseke, Laenen, Latunde-Dada, Motylinski, Nason, Oleari, Ridolfi, Webber; Krämer, Mrenna, Soper; Odaka, Kurihara; Giele, Kosower, Skands;

  – Parton shower based on the dipole formalism
    - ∗ Foundation: Nagy, Soper, ’05, ’06
    - ∗ For electron-positron annihilation very similar to Ariadne.
    - ∗ Implementation: Schumann, Krauss ’07 and Dinsdale, Ternick, SW. ’07

Parton shower based on the dipole formalism

$2 \longrightarrow 3$ splittings: An emitter-spectator pair radiates off an additional particle. Can satisfy momentum conservation and on-shell conditions.

Splitting kernels of the Sudakov factors are given by the dipole splitting functions. Correct behaviour in the collinear and the soft limit.

No conceptional distinction between initial- and final-state shower.

Natural choice to combine with NLO

Technical details

- 4 cases for emitter-spectator-pair: final-final, final-initial, initial-final, initial-initial.
- Only the singular terms of the dipole splitting functions are unique.
- Freedom to choose the finite terms.
- For a parton shower we would like to have a probabilistic interpretation:
  The splitting functions have to be non-negative everywhere.
  - Adjust finite terms
  - Rearrange terms between $\mathcal{D}_{ij,k}$ and $\mathcal{D}_{kj,i}$

Part II : The DGLAP equation

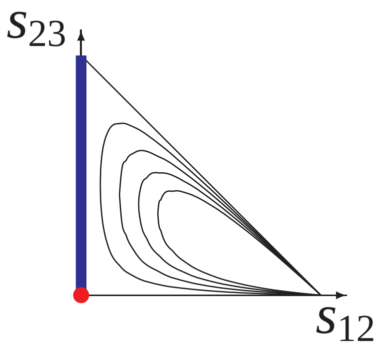
Motivation: A recent paper by Y. Dokshitzer and G. Marchesini

- Infrared sensible evolution variable
- The quark fragmentation function
- Numerical results

Infrared sensible evolution variable

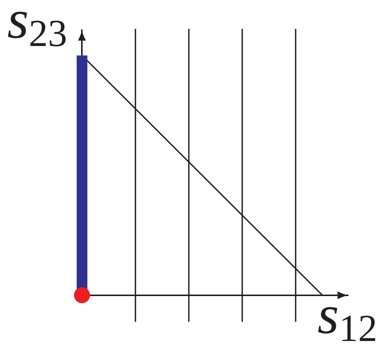
Infrared sensible: Both infinitely soft and collinear emissions should be classified as unresolved for any finite value of the evolution variable.

Examples:



$k_\perp$-ordered

$$e^{-t} = \frac{-k_\perp^2}{Q^2}$$



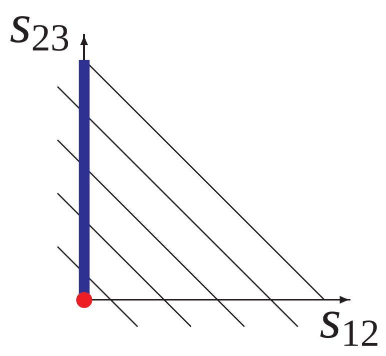
virtuality-ordered

$$e^{-t} = \frac{s_{12}}{Q^2}$$

Energy-ordering as evolution variable

Energy of emitted gluon as evolution variable:

$$e^{-t} = \frac{E_g^2}{Q^2}$$



This is not a sensible choice:

- For finite shower time t the splitting probability is infinite.
- In the limit $t \to \infty$ the integration over y_{23} reduces to a point and not to the integral over the splitting function.
- No resemblance in the collinear limit with Altarelli-Parisi splitting function.

The non-singlet quark fragmentation function

DGLAP evolution equation:

$$Q^2 \frac{d}{dQ^2} d(x, Q^2) = -\int_x^1 \frac{dz}{z} \frac{\alpha_s}{2\pi} C_F P_{qq}(z) d\left(\frac{x}{z}, Q^2\right)$$

Moments:

$$\tilde{d}(N, Q^2) = \int_0^1 dx\, x^{N-1} d(x, Q^2)$$

In Mellin space evolution equation factorises:

$$Q^2 \frac{d}{dQ^2} \tilde{d}(N, Q^2) = -\frac{\alpha_s}{2\pi} C_F \tilde{P}_{qq}(N) \tilde{d}(N, Q^2)$$

The non-singlet quark fragmentation function

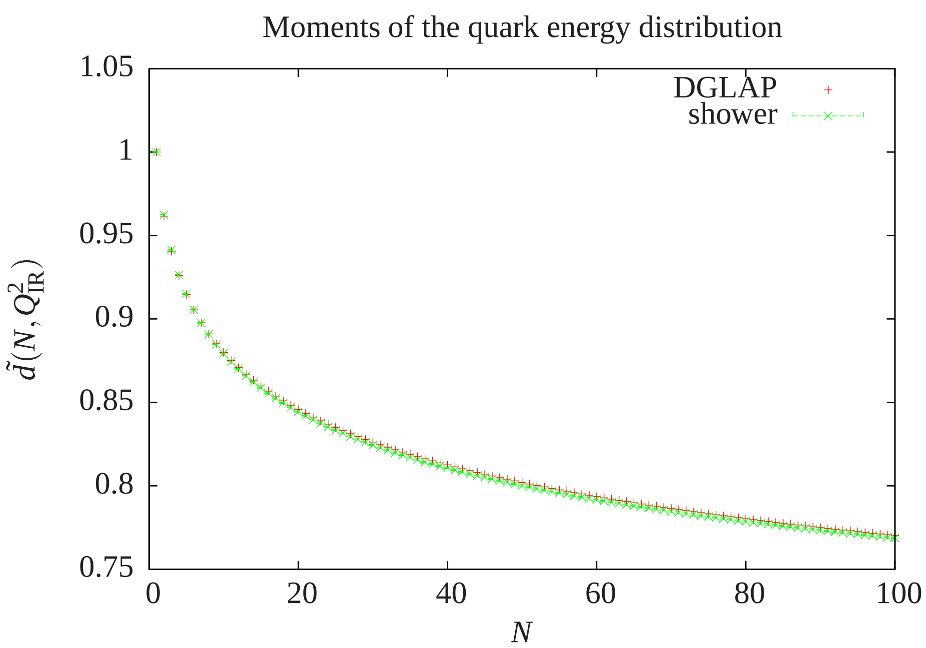
Testing single emissions in the strongly-ordered limit:

Centre-of-mass energy: $Q = m_Z$

Shower starting scale: $Q_0 = 2\text{GeV}$

Shower cut-off scale: $Q_{IR} = 1\text{GeV}$

Perfect agreement!



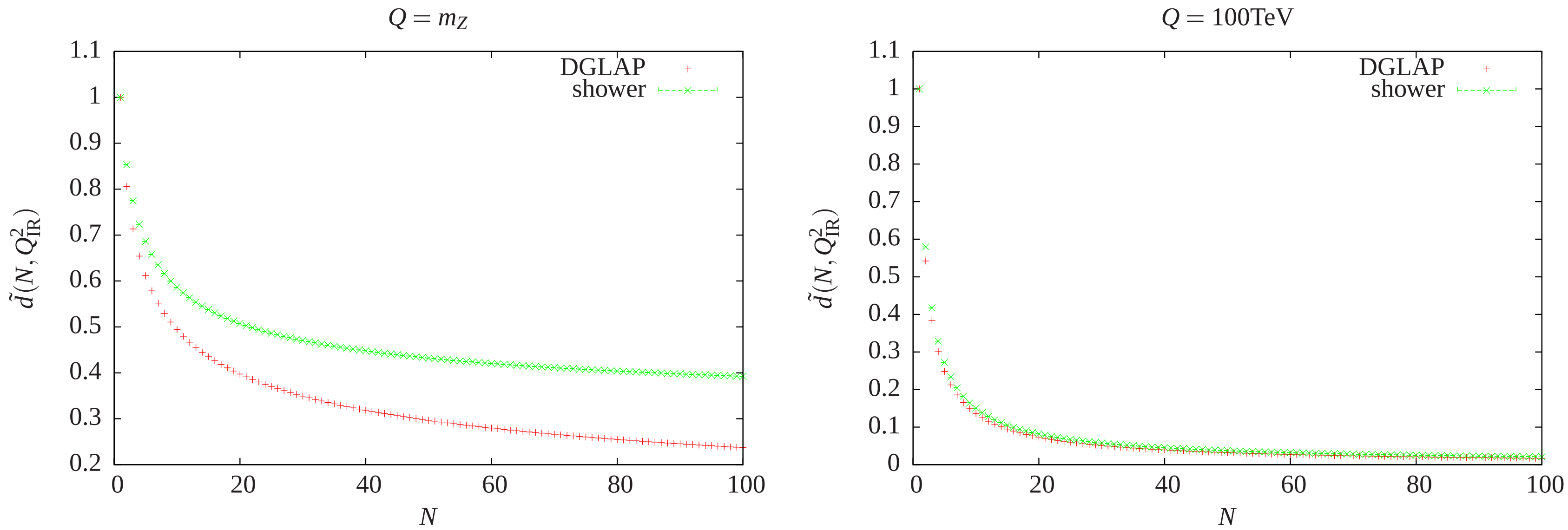
Analytic derivation: Nagy, Soper, '09

The non-singlet quark fragmentation function

Testing the leading logarithm of multiple emissions:

Shower starting scale: $Q_0 = Q$

Shower cut-off scale: $Q_{IR} = 1\text{GeV}$

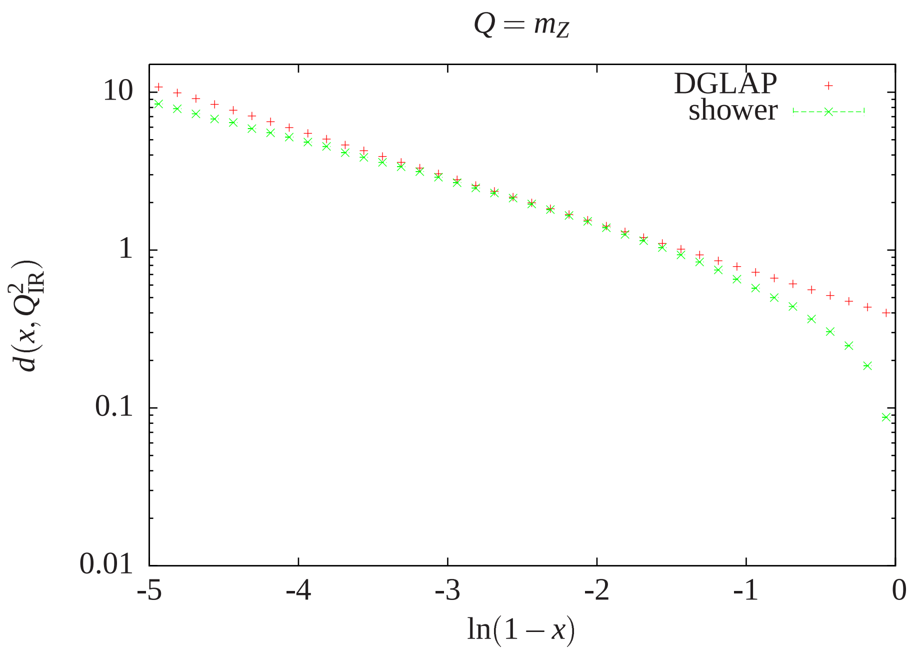


The non-singlet quark fragmentation function

Testing soft emissions in x-space:

Approximate analytical solution:

$$\ln d(x, Q^2) \quad = \quad A \ln(1-x) + B,$$



Summary

Implementation of a new parton shower algorithm based on the dipole formalism.

Transverse momentum as evolution variable.

Momentum conservation and “angular ordering” are inherent.

Initial- and final-state partons are treated on the same footing.

Natural choice to combine with NLO.

Detailed study that this shower reproduces the DGLAP equation